MATHEMATICAL MODELLING OF JOINTED ROCK MASSES

by

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DOCTORAL THESIS 1985:42 D

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MATHEMATICAL MODELLING
OF
JOINTED ROCK MASSES

av

Thomas Olofsson

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MATHEMATICAL MODELLING
OF
JOINTED ROCK MASSES

Thomas Olofsson

Division of Rock Mechanics
Division of Structural Engineering

Luleå 1985
PREFACE

The work presented in this thesis was carried out between 1980 and 1985 at the Divisions of Rock Mechanics and Structural Engineering, Luleå University of Technology.

I wish to express my thanks to my supervisor Professor Ove Stephansson for his support and guidance during the accomplishment of this work and for proposing several improvements of the manuscript.

Special acknowledgements are given to the recently deceased Res. Ass. Hans Larsson who has contributed to this work by many valuable suggestions and who shared the work of the theoretical model in the initial part of the project.

Further, I would like to thank Professor Lennart Elfgren for his keen reading and proposals of improvements in the manuscript. The discussions with Dr Charles Gerrard, CSIRO, Australia, are greatly acknowledged. I would like to thank Res. Ass. James Mathis for his critical review and linguistic correction, Miss Carina Hannu for typing the manuscript and Mrs Marianne Johansson for drawing the figures and to all others at the Luleå University of Technology that have been helpful during the accomplishment of the work. Finally, I would like to thank my wife Eva and our two children Ida and Peter for their encouragement and patience during this time.

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Luleå, May 1985

Thomas Olofsson
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SUMMARY

In this thesis, a theoretical model of the mechanical behaviour of jointed rock masses is developed. An equivalent material approach is used to formulate the constitutive equations, where the structural components, intact rock and joints are assigned continuous material properties.

The elastic and inelastic properties of the joints are modelled by an elasto-viscoplastic formulation. The model can be used to study general stress and strain paths for both two- and three-dimensional structures based on constitutive equations, i.e. stress-strain relations or in finite element codes. The rock mass model using the equivalent material approach can be applied to hard rock masses with several sets of intersecting continuous joints.

The theoretical model developed for a single joint can also be used for discrete formulation of joint elements in finite element codes, cf. chapter 3.

The intact rock is treated as a linearly elastic material. The elastic behaviour of the joint is modelled with a constant stiffness matrix. The onset of plastic flow is initiated when

- the normal stress exceeds the normal compressive strength of the joint asperities or the tensile normal strength of the joint.
- the shear stress exceeds the cohesive strength and frictional resistance of the joint surface.

The normal tensile strength and the cohesion of the joints are assumed to be constant material properties.
The frictional parameters the dilation rate, and the shear asperity angle, and the compressive normal strength are functions of the applied stress field and joint displacement. Simple relations based on Barton's constants joint roughness coefficient, JRC, joint compressive strength, JCS, and the residual friction angle, \( \Phi_r \), simple relations are fitted to these parameters. This implies that input data to the model can be extracted from the Rock Mechanics literature for a wide variety of joints.

Results from laboratory shear box test and numerical calculations have been made for a number of different joints. Good agreement was obtained. It shows, that peak shear strength behaviour of joint in principal is a function of dilation rate. Further, the calculations indicated that the elastic off-diagonal behaviour of joints, reported in the Rock Mechanics literature, is related to the dilation angle at the asperities in contact.

By means of finite element technique the model is applied to a circular opening in a jointed rock mass. It is concluded that the model offers several advantages over a discrete formulation.
1 INTRODUCTION

Many of the problems encountered in the field of Rock Mechanics involve the determination of rock mass behaviour. Foundation design of bridges, dams, large buildings etc requires a firm knowledge of the ground settlements. Slopes for road cuts and open pits have to be designed safely. The economic success and safety of an open pit are highly dependent on optimum design of the pit slope angle. For all types of underground construction knowledge of the rock mass behaviour is essential for rational and safe design.

Several approaches such as large scale field testing programs, empirical and analytical techniques have been used to predict rock mass deformability. The theoretical modelling of rock mechanical problems has grown in importance along with the rapid development of computers and numerical methods.

The analytical and numerical models for rock mechanics application are often taken from other disciplines, e.g. classical mechanics.

In recent years several attempts have been made to incorporate discrete joints and joint systems into theoretical models of rock masses. In principle two main approaches have been used. The first implies that jointed rock masses can be modelled by describing the response of every single joint separately - the discrete approach. The other approach defines an equivalent material in which the properties of the joint system are "smeared out" over a unit volume of rock - the equivalent material approach.

The research carried out in this dissertation aims to present a constitutive model for the joints and joint systems in rock masses. These properties are superimposed to the intact rock in order to formulate an equivalent material for the jointed rock mass.
The thesis starts with a brief review of the "state of the art" in rock mass modelling. Chapter 3 and 4 describes the properties of the individual components in a jointed rock mass. In chapter 5 an elasto-viscoplastic model for a jointed rock mass is developed. The model is compared with experimental results in chapter 6, and in chapter 7 a few finite element calculation have been made to demonstrate the capabilities of the equivalent material formulation. The thesis ends with a discussion of the presented model and some conclusions of the main advantages and limitations of the equivalent material formulation.
2 STATE OF THE ART IN ROCK MASS MODELLING

A brief historical review will be given over the different methods being used to model rock mass behaviour. The review is not intended to cover all the work on the subject of jointed rock mass.

The different methods of analysis are categorized in three groups:

- Observational models
- Physical models
- Theoretical models

The division is not distinct, in fact many of the models or methods of analysis include some features from all groups.

2.1 Observational models

The first attempts that were made to predict rock mass behaviour were usually based on observations. Information was collected from existing construction sites to postulate the behaviour for other types of rock mechanical problem. Bierbaumer (1913) used this approach to design tunnel supports. He estimated the rock pressure in tunnels from the observation of failure in timber supports.

Terzaghi (1946) postulated support loads for different rock mass conditions from tests carried out in railroad tunnels. He inserted wooden blocks of known strength in between timber sets and estimated the magnitude of the rock load acting on the support.

Deere et al (1970) introduced a field index property called the Rock Quality Design (RQD-index) to Terzaghi's classification scheme and thus provided a more "objective" way of selecting the proper classes of loading.

The "New Austrian Tunneling Method", NATM, Rabcewicz (1964), is a construction technique for minimizing the loads on the tunnel support. The method requires detailed knowledge of the force/deformation characteristics of the surrounding rock. This information is often collected from observations in smaller test drives of the project.
More sophisticated empirical rock mass models have recently been developed, Bieniawski (1973, 1976), Barton et al (1974) and Kendorski et al (1983). These models rate different factors such as rock joint properties, rock material, ground water condition, ground stresses, geometry etc. and combine them to obtain a single index. These indices are taken as an overall measure of the rock mass quality and are often used to estimate the support requirements.

2.2 Physical models

Physical modelling is still one of the most powerful methods to investigate and visualize the complex behaviour of jointed rock.

Physical models have often been used directly in the design of underground construction. Lang (1964) studied the behaviour of underground power stations in a model with discontinuities. Borg et al (1977) and Stephansson et al (1978) studied the caving of the hanging wall in two mines in Sweden with a 2D-basic friction model and a 3D-vibrating model.

Stillborg (1980) studied the effect of concrete ribs and cable bolts around large underground openings in a 3D-pressurized model chamber with ability to model the excavation process.

Physical models have also been used to examine the deformation characteristics of joints and jointed rock. Goldstein et al (1966) investigated the different failure conditions of slopes cut in jointed rock using a centrifuge. Barton (1974) examined the properties of blocky rock masses with models consisting of approximately 40000 blocks of heavy model material. Bandis et al (1981) studied scale effects on the shear behaviour of joints with specimens where the roughness of natural joints was replicated through a process of casting.

Photoelastic models have mainly been used to calculate stresses around underground openings under the assumption of linear elastic conditions. However, Lang (1961) used photoelastic models to study the effect of joints in the roof of an underground opening. Chappel (1973) investigated underground openings in jointed media with a photoelastic technique.
2.3 Theoretical models

The theoretical models of rock mass behaviour can be divided into two major groups, continuum models and discrete models.

2.3.1 Discrete models

A large portion of the theoretical models for stability of rock slopes uses the principal of limit equilibrium analysis.

Analysis of sliding rock wedges using stereographic projection is an important tool in rock slope engineering, John (1968). Bray (1966) used the equations for principal stress determination in the Mohr-Coulomb-Navier relation. He developed failure envelopes for sliding as a function of the stress orientation and the coefficient of friction.

The CANMET work on pit slope design, see Sage (1976), used limit equilibrium methods in combination with statistical methods. The probability of instability due to uncertainties in rock slope parameters such as length, spacing, shear strength properties of the joint system, etc. was analyzed. The analysis included support of the pit walls and allowed the opportunity to economically evaluate the hazards of alternative support systems and the probability of failure. A comprehensive textbook on the subject of stability of rock slopes has been presented by Hoek and Bray (1981).

The limit equilibrium method has recently been extended by Goodman and Shi (1982) and Shi (1983). They have developed a theory known as the "key-block" method. From joint orientation and excavation geometry potentially moveable and non-moveable blocks are identified. The removeable blocks are analysed using equilibrium analysis and the instable blocks are termed as "key-blocks". If the keyblocks are stabilized with support the overall stability of the excavation is ensured.

A more generalized discrete model of jointed media was presented by Cundall (1971), (1974) and Voegele (1978). The "Distinct Element Method" (DEM) as it is called calculates the total force-displacement interaction of a jointed rock mass consisting of rigid blocks separated by discontinuities. The method is based on Newton's second law of motion and the force-displacement law at the points of contact between
the rigid blocks. If the time step between each cycle of calculation is sufficiently small velocities and accelerations can be considered as constant and force propagation from any block is restricted to its nearest neighbours. Cundall et al (1978) improved the DEM method to include the deformation characteristics of the "rigid" blocks. The main advantage of the DEM model is that dynamic behaviour is implicit. There is no restrictions on deformation since the geometry is continuously updated.

2.3.2 Continuum models

For the category of classical continuum elastic approaches, beam and plate theories have been used in the design of underground openings in horizontally stratified rock, Obert et al (1960), Stephansson (1969), Fairhurst and Singh (1974).

The Finite Element Method (FEM) has become a useful tool to solve rock mechanical problems. Zienkiewicz et al (1968) used the FE-method to calculate the stress distribution around underground openings in a rock mass with "no tension" behaviour. The tensile stresses were redistributed through a iteration scheme which made it possible to obtain solutions with no tensile stresses present in the rock mass.


An alternative method of incorporating joint deformation into rock mass models is to derive equivalent material properties for the combined rock-joint material. Goodman and Duncan (1971) derived the effective elastic modulus for a rock intersected with a single set of uniformly distributed joints. This approach has been extended to include non-linear elastic behaviour with several sets of joints arbitrarily oriented, Olofsson (1981), Amadei and Goodman (1981). Gerrard (1982) introduced arbitrarily oriented sets of bolts in an equivalent rock joint model. The same approach has been proposed for reinforced cracked concrete, Bazant and Gambarova (1980), Bazant and Oh (1983). However, these models are restricted to problems with monotonic loading conditions.
An elasto-viscoplastic multi-laminate model for jointed rock was introduced by Zienkiewicz and Pande (1977) assuming a Mohr-Coulomb shear failure criterion for the joint system. Pande and Gerrard (1983), Larsson and Olofsson (1983) extended this approach to include set of bolts with strain hardening yield criterion. The elasto-viscoplastic formulation allows the modelling of both rate effects in the deformation process and quasi static response. In the latter case the elasto-viscoplastic formulation offers some advantages over the classical theory of plasticity, using time only as a computation parameter.

The equivalent material model together with finite element method offers several advantages over the traditional method of using discrete joint elements. The construction of the FE-mesh is simplified. Changing the joint orientations needs no reconstruction of the FE-mesh. An underground construction in a jointed rock mass can easily be analyzed with different joint systems, joint orientations and joint spacings.
3 JOINT PROPERTIES

Changes in the stress field due to underground construction or excavation will result in recoverable and non-recoverable deformations in the rock mass. In some cases the deformations will also be a function of the loading rate. The behaviour of a rock mass is determined by the mechanical properties of the intact rock and the properties of the different sets of discontinuities. This chapter deals with the formulation of a constitutive model for a single joint.

3.1 Definitions

A joint is a discontinuous planar structure in a rock mass i.e. two surfaces in contact. Joints are often filled or coated with clay minerals and the rock near the surfaces are often weathered and have different properties than the surrounding rock. Although this is a contact problem, joints have generally been modelled as a continuum with its own properties. This is most likely due to difficulties of determining the true contact area and surface properties.

Only stresses directed normal and tangential to the joint surfaces affect the displacement, Fig 3.1.

![Diagram of joint surfaces and stresses](image)

Fig 3.1 Notation of stresses and displacements on a joint surface in a cartesian coordinate system
Stresses and displacement in the $j_{x_1}$-direction coincide with the direction normal to the plane of the joint. $j_{x_2}$ and $j_{x_3}$ coincide with the direction of dip and strike respectively.

The properties of the joint in the $j_{x_2}$-$j_{x_3}$ plane are assumed to be isotropic. Hence starting from mated position, Fig 3.2, the stress-displacement relation is independent of the direction of the shear stress. Thus the properties will be reduced to the dimensions normal and parallel to the joint surface.

The following notation will be used:

$$\sigma_n = j_\sigma_{11}$$  \hspace{1cm} Normal stress acting on the joint surface. Compressive stresses are defined negative.

$$\sigma_s = \pm \left[ (j_\sigma_{12})^2 + (j_\sigma_{13})^2 \right]^{1/2}$$  \hspace{1cm} Shear stress acting on the joint surface.

$$u_n = j_u_1$$  \hspace{1cm} Normal deformation across the joint surface.

$$u_s = \pm \left[ (j_u_2)^2 + (j_u_3)^2 \right]^{1/2}$$  \hspace{1cm} Shear deformation on the joint surface.

Compressive stresses are normally defined positive in Rock Mechanics. To avoid confusion the notation $|\sigma_n|$ has been used in some equations to represent the compressive normal stress.

Since the properties of the joint are dependent on the way the two opposite surfaces are in contact, three regions of contacts will be defined.

The first region will be related to the position where the upper joint surface matches the lower surface, the so called mated position.
The second region defines the area of contact from almost mated position to the point where the two surfaces become totally unrelated. This region will be referred to as the almost mated position.

The third region of contact is the unmated position, i.e. the two matching surfaces are now totally unrelated.

The three positions can be illustrated by a simple shear test, see Fig 3.2. If the upper block of two mated surfaces is moved along the joint plane the normal versus shear deformation log will follow a curve that reflects the roughness of surface.

![Diagram](image)

**Fig 3.2**  
A) Deformation of a single joint in a block  
B) Normal versus shear displacement for three positions of mating of the joint, see text

The mated position 1, in Fig 3.2 will only be a point on the curve. The almost mated position, 2, is extended to the point where the rate of dilation, \(\frac{du_n}{du_s}\), approaches zero. Finally, in the unmated position, 3, the rate of dilation can be positive, negative or zero depending on how the asperities on the two surfaces contact.

### 3.2 Elastic properties

The elastic deformations of the joint are related to the applied normal and shear stress according to the following equation
\[
\begin{bmatrix}
\sigma_n \\
\sigma_s
\end{bmatrix} =
\begin{bmatrix}
k_{nn} & k_{ns} \\
k_{sn} & k_{ss}
\end{bmatrix}
\begin{bmatrix}
u_n \\
u_s
\end{bmatrix}
\] (3.1)

where \( k_{nn} \) is the normal stiffness, \( k_{ss} \) the shear stiffness and \( k_{ns} \) and \( k_{sn} \) are referred to as the off-diagonal stiffnesses. Sometimes the inverse to the stiffness matrix is used, i.e. \( k = (c)^{-1} \). The relationship is now written as

\[
\begin{bmatrix}
u_n \\
u_s
\end{bmatrix} =
\begin{bmatrix}
c_{nn} & c_{ns} \\
c_{sn} & c_{ss}
\end{bmatrix}
\begin{bmatrix}
\sigma_n \\
\sigma_s
\end{bmatrix}
\] (3.2)

3.2.1 Normal stiffness/compliance

---

**Fig 3.3** Normal compressive stress versus displacement for an extension fracture of a granodiorit specimen. After Goodman, 1976.
The normal stiffness can be measured by compressing a joint in the normal direction. Goodman (1976) proposed a hyperbolic relation between the normal stress, $\sigma_n$, and the normal displacement, $u_n$, Fig 3.3, as

$$\frac{(\sigma_n - \xi)}{\xi} = \frac{u_n}{(u_{mc} - u_n)t} \quad (3.3)$$

where

$\xi$ = is the seating pressure when $u_n = 0$

$u_{mc}$ = maximum joint closure

$t$ = constant

Further Goodman stated that the normal deformation in compression of joints is essentially unrecoverable.

Hungr and Coates (1978) measured the normal stress versus displacement of joints in bedded limestone and sandstone. The test was performed for mated samples. The stress-displacement curves were all found to be near linear. The linear relationship was explained by precompression in the past by pressures considerably greater than those applied in the test, see Fig 3.4. Hence, for pressures smaller than the precompression stress the mated joint has a linear elastic behaviour.
Fig 3.4 Effect of precompression on normal stress-displacement behaviour of a joint. From Hungr and Coates, 1978.

Swan (1981), (1983) and later Sun (1983) applied the theory of tribology to study the mechanism of normal compliances of rough joint surfaces. They concluded that a large portion of the nonlinear behaviour of joint compression is due to elastic compression of asperities - Swan (1983) suggested three types of asperity peak height distribution functions, the exponential, the power law and the Gaussian distribution.

From the (i) exponential and (ii) power law distribution function the following relationship can be derived

\[ |u_n| = a + b \ln |\sigma_n| \quad (i) \]
\[ |u_n| = n |\sigma_n|^m \quad (ii) \]

where \( u_n \) is the normal displacement, \( \sigma_n \) the normal stress and \( a, b, n, \) and \( m \) are constants.
An empirical relation between the constant, \( b \), in the exponential law and the joint aperture was established as

\[
u_n = a + f e_0 \ln |\sigma_n| \quad (3.5)
\]

where \( a, f \) are constants and \( e_0 \) the starting aperture. Differentiating equation (3.5) with respect to the normal stress gives the joint normal compliance \( c_{nn} \)

\[
c_{nn} = f e_0 / |\sigma_n| \quad (3.6)
\]

Since \( e_0 \) varies with the contact position, the normal compliance, \( c_{nn} \), will also change.

However, considering the scatter and uncertainties in the determination of rock mass properties a linear elastic joint stiffness is adequate, i.e. \( c_{nn} \) and \( k_{nn} \) can be taken as constants in the applied range of stress.

### 3.2.2 Shear stiffness/compliance

The slope \( (d\sigma_s / d\sigma_n) \) of the elastic part of the shear stress displacement curve at constant normal stress is referred to as the shear stiffness of the joint. This is only valid if the off-diagonal stiffness, \( k_{sn} \) is zero. However, since the test is conducted under constant normal stress the off-diagonal compliance \( (c_{sn}) \), if any, will not contribute to the shear displacement. Hence, the slope \( (d\sigma_s / d\sigma_n) \) can be evaluated as the shear compliance, \( c_{ss} \), of the joint. In the following text the off-diagonal stiffness is assumed to be zero and \( k_{ss} = 1/c_{ss} \).

Goodman (1976) simplified the elastic shear behaviour of joints by the introduction of two models, (A) the constant stiffness model and (B) the constant peak displacement model, Fig 3.5.
Fig 3.5  Models of joint shear stiffness $k_{ss}$
A) Constant stiffness model
B) Constant peak displacement model

Barton (1977) pointed out that the shear stiffness displays a strong scale effect. He suggested a constant peak displacement model to calculate the shear stiffness where the peak displacement is a function of scale, i.e. joint length:

$$k_{ss} = \frac{100}{L} \sigma_p$$  \hspace{1cm} (3.7)

where

- $L$ = joint length
- $\sigma_p$ = peak shear stress

The joint length, $L$, should not exceed the maximum spacing of cross-joints intersecting the joint of interest.
Hungr and Coates (1978) proposed a hyperbolic relation to represent the pre-peak portion of the stress-displacement relationship. They determined the tangent shear stiffness with the expression

\[ k_{ss} = \frac{d\sigma_s}{du_s} = \frac{a b (0.9 f \sigma_n)^2}{(a|\sigma_n| - b)(0.9 f b - u_s a(a|\sigma_n| - b))} \]  (3.8)

where

\[ f = \text{average friction coefficient} \]

\[ a = \text{the coefficient which determines the linear relationship between the secant shear stiffness at the yield point and the normal compressive stress.} \]

\[ b = \text{scale coefficient} \]

The yield point was taken as 0.9 \( \sigma_p \). The shear stiffness as determined from equation (3.8) is dependent both on the normal stress and shear displacement. Furthermore the model predicts a constant peak displacement, i.e. it is of type (B) in Fig 3.5.

Sun (1983) used an expression of the form

\[ \sigma_s = \frac{u_s}{A u_s + B} + C \]  (3.9)

to describe the shear stress-displacement curve of joints in two types of granite. \( A, B \) and \( C \) are constants. From equation (3.9) he evaluated the tangent and secant shear compliances, see Fig 3.6.
Fig 3.6  Tangent and secant compliance versus shear displacement for A) unmated joint in red granite and B) Almost mated joint in grey granite. After Sun, 1983.

The grey granite, Fig 3.6B, was tested in the almost mated position while the red granite, Fig 3.6A, was tested in the unmated position. The secant compliances vary both with the normal stress and shear displacement. However, the tangent compliances are almost independent of the normal stress for the mated and unmated joint. The non-linearity of the pre-peak portion of the curve showing the shear stress versus shear displacement indicates that sliding initiates long before peak displacement is reached. Sun (1983) divided the shear curves into three types, see Fig 3.7.
Fig 3.7 Different types of shear force versus shear displacement curves. After Sun, 1983.

He associated the different types with closure or dilatancy during shearing. If the joint dilates when sliding starts then the first curve, type 1, is obtained. The type 1 curve is associated with mated or almost mated position of the joint.

Progressive closure of the joint in the initial phase of sliding produce the third type of curve, type 3. The second type, type 2, is associated with zero normal displacement. Hence, both type two and type three can be associated with an unmated position of a joint.

3.2.3 Off-diagonal stiffness/compliances

Very few attempts have been made to measure the off-diagonal stiffness/compliance terms, $k_{sn}/c_{sn}$ and $k_{ns}/c_{ns}$.

Hungr and Coates (1978) reported results of a few measurements of joints in limestone and sandstone. They measured the off-diagonal stiffnesses at 50% and 100% of the peak shear load and found a large scatter in the data. The off-diagonal stiffnesses measured on joints in limestone ranged from -143 to +580 GPa. One explanation for this scatter could be that non-recoverable deformation was included in the calculation and that the sign of the off-diagonal stiffness term is dependent on whether the joint is dilating or closing. However, the calculations made at 50% of peak load also showed a considerable scatter of the off-diagonal stiffness.
Sun (1983) measured the normal deformation during shear testing with high accuracy, see Fig 3.8. His results indicate that there is a tendency for joint closure as the shear stress increases in the elastic region, part I, especially for the mated joint in the grey granite. On the onset of sliding the mated joint starts to dilate. For the unmated red granite the coupling between shear stress and normal deformation is much less pronounced.

Assuming that the off-diagonal components are zero for the case of an unmated joint, i.e. no normal deformation during shearing. The off-diagonal terms in the case of a mated joint can then be explained by the dilation rate.
Shearing a joint with a dilatancy angle of $\phi_i$ in the mated or almost mated position can be idealized with an inclined surface as shown in Fig 3.9. Note that the "inclined surface" only represents the direction of movement, the nominal contact area does not change as the joint dilates.

![Diagram A](image)

![Diagram B](image)

**Fig 3.9** Shearing of a joint in mated or almost mated position.
A. Natural situation
B. Idealized situation with an inclined surface

The elastic stress-displacement relationship for the inclined surface can then be written as

$$u' = c' \sigma'$$

$$\begin{bmatrix}
  u'_n \\
  u'_s
\end{bmatrix} = \begin{bmatrix}
  c'_{nn} & 0 \\
  0 & c'_{ss}
\end{bmatrix} \begin{bmatrix}
  \sigma'_n \\
  \sigma'_s
\end{bmatrix}$$

(3.10)

Equation (3.10) can now be transformed to represent the situation at the joint plane. Since the area in contact is assumed to be constant when the joint dilates, stresses and displacements are transformed through ordinary coordinate transformation laws, i.e. $u' = r u$, $\sigma' = r \sigma$ giving

$$c = r^T c' r$$

(3.11)
where

\[
    r = \begin{bmatrix}
        \cos \phi_i & -\sin \phi_i \\
        \sin \phi_i & \cos \phi_i
    \end{bmatrix}
\]

The compliance, \( c \), on the plane of the joint in terms of \( c'_{nn} \) and \( c'_{ss} \) can then be calculated as

\[
    c = \begin{bmatrix}
        c'_{nn} \cos^2 \phi_i + c'_{ss} \sin^2 \phi_i & (c'_{ss} - c'_{nn}) \sin \phi_i \cos \phi_i \\
        (c'_{ss} - c'_{nn}) \sin \phi_i \cos \phi_i & c'_{ss} \cos^2 \phi_i + c'_{nn} \sin^2 \phi_i
    \end{bmatrix}
\]

(3.12)

The off-diagonal compliance, \( c_{sn} = c_{ns} \), is determined by the relation between the normal and the shear compliance. If \( c'_{nn} > c'_{ss} \), the joint contracts elastically due to shearing and if \( c'_{nn} < c'_{ss} \) the joint dilates. The effective normal stress, \( |\sigma'_n| \), increases with increasing \( \sigma_s \). The normal compliance \( c'_{nn} \) decreases while the shear compliance, \( c'_{ss} \), in general increases with increasing stress level. So the off-diagonal compliances will increase with increasing shear stress, \( \sigma_s \), implicating that the elastic contraction - if any - will decrease and eventually turn into dilation.

In Fig 3.10 the off-diagonal compliance together with the shear and normal compliances are shown as a function of shear stress for mated and unmated joints in granite, Sun (1983). Note that the off-diagonal compliances shown in Fig 3.10 are negative using the sign conventions in this thesis.

The off-diagonal compliance for the mated joint is much more pronounced than for the unmated joint. Also the off-diagonal compliance increases (\( c_{sn} \) gets less negative) with increasing shear stress. Equation (3.12) can also explain the scatter in the off-diagonal stiffnesses found by Hungr and Coates (1978).
Fig 3.10 Normal, shear and off-diagonal compliances for an unmated and mated joint. After Sun, 1983.
A. Unmated joint in red granit
B. Almost mated joint in grey granit
3.3 Joint failure criterion and postfailure properties

In the mechanics of rock joints the failure stress is often assumed to be equivalent to the peak shear strength. However, in this thesis a wider definition of the failure stress will be used.

The failure stress determines the onset of plastic flow, i.e. the stress level that initiates unrecoverable deformations. This means that the failure stress is not equivalent with the peak stress. In fact, for most materials the onset of plastic flow is reached long before the peak stress.

A joint has two basic modes of failure

(i) Shear failure
(ii) Normal failure

The shear failure is initiated when the shear stress exceed the frictional forces in the joint plane. The two surfaces slip or slide along the plane of the joint.

Normal failure can either be tensile or compressive. A tensile failure is initiated when the normal stress exceeds the tensile strength of the joint. For an unfilled joint failure is initiated when $\sigma_n$ becomes positive. Tensile failure causes separation of the two joint surfaces.

Compressive normal failure begins when the stresses in the contacting asperities exceed the compressive strength of the joint surface. Compressive failure leads to non-recoverable joint contraction.

3.3.1 Existing theories and models

Shear failure

Most of the experimental and theoretical work on joint properties so far have been devoted to determining the peak shear strength for different types of joints under various conditions. It is an important parameter in limit equilibrium analysis where the overall stability in many situations is controlled by the peak shear strength of the critical joint set.
A number of peak shear strength models have been suggested for rock joints.

Patton (1966) proposed a bilinear relationship between the normal stress and peak shear strength. He showed that the friction angle of a rough surface can be calculated by adding the peak dilatancy angle, $\phi_i^D$ to the basic friction angle, $\phi_b$.

\[
\sigma_p = |\sigma_n| \tan(\phi_b + \phi_i^D) \quad \sigma_n < \sigma_T
\]

\[
\sigma_p = |\sigma_n| \tan \phi_r + c_p \quad \sigma_n > \sigma_T
\]

(3.13)

where $\phi_r$ is the residual friction angle and $c_p$ is the shear stress intercept. $\sigma_T$ can be calculated from

\[
\sigma_T = c_p/(\tan(\phi_b + \phi_i^D) - \tan \phi_r)
\]

(3.14)

Jaeger (1971) proposed a continuously variable peak shear strength

\[
\sigma_p = c_j(1 - e^{-b|\sigma_n|}) + |\sigma_n| \tan \phi_r
\]

(3.15)

where $c_j$ and $b$ are constants and $\phi_r$ is the residual friction angle.

Fig 3.11 shows the Patton and Jaeger shear strength criterion for $c_p = c_j$ and $\tan(\phi_b + \phi_i^D) = \tan \phi_r + c_j b$.

The relation by Jaeger is a continuous formulation of Patton's bilinear peak shear strength criterion.
Fig 3.11  Pattons and Jaegers peak shear strength criteria for rock joints

At this point we will study an equation based on identified properties of the joint and wall rock. Ladanyi and Archambault (1970) combined the friction, dilatancy and the shear strength of the asperities to derive a general expression for the peak shear strength:

\[ \sigma_p = \frac{|\sigma_n| (1 - a_s) (\dot{v} + \tan \phi) + a_s s_R}{1 - (1 - a_s) \nu \tan \phi} \]  \hspace{1cm} (3.16)

where

- \( a_s \) = proportion of joint area sheared through the asperities
- \( \dot{v} \) = dilation rate, \( \tan \phi_p \), at peak shear stress
- \( s_R \) = shear strength of asperities

Ladanyi and Archambault proposed the following relation for \( a_s \) and \( \dot{v} \):

\[ a_s = 1 - (1 - \frac{\sigma_n}{\sigma_T})^{k_1} \]
\[ \dot{v} = (1 - \frac{\sigma_n}{\sigma_T})^{k_2} \tan \phi_{10} \]  \hspace{1cm} (3.17)

where \( k_1 \) and \( k_2 \) are constants and \( \sigma_T \) is the uniaxial compressive strength of the wall rock material. \( \phi_{10} \) is the initial peak dilation angle for \( \sigma_n = 0 \).
They used the Fairhurst parabolic failure criterion for $s_R$ according to the equation

$$s_R = \sigma_T \left( \frac{1 + n}{n} \right)^{1/2} \left( \frac{q_n}{|\sigma_T|} \right)^{1/2}$$

(3.18)

where $n$ is the ratio of the uniaxial compressive to the uniaxial tensile strength of the wall rock.

Barton (1973) derived an empirical relationship between the peak shear strength and the normal stress from test on artificially constructed rough joints and derived the equation

$$\sigma_p = |q_n| \tan(\text{JRC} \log_{10} \frac{\text{JCS}}{|q_n|} + \phi_r)$$

(3.19)

where

\begin{align*}
\text{JRC} & = \text{joint roughness coefficient} \\
\text{JCS} & = \text{joint compressive strength} \\
\phi_r & = \text{residual friction angle}
\end{align*}

Note that as $q_n$/JCS approaches zero the logarithmic term in Barton's equation approaches infinity and the equation is no longer valid. To overcome this difficulty Barton suggests a maximum value for

$$(\text{JRC} \log_{10} \frac{\text{JCS}}{|q_n|} + \phi_r) = 70^\circ.$$  

Fig 3.18 shows the Ladanyi - Archambault's relationship for $k_1 = 4$, $k_2 = 1.5$, $\phi_{10} = 20^\circ$, $\phi = 30^\circ$ and $n = 10$, Fairhurst parabolic criterion and Barton's empirical relationship for $\phi_r = \phi$ and JCS = $\sigma_T$.

At low normal stresses Barton’s equation is in close agreement with Ladanyi and Archambault’s equation at $\phi_{10} = \text{JRC} = 20$. At higher normal stresses the curves diverge since Barton’s equation reduces to $\sigma_p = |q_n| \tan \phi_r$ when $|q_n| = \text{JCS} = \sigma_T$ whereas for the Ladanyi - Archambault’s equation $\sigma_p$, approaches $s_R$. Barton’s original studies were carried out at very low normal stresses and therefore his equation is more applicable for situations when the normal stress is low. For most practical situations in rock mechanics the normal stresses are low compared to the compressive strength of the wall rock.
Fig 3.12  Ladanyi - Archambault's and Barton's peak shear strength equation for joints and Fairhurst failure criterion for intact rock.

Equation (3.19) have later been fitted to a range of different types of joints and simple tests have been proposed to determine the three variables JCS, JRC, and $\phi_r$, Barton and Choubey (1977).

Rengers (1974), Bandis et al (1981) and Sun (1983) divided the shear strength of rock joints into three components

$\phi_i$ - dilation angle, i.e. the angle between the direction of relative movement and the joint plane (geometric component)

$\phi_b$ - basic friction angle, i.e. the friction angle for the smooth rock surface

$\phi_s$ - shear asperity angle that represent the effect of surface damage such as wear and asperity fractures.
Several equations relating the peak dilation rate to the normal pressure have been proposed. Ladanyi and Archambault proposed a power expression on the peak dilation rate, see equation (3.17). Barton and Choubey (1977) proposed the following equation for the dilation angle, $\phi_1^P$, at peak shear resistance

$$\phi_1^P = \frac{12 \text{JRC} (\log_{10}(|\text{JCS}/\sigma_n|))^2}{\text{JRC} + 8.4 \log_{10}(|\text{JCS}/\sigma_n|)} \quad (3.20)$$

Pande and Xiong (1982) analysed the Barton and Choubey work further and found that the best fit to the experimental data was given by

$$\tan \phi_1^P = \frac{\tan \lambda - (\tan \Phi_r + K_2 \tan^2 \lambda - K_2 (1 - |\text{JCS}/\sigma_n|) K_2^2)}{K_1}$$

$$\quad (3.21)$$

where

$$\tan \lambda = \tan(\text{JRC} \log_{10}(|\text{JCS}/\sigma_n|))$$

$$K_1 = 1 - \tan \lambda \tan \Phi_r$$

$$K_2 = \frac{180}{\pi} \ln 10 \approx 131.93$$

The three models on peak dilation angle discussed in this chapter are shown in Fig 3.13.
Fig 3.13  Peak dilation angle versus normal stress for joints according to the expressions by Landanyi and Archambault 1970, Barton and Choubey 1977, and Pande and Xiong 1982.

The Pande and Xiong equation gives results quite similar result to the Barton and Choubey expression. This is not suprisingly since they used the same shear test data for different rock with ratios of $|\sigma_n/JCS|$ in the range 0.001 - 0.05. Both expressions are singular at $\sigma_n = 0$, i.e. $\phi_p^p + 90^0$ when $\sigma_n = 0$. Equation (3.21) also have a singularity when $\tan \lambda = 1/\tan \phi_p$, i.e. when $\lambda + \phi_p = 90^0$. For the example shown in Fig 3.21 this occurs when $|\sigma_n/JCS| = 0.001$. 
Once knowing the component of dilation the basic friction and the shear asperity component can be evaluated by subtracting the component of dilation from the total shear resistance. However, the expressions for the total shear resistance gives only the peak shear value, so

$$\tan (\phi_s^p + \phi_b) = \frac{\sigma_p}{\sigma_n} - \tan \phi_i^p$$ \hspace{1cm} (3.22)

using Barton's equations (3.19) and (3.20) for $\sigma_p$ and on $\phi_i^p$, equation (3.22) can be written as

$$\Phi_s^p + \Phi_b = JRC \log_{10}(|JCS/\sigma_n|) + \Phi_r - \frac{12 JRC \left(\log_{10}(|JCS/\sigma_n|)\right)^2}{JRC + 8.4 \log_{10}(|JCS/\sigma_n|)}$$ \hspace{1cm} (3.23)

Barton and Choubey (1977) calculated the shear asperity component as a factor $M$ times the peak dilation angle, i.e. $M \phi_i^p = \phi_i^p + \phi_s^p$, using the same technique assuming that $\phi_r = \phi_b$. They found that the shear asperity component varied from 1 for smooth surfaces to 1.5 - 2.0 for very rough surfaces, i.e. $0^0 \leq \phi_s^p \leq \phi_i^p$.

Bandis et al (1981) investigated scale effects on model replicas of joint surfaces. They found that both the asperity and the dilation component decreased with increasing size of the joints. The scale effect was much more pronounced for rough surfaces, see Fig 3.14.
Fig 3.14 Effect of scale on joint properties, Bandis et al 1981.

A. Principal effects on the shear strength components by varying sample size.

B. Scale effects on the joint roughness coefficient, JRC, and joint compressive strength, JCS, in dimensionless form.

Normal failure

As discussed earlier in chapter 3.3.1 compression of a joint surface involving damage of asperities leads to non-recoverable deformations. Goodman (1976) proposed a hyperbolic expression of joint closure, see equation (3.3). He also stated that normal deformation in compression of joints is essentially non-recoverable.
Bandis et al (1983) studied the normal deformability in compression by conducting repeated load cycles on fresh and weathered joints in five different rock types. They also studied the difference between interlocked and dislocated joints. They found that hyperbolic functions enable accurate representation of normal stress ($\sigma_n$) versus closure ($\Delta u_n$), equation (3.24), for interlocked joints while semilogarithmic functions, equation (3.25), best described the normal behaviour in compression for the dislocated joints.

\[
|\sigma_n| = \frac{\Delta u_n}{a - b \Delta u_n} \quad (3.24)
\]

\[
\log|\sigma_n| = p + q \Delta u_n \quad (3.25)
\]

where $a$, $b$, $p$ and $q$ are constants. A comparison with Goodman's equation (3.3) gives that $a/b$ in equation (3.24) is equal to the maximum joint closure, $u_{mc}$. They also proposed quantitative relations for initial joint stiffness, aperture and maximum joint closure as a function of wall strength (JCS) and joint roughness (JRC). In doing this, they included the non-recoverable deformations in evaluating the initial joint stiffness.

However, the normal stress-deformation behaviour cannot be treated separately since permanent changes on the joint surface will effect the geometrical conditions on the surface and change parameters such as $\phi$ etc.

3.3.2 A general failure condition for a rough joint

A failure model for a rough joint must describe the different modes of failure correctly. Therefore, a yield function or failure condition, $F$, will be defined in the $\sigma_n$, $\sigma_s$ stress space such that $F = 0$ indicates failure and the onset of plastic flow. When $F < 0$ the joint deforms elastically.

First a basic yield function, $F^b$, will be defined for a planar joint, see Fig 3.15.
The basic yield function is divided in two regions. Region I is a point in the $\sigma_n$-$\sigma_s$ stress space representing zero tensile strength behaviour, i.e.

$$F^b_I = \sigma_n$$  \hfill (3.26)

The yield function in region II is equivalent to the simple Mohr-Coulomb equation, i.e.

$$F^b_{II} = |\sigma_s| + \sigma_n \tan \phi_b$$  \hfill (3.27)

where $\phi_b$ is the basic friction angle.
However, if the behaviour of a natural joint is to be modelled, several modifications of the basic yield function have to be introduced to account for

- dilantancy, i.e. sliding uphill or sliding downhill
- increased/decreased frictional resistance as asperities are introduced
- cohesion from joint filling material
- surface damage of the joint surface

The dilatancy angle $\phi_i$ will influence the frictional resistance. Shearing in one direction will increase the resistance and shearing in the reversed direction will decrease the resistance. The effect will be a rotation of the basic yield function, $f_i^b$, around the point $\sigma_n = \sigma_s = 0$ with the angle $\phi_i$ as shown in Fig 3.16.

![Diagram of the effect of dilatancy on the yield function](image)

**Fig 3.16** The effect of dilatancy, $\phi_i$, on the basic yield function
The frictional resistance due to asperities, $\phi_s$, on the joint surface will influence the basic friction angle, $\phi_b$. The influence of filling material is modelled by a cohesive component $c_0$. The filling material is assumed to have zero tensile strength.

Finally the introduction of surface damage normal to the joint plane is controlled by a "cap" located at the axis of the normal stress, $N$. Fig 3.17 shows the modification of the basic yield function due to $\phi_s$, $c_0$ and $N$.

![Diagram](image)

**Figure 3.17** The effect of asperities, cohesion and surface damage on joint behaviour

The dilatancy, $\phi_i$, friction angle $\phi_s$ and the N-cap are functions of both stresses and deformations. Adding the four components basic friction, dilatancy, asperity friction angle and cohesion for a joint, the final yield function can be evaluated, see Fig 3.18.
Fig 3.18  General yield function, $F$, for a filled joint with dilatancy and asperities

At the initial state of stress, $\Phi_i = 0$, we obtain the following yield function for the different regions as

\[
\begin{align*}
F_{I}^I &= \sigma_n^i \\
F_{IIa}^I &= \sigma_s^i - c_o^i + \sigma_n^i \tan(\phi_b + \phi_s) \\
F_{IIb}^I &= -\sigma_s^i - c_o^i + \sigma_n^i \tan(\phi_b + \phi_s) \\
F_{III}^I &= N - \sigma_n^i
\end{align*}
\]

(3.28)

Note that $N$ is defined negative.
Assuming that the surface area of the joint in contact does not change when the joint dilate the $F_I$, $F_{II}$ and $F_{III}$ regions can be calculated by ordinary coordinate transformation, i.e.

\[
\sigma'_n = \sigma_n \cos \phi_i - \sigma_s \sin \phi_i \\
\sigma'_s = \sigma_n \sin \phi_i + \sigma_s \cos \phi_i
\]  

(3.29)

Inserting equation (3.29) in (3.28) gives

\[
F'_I/\cos \phi_i = \sigma_n - \sigma_s \tan \phi_i
\]

\[
F'_{IIa}/A = \sigma_s - c'_o/A + \sigma_n \tan(\phi_b + \phi_s + \phi_i)
\]

(3.30)

\[
F'_{IIb}/B = -\sigma_s - c'_o/B + \sigma_n \tan(\phi_b + \phi_s - \phi_i)
\]

\[
F'_{III}/\cos \phi_i = N'/\cos \phi_i + \sigma_s \tan \phi_i - \sigma_n
\]

where

\[
A = \cos \phi_i[1 - \tan \phi_i \tan(\phi_b + \phi_s)]
\]

\[
B = \cos \phi_i[1 + \tan \phi_i \tan(\phi_b + \phi_s)]
\]

using $F_I = F'_I/\cos \phi_i$, $F_{IIa} = F'_{IIa}/A$, $F_{IIb} = F'_{IIb}/B$ and $F_{III} = F'_{III}/\cos \phi_i$ equation (3.30) can be written as

\[
F_I = \sigma_n - \sigma_s \tan \phi_i
\]

\[
F_{IIa} = \sigma_s - c_a + \sigma_n \tan(\phi_b + \phi_s + \phi_i)
\]

(3.31)

\[
F_{IIb} = -\sigma_s - c_b + \sigma_n \tan(\phi_b + \phi_s - \phi_i)
\]

\[
F_{III} = N' - \sigma_n + \sigma_s \tan \phi_i
\]
where

\[ c_a = c'_o / A \]

\[ c_b = c'_o / B \]

\[ N = N' / \cos \phi_i \]

On the yield surface, \( F = F' = 0 \), equation (3.31) reduces to

\[ \sigma_n = \sigma_s \tan \phi_i \quad (I) \]

\[ \sigma_s = c_a - \sigma_n \tan(\phi_b + \phi_s + \phi_i) \quad (II_a) \]

\[ \sigma_s = -c_b + \sigma_n \tan(\phi_b + \phi_s - \phi_i) \quad (II_b) \]

\[ \sigma_n = N + \sigma_s \tan \phi_i \quad (III) \]

For joints with no cohesion, \( c'_o = 0 \), the failure condition, \( F = 0 \), can be written as

\[ \sigma_n = 0 \]

\[ \sigma_s = -\sigma_n \tan(\phi_b + \phi_s + \phi_i) \quad (3.33) \]

\[ \sigma_s = \sigma_n \tan(\phi_b + \phi_s - \phi_i) \]

\[ \sigma_n = N + \sigma_s \tan \phi_i \]

For a natural joint \( \phi_s, \phi_i \) and \( N \) are functions of both stresses and displacements. Only the basic friction angle, \( \phi_b \), and the cohesion, \( c'_o \), are assumed to be constant properties. So if the different modes of failure for a rough joint are to be correctly described, the following functions must be determined.
\[ \phi_i = \phi_i(g, y) \]
\[ \phi_s + \phi_b = \phi_s(g, y) + \phi_b \]  
(3.34)

\[ N = N(g, y) \]

The friction angle, \( \phi_b + \phi_s \), and the dilatancy angle, \( \phi_i \), determine the shear failure condition while the onset of plastic flow normal to the joint is determined by the normal cap, \( N \).

3.3.3 Failure properties for a rough joint

The failure properties for a rough joint are determined by the dilatancy angle \( \phi_i \), the frictional component \( \phi_b + \phi_s \) and the normal cap \( N \). Only unfilled open joints with zero cohesion will be treated, equation (3.33). Before the different components are examined in detail some basic definitions and assumptions will be made.

Consider the shear stress-deformation curve for an open discontinuity undergoing shear at constant normal pressure, see Fig 3.19.

![Shear stress and normal deformation versus shear deformation for a mated joint under constant normal pressure.](image-url)
Starting from the mated position the joint will deform elastically before sliding initiates and dilation starts at $\sigma_S = \sigma_e$ and $u_S = u_e$. At the shear displacement $u_e + u_p$ the peak shear strength, $\sigma_p$ is reached. Barton and Choubey (1977), Bandis et al (1981) associated the peak shear strength with the maximum dilation angle, $\phi_p^D$. The post-peak behaviour is often characterized by a drop in shear stress to a constant value called the residual shear strength, $\sigma_r$, at the total shear displacement of $u_e + u_r$. At the same stage of deformation the maximum dilation or joint aperture is reached, $u_a$.

The shear failure condition is usually defined as the peak shear strength, $\sigma_p$, of the joint. However, since the shear failure is coupled to the onset of plastic flow the shear strength, $\sigma_S(F = 0)$, varies with the shear deformation, see Fig 3.20. $\sigma_p$ is only the shear resistance at a given shear deformation, $u_p + u_e$.

Increasing the normal stress will increase the shear resistance and decrease the dilatancy.

![Diagram](image)

**Fig 3.20** Shear strength, $\sigma_S(F = 0)$, versus shear deformation for a mated joint under normal constant stress.
Dilation angle and normal compressive strength

The dilation of a rough surface being sheared is a function of both shear displacement and normal stress.

The dilation rate, \( du_n/du_s \), will reach a maximum value, \( \tan \phi_i^0 \), often at the peak shear strength of the joint. After the peak is reached the dilation rate will decrease to zero as the joint approaches the un-mated position. Increasing normal stress decreases the dilation rate according to Barton and Choubey (1977).

Fig 3.21, shows the non-recoverable normal and shear deformation, i.e. the actual surface profile of the joint, cf fig 3.20

![Diagram of normal dilation versus shear deformation](image)

Fig 3.21 Normal dilation versus shear deformation. Only the non-recoverable part of the deformations are shown.

The maximum dilatancy angle, \( \phi_i^0 \), is reached at the peak shear deformation, \( u_p \). The dilation, \( u_n \), continues to increase with increasing shear deformation until the maximum joint aperture, \( u_a \), is reached at the point \( u_s = u_r \).

To obtain a correct physical description of the normal dilation, the dilation angle used in the shear resistance equation has to be derived from the normal vs shear displacement curve or vice versa. Although this is easily understood, many authors, including myself, have
proposed models where the component of dilation in the shear resistance is not related to the actual dilatancy of the model. Pande and Gerrad (1983), Carol and Alonso (1983), Larsson and Olofsson (1983), all proposed models with a constant dilation angle for the normal dilation while the shear resistance decreased linearly to a residual value. In Fig 3.22 the proposed models are compared with a more correct physical interpretation of a constant dilation angle, assuming a constant friction angle, $\phi_d + \phi_s$.

![Graph A](image1)

![Graph B](image2)

**Fig 3.22**  Relation between shear resistance and dilation assuming a constant friction angle, $\phi_s + \phi_d$ vs $u_s$.

A. Commonly proposed model.

B. Correct physical description.

The dilation rate vs shear displacement will be modelled by two equations. The pre-peak portion of the dilation rate will be modelled by an ellipsoidal function to ensure the continuity at the onset of failure.
From the peak shear displacement, $u_p$, the dilation rate will decrease linearly down to zero at $u_s = u_r$.

$$\tan \phi_1 = \tan \phi_1^p \left[ 1 - \left( \frac{|u_s| - u_p}{u_p} \right)^2 \right]^{1/2} \quad |u_s| < u_p$$

$$\tan \phi_1 = \tan \phi_1^p \frac{u_r - |u_s|}{u_r - u_p} \quad u_p < |u_s| < u_r \quad (3.35)$$

$$\tan \phi_1 = 0 \quad |u_s| \geq u_r$$

If $u_s < 0$ then $\tan \phi_1 = -\tan \phi_1$.

The normal dilation versus shear displacement can now be resolved by integrating the dilation rate.

$$u_n = \frac{\tan \phi_1^p}{2} \left[ u_x (1 - \left( \frac{u_x}{u_p} \right)^2) \right]^{1/2} + u_p \frac{\Pi}{2} + \sin^{-1} \left( \frac{u_x}{u_p} \right) \quad |u_s| < u_p$$

$$u_n = \frac{\tan \phi_1^p}{2} (2u_x - \frac{u_x}{u_r - u_p} + u_p \frac{\Pi}{2}) \quad u_p < |u_s| < u_r$$

$$u_n = \frac{\tan \phi_1^p}{2} [u_r - u_p (1 - \frac{\Pi}{2})] = u_a \quad |u_s| \geq u_r$$

(3.36)

where $u_x = |u_s| - u_p$.

Fig 3.23 shows the dilation rate, equation (3.35), and normal displacement, equation (3.36), as a function of shear displacement in dimensionless form.
Fig 3.23

A. The dilation rate, $\tan \phi_i / \tan \phi_i^p$, versus shear displacement $u_s / u_r$ according to equation (3.35).

B. Normal displacement $u_n / u_a$ versus shear displacement $u_s / u_r$ according to equation (3.36).
As shown earlier in chapter 3.3.1 the peak dilation rate, \( \tan \Phi_p \), is a function of the normal stress. However, since the shear-normal displacement log of the joint surfaces is related to the dilation rate, equation (3.35) and (3.36), the expression for the peak dilation rate versus normal stress will also affect the plastic behaviour normal to the joint or vice versa.

Let us take the hyperbolic functions that Goodman (1976) and Bandis et al (1983) suggested for the normal compression mode of failure.

\[
| \sigma_n | = \frac{1}{b} \frac{\Delta u_n}{a/b - \Delta u_n} = C \frac{\Delta u_n}{u_{mc} - \Delta u_n} \tag{3.37}
\]

where \( C \) is a constant and \( u_{mc} \) maximum joint closure. For an unmated joint the maximum joint closure must be less or equal to the maximum joint aperture for zero normal pressure, \( u_{ao} \). However, if the joint is moved back to its mated position the aperture decreases and eventually become "zero" when the two surfaces matches perfectly indicating that no plastic deformation can occur in the mated position. In practice however, the zero joint aperture does not exist even on a fresh joint. The fracturing process, (creating the joint), damages the surfaces and makes it impossible to get a "perfect match".

If \( N = \sigma_n \) at failure and \( u_{mc} = u_{no} \) and \( \Delta u_n = u_{no} - u_n \) then equation (3.37) can be written as

\[
N = -C \frac{u_{no} - u_n}{u_n} = -C (\frac{u_{no}}{u_n} - 1) \tag{3.38}
\]

where \( u_{no} \) is the joint aperture for \( \sigma_n = 0 \) and \( u_n \) is the actual aperture for the existing normal pressure. Note that both \( u_{no} \) and \( u_n \) is functions of shear displacements, equation (3.36).
If the normal stress in shear mode causes approximately the same amount of damage to the joint surface as in compression the peak dilation rate versus normal stress can be found by inserting the normal dilation, equation (3.36), in equation (3.38), i.e.

$$\tan \Phi_{10}^p = C \left( \frac{|\sigma_n|}{\tan \Phi_i} - 1 \right) \quad (3.39)$$

where $\tan \Phi_{10}^p$ is the peak dilation rate for zero normal stress. Equation (3.39) is valid only if the peak and residual shear displacements, $u_p$ and $u_r$, are constant with respect to the normal stress.

Evaluating $\tan \Phi_i^p$ from equation (3.39) gives

$$\tan \Phi_i^p = \tan \Phi_{10}^p / (1 + |\sigma_n|/C) \quad (3.40)$$

The two parameters, $\tan \Phi_{10}^p$ and $C$ have been evaluated by fitting equation (3.40) to Bandis (1981) experimental results on model replicas of joint surfaces, see Fig 3.24.

Fig 3.24 The peak dilation angle versus normal stress. Equation (3.40) compared with Bandis (1981) experimental result on model replicas of joints.
The parameters $\tan\Phi_{10}^p$ and $C$ were evaluated as

$$
\Phi_{10}^p = 0.14 \text{ JRC}^2 \\
C = \text{JCS/7JRC}
$$

(3.41)

In Fig 3.25, equation (3.38) with $C = \text{JCS/7JCR}$ is compared with reported data by Bandis et al. The initial aperture, $u_{\text{no}}$, was measured as 0.2 mm and 0.55 for the two tests on limestone bedding with JCS = 157 MPa and JRC = 7.6.

Fig 3.25  Measured and predicted normal deformation response for a bedding surface in limestone. Experimental data from Bandis, 1983.

A. Maximum closure $u_{\text{mc}} = u_{\text{no}}$, initial aperture.

B. Maximum closure $u_{\text{mc}} = u_{\text{no}}/2$. 
If $u_{mc}$ is set equal to the measured initial joint aperture the predicted response is to soft. Physically this would indicate that shear stresses causes more extensive wear on the joint surface compared with normal stresses, since the constant C in equation (3.38) was evaluated from shear tests. Setting $u_{mc} = u_{no}/2$ produces a more realistic prediction.

The remaining two parameters that have to be evaluated are the peak shear displacement, $u_p$, and the residual shear deformation, $u_r$, or the initial joint aperture, $u_{ao}$, for an unmated joint.

Barton (1971) suggested that $u_p$ was reached after shearing 1% of the sample length, i.e. the length of the joint between crossing sets of joints. Barton and Choubey (1977) and Bandis et al (1981) found that this "rule of thumb" had to be adjusted for greater joint length. Bandis et al suggested a limiting joint length of about 5 m for undulating joints and 3 m for joints with less waviness to planar surfaces.

It is reasonable to believe that the residual shear deformation is also a factor of the joint length. The experimental results presented by Bandis (1981) suggest that $u_r = 5\% - 10\% L$, where $L$ is the joint length.

Shear asperity and basic friction angle

Once knowing the component of dilation, the shear asperity and the basic friction angle can be evaluated by subtracting the dilation component from the total shear resistance. Fig 3.26 shows the total shear resistance divided into its three components, $\Phi_b$, $\Phi_s$ and $\Phi_i$ at peak resistance and after-peak shear conditions. The after-peak shear condition has been defined when $\Phi_i$ approaches zero, i.e. when $\sigma_s = \sigma_r$. The values have been derived from results on model joints (9 x 5 cm) with different surface roughness and normal pressures, Bandis et al (1981).
Fig 3.26  Dilation, $\phi_i$, shear asperity, $\phi_s$, and basic friction angle, $\phi_b$ at peak shear resistance and after the peak, $\phi_i \approx 0^\circ$. The results has been derived from test on model replicas of joint surfaces. After Bandis et al, 1981.

Since, $\phi_s$ has practically the same value at the peak as after the peak, the peak shear behaviour is caused mainly by dilation.

The conclusion is that the sum, $\phi_s + \phi_b$, is constant with respect to the shear displacement, $u_s$. Using Bartons equation on the peak shear strength, $\phi_b + \phi_s$ can be evaluated as

$$\phi_s + \phi_b = JRC \log_{10} \left( \frac{JCS}{|\sigma_n|} \right) + \phi_r - \phi_i^p$$  \hspace{1cm} (3.42)

where $\phi_i^p$ is calculated from equation (3.40).
In Fig 3.27 equation (3.42) is compared with the evaluated values on \( \Phi_s + \Phi_b \) for the model joints subtracting the measured peak dilation angle, see Fig 3.27, from the total peak friction angle. Note that the condition JRC log\(_{10}\) \((JCS/|\sigma_n|) + \Phi_f \leq 70^\circ\) effects the curve for JRC = 16.6.

Fig 3.27 Peak asperity and basic friction angle, \( \Phi_s + \Phi_b \), as a function of normal stress for various joint roughness coefficients.

3.4 Summary of joint properties

Up to the onset of failure the joint behaves elastically according to

\[
\begin{bmatrix}
  u_n \\
  u_s
\end{bmatrix} =
\begin{bmatrix}
  c_{nn} & c_{ns} \\
  c_{sn} & c_{ss}
\end{bmatrix}
\begin{bmatrix}
  q_n \\
  q_s
\end{bmatrix}
\]

(3.43)

The off-diagonal compliances \( c_{ns} \) and \( c_{sn} \) were assumed to be zero for the case of an unmated joint, i.e. no dilation. To obtain the stiffness matrix for the mated case, \( \Phi_i \neq 0 \), the stiffness matrix for an unmated joint was transformed from a plane with the inclination \( \Phi_i \) to the plane of the joint. This showed that \( c_{ns} \) and \( c_{sn} \) could be evaluated through

\[
c_{ns} = c_{sn} = (c_{ss}' - c_{nn}') \cos\Phi_i \sin\Phi_i
\]

(3.44)
Since $c_{ss}'$ and $c_{nn}'$ in general are functions of the applied stress field the sign and magnitude of the off-diagonal compliances will be strongly dependent on loading condition and joint geometry. However, considering that the elastic deformations are small compared with the unrecoverable displacements the use of a constant decoupled stiffness or compliance matrix will be recommended.

\[
\begin{align*}
\mathbf{u}_n &= c_{nn} \sigma_n = \frac{\sigma_n}{k_{nn}} \\
\mathbf{u}_s &= c_{ss} \sigma_s = \frac{\sigma_s}{k_{ss}}
\end{align*}
\]  

\[ (3.45) \]

The onset of plastic flow, i.e. failure, is initiated when

- $\sigma_n \geq 0$, separation of joint surfaces
- $\sigma_s \geq -\sigma_n \tan (\phi_b + \phi_s + \phi_i)$, shear failure
- $\sigma_s \leq \sigma_n \tan (\phi_b - \phi_s - \phi_i)$, shear failure in the opposite direction
- $\sigma_n \leq N + \sigma_s \tan \phi_i$, compressive failure

The dilation rate is a function of shear displacement according to

\[
\begin{align*}
\tan \phi_i &= \tan \phi_i^p \left[ 1 - \left( \frac{|u_s| - u_p}{u_p} \right)^2 \right]^{1/2} \quad |u_s| < u_p \\
\tan \phi_i &= \tan \phi_i^p \frac{u_r - |u_s|}{u_r - u_p} \quad u_p < |u_s| < u_r \\
\tan \phi_i &= 0 \quad |u_s| > u_r
\end{align*}
\]

\[ (3.46) \]

Integrating the dilation rate gives the dilation as a function of shear displacement

\[
\begin{align*}
\mathbf{u}_n &= \frac{\tan \phi_i^p}{2} \left[ u_x (1 - \frac{u_x}{u_p})^{1/2} + u_p \frac{\Pi}{2} + \sin^{-1} \left( \frac{u_x}{u_p} \right) \right] \quad |u_s| < u_p \\
\mathbf{u}_n &= \frac{\tan \phi_i^p}{2} \left( 2 u_x - \frac{u_x^2}{u_r - u_p} + u_p \frac{\Pi}{2} \right) \quad u_p < |u_s| < u_r \\
\mathbf{u}_n &= \frac{\tan \phi_i^p}{2} \left[ u_r - u_p \left( 1 - \frac{\Pi}{2} \right) \right] = u_a \quad |u_s| > u_r
\end{align*}
\]

\[ (3.47) \]

where $u_x = |u_s| - u_p$. 
The friction angle is determined from \( \phi_b + \phi_s = \phi^p - \phi^p_i \) where \( \phi^p \) is the total shear resistance angle. Using Barton's peak shear equation, \( \phi_b + \phi_s \) can be calculated according to

\[
\phi_b + \phi_s = \text{JRC} \log_{10}(\text{JCS}/|\sigma_n|) + \phi_r - \phi^p_i \tag{3.48}
\]

Finally the normal strength \( N \) and peak dilation rate, \( \tan \phi^p_i \) are evaluated from

\[
N = -C \frac{\Delta u_n}{u_{mc} - \Delta u_n} = -C \frac{\Delta u_n}{u_{no}^{\frac{1}{2} - \Delta u_n}} = -C \frac{u_{no} - u_n}{u_n - u_{no}^{\frac{1}{2}}}
\]

\[
\tan \phi^p_i = \frac{\tan \phi^p_{10}}{1 - \sigma_n/C}
\tag{3.49}
\]

where \( u_{no} \) is the normal dilation for \( \phi^p_i = \phi^p_{10} \), i.e. \( \sigma_n = 0 \).

The unknown parameters have as an example been fitted to Barton's parameters JRC, JCS and the joint length, i.e.

\[
\phi^p_{10} = 0.14 \text{ JRC}^2
\]

\[
C = \text{JCS}/7\text{JRC}
\tag{3.50}
\]

\[
u_p = 0.01 L \quad \text{L} < \text{L}_c \quad \text{(critical joint length)}
\]

\[
0.05 \text{ L} < \nu_r < 0.1 \text{ L} \quad \text{L} < \text{L}_c
\]

The reason for choosing the constants JRC, JCS and L is that they have been fitted to a wide variety of joints, Barton (1973, 1974), Barton et al (1974, 1977), Bandis et al (1981, 1983) etc. However the parameters are measurable quantities and could be determined from experiments.
4 THE PROPERTIES OF INTACT ROCK

The deformability of a rock mass results predominantly from sets of joints that intersect the intact rock, at least the hard rock common in Scandinavia.

All non-recoverable deformations in the rock mass are assumed to be restricted to the joint system. Hence, only the elastic constitutive equations for the intact rock have to be considered.

4.1 Linear elastic properties

The intact rock is assumed to be a homogenous linear elastic material. The constitutive equation for a linear elastic material relates stress to strain through the compliance, $C$

$$
\varepsilon_{ij} = C_{ijkl} \sigma_{k1}
$$

(4.1)

or

$$
\sigma_{ij} = S_{ijkl} \varepsilon_{ij}
$$

(4.2)

where $S$ is the stiffness tensor. The stiffness matrix is the inverse of the compliance matrix $S = (C)^{-1}$.

The compliance or the stiffness matrix have 81 components. However due to symmetry of both stress and strain and the existence of a strain energy function the number of independent elastic constants are reduced to twenty-one at the most. Thus equation (4.1) can be written as
\[
\begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
2\varepsilon_{12} \\
2\varepsilon_{13} \\
2\varepsilon_{23}
\end{bmatrix}
= \begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{33} & C_{34} & C_{35} & C_{36} \\
Symmetric & C_{44} & C_{45} & C_{46} \\
& C_{55} & C_{56} \\
& & C_{66}
\end{bmatrix}
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{13} \\
\sigma_{23}
\end{bmatrix}
\] (4.3)

Rock often possesses at least three mutually perpendicular planes of symmetry. Such material is called orthotropic and the independent elastic constants are reduced to nine. If \(E_1, E_2, E_3\) are the elastic Young's modulus, \(v_{12}, v_{13}, v_{23}\) the Poisson's ratios and \(G_{12}, G_{13}, G_{23}\) the shear modulus associated with the planes of symmetry, the elastic compliance matrix \(C\) can be written as:

\[
C = \begin{bmatrix}
\frac{1}{E_1} & -\frac{v_{12}}{E_2} & -\frac{v_{13}}{E_3} & 0 & 0 & 0 \\
-\frac{v_{12}}{E_2} & \frac{1}{E_2} & -\frac{v_{23}}{E_3} & 0 & 0 & 0 \\
-\frac{v_{13}}{E_3} & -\frac{v_{23}}{E_3} & \frac{1}{E_3} & 0 & 0 & 0 \\
Symmetric & \frac{1}{G_{12}} & 0 & 0 \\
& & \frac{1}{G_{13}} & 0 \\
& & & \frac{1}{G_{23}}
\end{bmatrix}
\] (4.4)

A special type of orthotropic material is the transversely isotropic material. Stratified or bedded rock like sandstone or limestone are typical examples of transversely isotropic rocks. If the material is bedded perpendicularly to 1-direction, it is isotropic in the 2-3 plane, see Fig 4.1.
Fig 4.1  Transversely isotropic material

The following relationships then hold

\[ E_2 = E_3 \]
\[ \nu_{12} = \nu_{13} \]  \hspace{1cm} (4.5)
\[ G_{12} = G_{13} \]
\[ G_{23} = E_2 / 2(1 + \nu_{23}) \]

The compliance matrix \( C \) can thus be formulated as

\[
C = \begin{bmatrix}
\frac{1}{E_1} & -\frac{\nu_{12}}{E_2} & -\frac{\nu_{12}}{E_2} & 0 & 0 & 0 \\
-\frac{\nu_{12}}{E_2} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0 \\
\frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_2} & 0 & 0 & 0 \\
\frac{1}{G_{12}} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{G_{12}} & 0 & 0 & 0 & 0 & 0 \\
\frac{2(1+\nu_{23})}{E_2} & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]  \hspace{1cm} (4.6)
For an isotropic material the independent elastic constants are reduced to two, the Young's modulus $E$ and the Poisson's ratio $\nu$

\[
C = \begin{bmatrix}
\frac{1}{E} & \frac{-\nu}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\
\frac{1}{E} & \frac{-\nu}{E} & 0 & 0 & 0 \\
\frac{1}{E} & 0 & 0 & 0 & 0 \\
\frac{2(1+\nu)}{E} & 0 & 0 \\
\frac{2(1+\nu)}{E} & 0 \\
\frac{2(1+\nu)}{E}
\end{bmatrix}
\]

(4.7)

Symmetric
ELASTO-VISCO-PLASTIC MODEL FOR A JOINTED ROCK MASS

In this chapter an elasto-visco-plastic formulation will be used to model the stress-strain behaviour of a jointed rock mass. The discontinuous displacement in the joint system will be evenly distributed over a certain volume of rock. The constitutive equations will be formulated for an equivalent material consisting of intact rock separated by several sets of joints.

5.1 Formulation of an equivalent material

Consider a jointed rock mass of a certain volume, Fig 5.1. The load-deformation characteristics of that volume is governed by the material properties of the two components, i.e. the intact rock and the intersecting joints.

Fig 5.1 Basic model of the rock mass
A. A jointed rock mass
B. Idealized model

In many cases the rock mass can be idealized as volumes of intact rock bounded by sets of evenly spaced continuous parallel joints. It is further assumed that the joints occupy no volume so that an applied stress is distributed homogeneously over the jointed rock mass. The total strain, $\varepsilon$, for the jointed rock mass can thus be calculated by summing the strain components from the intact rock $R_{\varepsilon}$ and the intersecting joints, $J_{\varepsilon}$.
\( \varepsilon = R \varepsilon + J \varepsilon \) \hspace{1cm} (5.1)

The total stress \( \sigma \) is equal to the stress in the intact rock \( R \sigma \) and intersecting joints \( J \sigma \)

\( \sigma = R \sigma + J \sigma \) \hspace{1cm} (5.2)

Fig 5.2 demonstrates how the total stress and strain are calculated for a rock with a single joint in one dimension.

Fig 5.2 Model of a one-dimensional rock-joint system

If the total stress, \( \sigma \) is applied to the rock in Fig 5.2 the total strain, \( \varepsilon \), can be calculated as follows

\[ \varepsilon = R \varepsilon + J \varepsilon = \sigma/R_s + \sigma/J_s \]

\[ \varepsilon = \sigma(R_s + J_s)/R_s J_s \] \hspace{1cm} (5.3)

where \( R_s \) and \( J_s \) are the stiffness of the intact rock and joint respectively, see Fig 5.2.
The validity of adding the contribution from each component to a constitutive law assuming a continuum basis depends on the magnitude of the discrete displacement in the joint. Large discrete displacements make the equivalent material approach less valid. However, for rock masses with several sets of joints the equivalent material offers computational advantages compared with discrete systems, for example in the finite element method. Great care must be taken when interpreting the results from the model, especially when large displacements occur in intersecting joint sets.

5.2 Elasto-visco-plastic theory

Zienkiewicz & Pande (1977) were among the first to use an elasto-visco-plastic formulation for jointed rock masses. Later this formulation was extended to include reinforcement, Pande and Gerrard (1983), Larsson and Olofsson (1983).

The elasto-visco-plastic algorithm allows the modelling of both rate effects in the deformation process, and the quasistatic response. In the latter case the elasto-visco-plastic models offer some advantages over the classical theory of plasticity, where time is used as a computational parameter only.

An elasto-visco-plastic material is elastic only up to a certain level of stress. When the stress reaches this critical level, the material yields. The critical level or the yield condition can be expressed as a surface in the stress space where the coordinate axis coincide with the direction of the principal stresses.
Fig 5.3  Yield condition, $F$, in the principal stress space for an elasto-visco-plastic material

Every point in this space corresponds to a unique state of stress. Stresses that are inside the yield surface represent the elastic state of stress, $F < 0$. Stress points outside the surface represent the viscoplastic state of stress with the viscous stress proportional to the excess of stress over the yield stress, $F > 0$. Stress states on the surface represent the quasi-static response, i.e. the plastic state of stress, $F = 0$.

The elasto-visco-plastic model of the jointed rock mass can qualitatively be represented by the one-dimensional rheological model shown in Fig 5.4. The contribution from the individual components of the jointed rock mass are included in the scheme.
Fig 5.4  Rheological representation of the rock mass model

Axial force in the model represents the stress tensor in the continuum, while axial elongation represents the strain tensor.

The spring models instantaneous elastic response in the model. The friction glider becomes effective only when $F \geq 0$, where $F$ is the limiting yield function. The presence of the dashpot allows the stress level to exceed the yield stress, $F > 0$.

The one-dimensional rheological model is now extended to the case of a general continuum, with arbitrarily oriented sets of joints. The basic elasto-visco-plastic constitutive relations for the composite system can be formulated by considering the contribution from each system separately. Writing equations (5.1) and (5.2) in rate form for $m$ set of joints, we obtain
\[ \dot{\varepsilon} = R \dot{\varepsilon} + \sum_{i}^{m} (J_{i})_{ij} \]  

(5.4)

\[ \dot{\gamma} = R \dot{\gamma} = (J_{\gamma})_{1} = \ldots = (J_{\gamma})_{m} \]  

(5.5)

As usual in elasto-visco-plastic formulations the total strain rate is divided into an elastic part and a viscoplastic part

\[ \dot{\varepsilon} = \dot{\varepsilon}^{e} + \dot{\varepsilon}^{vp} \]  

(5.6)

The viscoplastic strain rate is determined by the viscoplastic flow rule

\[ \dot{\varepsilon}^{vp} = \gamma <\Phi(F)> \frac{\partial Q}{\partial \gamma} \]  

(5.7)

where \( \gamma \) controls the plastic flow rate. The term \( \Phi(F) \) is a positive monotonic increasing function for \( F > 0 \) and the notation \( < > \) implies that

\[ <\Phi(F)> = \Phi(F) \text{ for } F > 0 \]  

(5.8)

\[ <\Phi(F)> = 0 \text{ for } F \leq 0 \]

\( \frac{\partial Q}{\partial \gamma} \) defines the viscoplastic flow tensor. If \( Q \equiv F \) the restriction of associated plasticity is imposed.

The viscoplastic strain rate for the jointed rock mass can be determined by summation over the \( m \) sets of joints

\[ \dot{\varepsilon}^{vp} = \sum_{i}^{m} \gamma <\Phi(F)> \frac{\partial Q}{\partial \gamma}_{i} \]  

(5.9)
For an elasto-visco-plastic material the total stress rate depends only on the elastic strain rate, i.e.

\[ \dot{\varepsilon} = S \dot{\varepsilon}^e \] (5.10)

or

\[ \dot{\varepsilon}^e = C \dot{\varepsilon} \] (5.11)

where \( S \) and \( C \) are the elastic stiffness and compliance tensor for the combined rock joint system.

Using equations (5.4) and (5.5) in equation (5.11) gives

\[ \dot{\varepsilon}^e = R_C \dot{\varepsilon} + \sum_i^{m(JC)} \dot{\varepsilon} = [R_C + \sum_i^{m(JC)}] \dot{\varepsilon} \] (5.12)

Comparison with equation (5.11) gives

\[ C = R_C + \sum_i^{m(JC)} \] (5.13)

Using equation (5.6) together with the relation \( S = (C)^{-1} \), the stress rate for the jointed rock mass can be formulated as

\[ \dot{\varepsilon} = S \dot{\varepsilon}^e = S \dot{\varepsilon} - S \dot{\varepsilon}^{VP} \] (5.14)

Defining \( S \dot{\varepsilon}^{VP} \) as the viscoplastic stress rate, \( \dot{\varepsilon}^{VP} \), equation (5.14) can be rewritten as

\[ \dot{\varepsilon} + \dot{\varepsilon}^{VP} = S \dot{\varepsilon} \] (5.15)

The strain components from the individual sets of joints have to be transformed to the global coordinated system before adding them to the total strain rate of the rock mass, equation (5.16). Consider a set of joints in a rock mass, Fig 5.5.
Fig 5.5 A set of joints in a rock, with associated stresses and strains in a local coordinate system.

A local coordinate system is associated with a set of joints, $Jx$. $Jx_1$ coincides with the normal direction to the joint plane. $Jx_2$ and $Jx_3$ coincide with the dip and strike directions, respectively.

The strain rate for a set of joints is defined as

$$\dot{J}\varepsilon_{1j} = \frac{1}{2}(\delta_{1j}\dot{u}_j + \dot{u}_j)/d \tag{5.16}$$

where $\delta_{1j}$ is the Kronecker's delta. The stress field $J\tilde{\sigma}$ acting on the joint set in Fig 5.5 is calculated from the global stress field $\tilde{\sigma}$ by the stress transformation law

$$J\tilde{\sigma}_{1j} = r_{1p} r_{jq} \delta_{pq} \tag{5.17}$$

$$J\tilde{\sigma} = R \tilde{\sigma}$$

where $r_{ij}$ is the direction of cosines and
\[ J x_j = r_{ij} x_j \]  

(5.18)

The total strain rate over the joints, \( J_\xi \), caused by the stress field \( J_g \), is divided into an elastic and a viscoplastic part

\[ J_{\xi 1j} = J_{\xi e 1j} + J_{\xi vp 1j} \]  

(5.19)

The local strain rate is transformed to the global coordinate system, \( J_\xi \), before adding the strain over the joints set to the total strain rate over the jointed rock mass, cf equation (5.4)

\[ J_{\xi ij} = r_{1i} r_{qj} J_{\xi 1q} \]  

(5.20)

\[ J_\xi = R^T J_\xi \]

5.3 Elastic properties of the rock mass

The compliance matrix, \( C \), can be evaluated from equation (5.13) as

\[ C = R_C + \sum_i^m (J^i_C) \]  

(5.21)

where \( R_C \) is the compliance matrix for the intact rock and \( (J^i_C) \) is the compliance matrix for joint set \( i \) transformed to the global coordinate system.

In the local coordinate system the elastic compliance equation for a joint set can be written as

\[ \begin{bmatrix} J_{\xi e 11} \\ 2 J_{\xi e 12} \\ 2 J_{\xi e 13} \end{bmatrix} = J \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} \xi_{11} \\ \xi_{12} \\ \xi_{13} \end{bmatrix} \]  

(5.22)
or in compact notation

\[
\mathbf{j}_e = \mathbf{j}_c \mathbf{j}_g
\]  

(5.23)

Since the strain energy is an invariant quantity, i.e. independent of the coordinate system, we can write

\[
(\mathbf{q}^T \mathbf{j}_c \mathbf{e}) \mathbf{q}_e = (\mathbf{q}^T \mathbf{j}_c \mathbf{e}) \mathbf{q}_e
\]  

(5.24)

\[
(\mathbf{q}^T \mathbf{j}_c \mathbf{q}_c = (\mathbf{q}^T \mathbf{j}_c \mathbf{q}_c
\]  

Using the transformation tensor \( \mathbf{R} \), equation (5.17), the compliance matrix \( \mathbf{j}_c \) can be evaluated

\[
\mathbf{j}_c = \mathbf{R}^T \mathbf{j}_c \mathbf{R}
\]  

(5.25)

Equation (4.21) can thus be written as

\[
\mathbf{C} = \mathbf{R}_c + \sum_i^m (\mathbf{R}^T \mathbf{j}_c \mathbf{R})_i
\]  

(5.26)

The transformation equation (5.17) can be expressed in the dip angle \( \alpha \), and the strike angle, \( \beta \), of the joint set, see Fig. 5.6, as

\[
\begin{bmatrix}
j_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{13} \\
\sigma_{23}
\end{bmatrix} =
\begin{bmatrix}
c^2 \alpha & s^2 \alpha & c^2 \beta & s^2 \alpha & c^2 \beta & s^2 \alpha & c^2 \beta & s^2 \alpha & s^2 \beta \\
s^2 \alpha & c^2 \alpha & c^2 \beta & c^2 \alpha & c^2 \beta & -s^2 \alpha & c^2 \beta & -s^2 \alpha & s^2 \beta \\
0 & s^2 \beta & c^2 \beta & 0 & 0 & -s^2 \beta \\
-\frac{1}{2} s^2 \alpha & \frac{1}{2} s^2 \alpha & c^2 \beta & \frac{1}{2} s^2 \alpha & s^2 \beta & c^2 \alpha & c^2 \beta & \frac{1}{2} s^2 \alpha & s^2 \beta \\
0 & -\frac{1}{2} s^2 \alpha & s^2 \beta & \frac{1}{2} s^2 \alpha & s^2 \beta & -c^2 \alpha & c^2 \beta & s^2 \alpha & c^2 \beta \\
0 & -\frac{1}{2} s^2 \alpha & s^2 \beta & \frac{1}{2} c^2 \alpha & s^2 \beta & s^2 \alpha & -s^2 \alpha & c^2 \beta & c^2 \beta \\
\end{bmatrix}
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{13} \\
\sigma_{23}
\end{bmatrix}
\]

(5.27)

c = \cos
s = \sin
Fig 5.6  Definition of dip angle, $\alpha$, and strike angle $\beta$ of a set of joints
A. Lower hemisphere stereographic projection
B. Corresponding plane in the 3-D coordinate system
The \( j_x_1 \)-axis coincides with the normal to the joint plane. \( j_x_2 \) lies in the dip direction and \( j_x_3 \) in the strike direction of the joint.

For 2-D analysis equation (5.27) reduces to

\[
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12}
\end{bmatrix} =
\begin{bmatrix}
\cos^2 \alpha & \sin^2 \alpha & \sin \alpha \\
\sin^2 \alpha & \cos^2 \alpha & -\sin \alpha \\
-\frac{1}{2} \sin 2\alpha & \frac{1}{2} \sin 2\alpha & \cos 2\alpha
\end{bmatrix}
\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{12}
\end{bmatrix}
\] (5.28)

The angle \( \alpha \) is defined as in Fig 5.7.

Fig 5.7 Definition of joint angle, \( \alpha \) and joint spacing, \( d \) for 2-D analysis of jointed rock mass.

In the following section a few examples of application of equations (5.21) and (5.25) will be given.

5.3.1 Case 1, one set of joints, plane strain condition

Consider the case of one set of joints in a rock mass oriented in such a way that plane strain conditions can be assumed, Fig 5.8. The intact rock is assumed to be linear elastic and isotropic. The joint deformation is governed by the normal and shear compliances, \( c_{nn} = 1/dk_{nn} \), \( c_{ss} = 1/dk_{ss} \), respectively. The off-diagonal terms in the matrix are assumed to be zero.
Fig 5.8 One set of joints in a rock mass, $\alpha = 0^\circ$, plane strain conditions

Using equations (5.21) and (5.25)

$$C = R_C + R^T J_C R$$

$$R_C = \begin{bmatrix}
\frac{1-v^2}{E} & -\frac{\nu(1+\nu)}{E} & 0 \\
-\frac{\nu(1+\nu)}{E} & \frac{1-v^2}{E} & 0 \\
0 & 0 & \frac{2(1+\nu)}{E}
\end{bmatrix}$$

$$R^T J_C R = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
C_{nn} & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & C_{ss}
\end{bmatrix} \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}$$
giving the compliance for the rock mass as

\[
C = \begin{bmatrix}
\frac{1-v^2}{E} + \frac{1}{d k_{nn}} & -\frac{v(1+v)}{E} & 0 \\
-\frac{v(1+v)}{E} & \frac{1-v^2}{E} & 0 \\
0 & 0 & \frac{2(1+v)}{E} + \frac{1}{d k_{ss}}
\end{bmatrix}
\]

(5.29)

Next, the compliance for a rock mass, consisting of two joint sets oriented \( \alpha = \pm 30^0 \), will be calculated, see Fig 5.9.

Plane strain condition is assumed and all off-diagonal joint compliances are neglected.

![Diagram of two sets of joints](image)

**Fig 5.9** Two sets of joints oriented \( \alpha = \pm 30^0 \), plane strain conditions
The joint compliances, $J_C$, transformed to the global coordinate system are

$$J_{C1} = \begin{bmatrix} 3 & 1 & -\sqrt{3} \\ 4 & 4 & 0 \\ 1 & 1 & 4 \\ 0 & 0 & 0 \\ \sqrt{3} & \sqrt{3} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} c_{nn} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_{ss} \end{bmatrix} \begin{bmatrix} 3 & 1 & \sqrt{3} \\ 4 & 4 & 0 \\ 1 & 1 & 4 \\ 0 & 0 & 0 \\ \sqrt{3} & \sqrt{3} & \frac{1}{2} \end{bmatrix}$$

$$J_{C1} = \frac{1}{16} \begin{bmatrix} 9c_{nn} + 3c_{ss} & 3c_{nn} - 3c_{ss} & 6\sqrt{3}c_{nn} - 2\sqrt{3}c_{ss} \\ c_{nn} + 3c_{ss} & 2\sqrt{3}c_{nn} + 2\sqrt{3}c_{ss} & \text{Symmetric} \\ 12c_{nn} + 4c_{ss} \\ \end{bmatrix}$$

$$J_{C2} = \frac{1}{16} \begin{bmatrix} 9c_{nn} + 3c_{ss} & 3c_{nn} - 3c_{ss} & 6\sqrt{3}c_{nn} + 2\sqrt{3}c_{ss} \\ c_{nn} + 3c_{ss} & -2\sqrt{3}c_{nn} - 2\sqrt{3}c_{ss} & \text{Symmetric} \\ 12c_{nn} + 4c_{ss} \end{bmatrix}$$
If the stiffness properties are the same for both set of joints, i.e. \((k_{nn})_1 = (k_{nn})_2\), \((k_{ss})_1 = (k_{ss})_2\), then the compliance for the jointed rock mass can be written as

\[
C_{11} = \frac{1+\nu}{E} + \frac{3}{16} \left( \frac{3}{k_{nn}} + \frac{1}{k_{ss}} \right) \left( \frac{1}{d_1} + \frac{1}{d_2} \right)
\]

\[
C_{12} = C_{21} = -\sqrt{\frac{(1+\nu)}{E}} + \frac{3}{16} \left( \frac{1}{k_{nn}} - \frac{1}{k_{ss}} \right) \left( \frac{1}{d_1} + \frac{1}{d_2} \right)
\]

\[
C_{13} = C_{31} = \frac{2\sqrt{3}}{16} \left( \frac{3}{k_{nn}} - \frac{1}{k_{ss}} \right) \left( \frac{1}{d_1} - \frac{1}{d_2} \right)
\]

\[
C_{22} = \frac{1+\nu}{E} + \frac{1}{16} \left( \frac{1}{k_{nn}} + \frac{3}{k_{ss}} \right) \left( \frac{1}{d_1} - \frac{1}{d_2} \right)
\]

\[
C_{23} = C_{32} = \frac{2\sqrt{3}}{16} \left( \frac{1}{k_{nn}} + \frac{1}{k_{ss}} \right) \left( \frac{1}{d_1} - \frac{1}{d_2} \right)
\]

\[
C_{33} = \frac{2(1+\nu)}{E} + \frac{4}{16} \left( \frac{3}{k_{nn}} + \frac{1}{k_{ss}} \right) \left( \frac{1}{d_1} + \frac{1}{d_2} \right)
\]

\[\text{(5.30)}\]

5.3.2 Case 2, three sets of orthogonal joints

In Fig 5.10, three sets of joints mutually perpendicular intersect a rock mass.

As in case one, section 5.3.1, the intact rock is linear elastic and isotropic. Each set of joint has a normal stiffness and a shear stiffness that couple deformation to stresses. The shear stiffness in the plane of the joint surface is assumed to be isotropic.
Fig 5.10 Three sets of joints intersecting a rock mass, each set is perpendicular to the other two.

The compliance for the intact rock, $R_C$ is

$$R_C = \begin{bmatrix}
\frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\
\frac{1}{E} & \frac{1}{E} & 0 & 0 & 0 \\
\frac{1}{E} & 0 & 0 & 0 \\
\text{Symmetric} & \frac{1}{G} & 0 & 0 \\
\frac{1}{G} & 0 & 0 \\
\frac{1}{G} & 0 & 0 & 0
\end{bmatrix}$$

$$G = \frac{E}{2(1+\nu)}$$
Joint set 1, $\alpha = 90^0$, $\beta = 0$

$$J_{C_1} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{d_1 k_{nn}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{d_1 k_{ss}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{d_1 k_{ss}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix} = \frac{1}{d_1} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{k_{nn}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{k_{ss}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{k_{ss}} \end{bmatrix}$$
Joint set 2, \( \alpha = 90^\circ, \beta = 90^\circ \)

\[
J_{c2} = \frac{1}{d_2}
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{k_{nn}} & 0 & 0 & 0 & 0 & 0 \\
\text{Symmetric} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{k_{ss}} & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

Joint set 3, \( \alpha = 0^\circ, \beta = 0^\circ \)

\[
J_{c3} = \frac{1}{d_3}
\begin{bmatrix}
\frac{1}{k_{nn}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\text{Symmetric} & \frac{1}{k_{ss}} & 0 & 0 & 0 & 0 \\
\frac{1}{k_{ss}} & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
Adding $R_C$, $J_{C_1}$, $J_{C_2}$ and $J_{C_3}$ gives the total compliance as

$$C = \begin{bmatrix}
\frac{1}{E} + \frac{1}{d_{3}k_{nn}} & \frac{\nu}{E} & \frac{\nu}{E} & 0 & 0 & 0 \\
\frac{1}{E} + \frac{1}{d_{1}k_{nn}} & \frac{\nu}{E} & 0 & 0 & 0 & 0 \\
\frac{1}{E} + \frac{1}{d_{3}k_{nn}} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{G} + \frac{1}{d_{1}k_{ss}} + \frac{1}{d_{2}k_{ss}} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{G} + \frac{1}{d_{1}k_{ss}} + \frac{1}{d_{2}k_{ss}} & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{G} + \frac{1}{d_{1}k_{ss}} + \frac{1}{d_{2}k_{ss}} & 0 & 0 & 0 & 0 & 0 
\end{bmatrix}$$

5.3.3 Randomly distributed joints

If the joints are randomly distributed throughout the rock mass, i.e. they have no preferred direction, then the rock mass will remain isotropic if the intact rock itself is isotropic.

Instead of using equations (5.21) and (5.25), two simple load conditions will be analysed in order to evaluate the elastic constants of the rock mass. Budiansky used this approach to evaluate the elastic properties of material with randomly distributed penny shaped cracks, Budiansky et al (1976) and Olofsson (1980).

The intact rock is assumed to be linear elastic and isotropic. The off-diagonal compliances of the joint are neglected. Further, the shear stiffness is assumed to be isotropic in the plane of the joint surface.
Consider a randomly jointed rock mass under influence of a uniaxial stress field, \( \sigma_u \). The strain energy can be written as

\[
\frac{\sigma_u^2}{2} \frac{V}{E_M} = \frac{\sigma_u^2}{2E_R} + \sum_i \Delta \psi_i
\]  

(5.32)

where \( V \) is the volume of the rock mass, \( E_M \) is the elastic modulus of the rock mass, \( E_R \) is the elastic modulus of the intact rock and \( \Delta \psi_i \) the strain energy for joint \( i \).

The strain energy for one joint can be written as, see Fig 5.11,

\[
\Delta \psi = \frac{A_j}{2} \left( \frac{\sigma_{11}}{k_{nn}} + \frac{\sigma_{12}}{k_{ss}} + \frac{\sigma_{13}}{k_{ss}} \right)
\]  

(5.33)

but

\[
\sigma_{11} = \sigma_u \cos^2 \alpha
\]

\[
\sigma_{12} = \sigma_u \sin \alpha \cos \alpha \cos \beta
\]  

(5.34)

\[
\sigma_{13} = \sigma_u \sin \alpha \cos \alpha \sin \beta
\]

Equation (5.34) inserted in equation (5.33) gives

\[
\Delta \psi = \frac{\sigma_u^2}{2} A_j \left( \frac{\cos^4 \alpha}{k_{nn}} + \frac{\sin^2 \alpha \cos^2 \alpha}{k_{ss}} \right)
\]  

(5.35)

where \( A_j \) is the area of the joint surface.
Fig 5.11  An arbitrarily oriented joint loaded with a uniaxial stress field

For m-joints equation (5.32) can be written as

\[
\frac{\sigma_u^2}{2} \frac{V}{E_M} = \frac{\sigma_u^2}{2} \frac{V}{E_R} + \frac{\sigma_u^2}{2} \sum \frac{m}{A_j} \left[ \frac{\cos^4 \alpha + \sin^2 \alpha \cos^2 \alpha}{k_{nn}} + \frac{\sin^2 \alpha \cos^2 \alpha}{k_{ss}} \right]
\]  (5.36)

If \( \{ \} \) denotes the mean values, equation (5.36) can be formulated as

\[
\frac{1}{E_M} = \frac{1}{E_R} + \left( \frac{\cos^4 \alpha}{[k_{nn}]} + \frac{\sin^2 \alpha \cos^2 \alpha}{[k_{ss}]} \right) m[A_j] \tag{5.37}
\]

The mean value of \( \cos^4 \alpha \) and \( \sin^2 \alpha \cos^2 \alpha \) over all angles of \( \alpha \) and \( \beta \) can be calculated with the formula

\[
\{ f(\alpha) \} = \int_0^\pi f(\alpha) \, d\alpha / \int_0^\pi \sin \beta \, d\beta \tag{5.38}
\]

which gives \( \{ \cos^4 \alpha \} = 3/15 \) and \( \{ \sin^2 \alpha \cos^2 \alpha \} = 2/15 \).
\[ \frac{A_j}{V} \] is in fact the joint frequency, \( f \) (joints per m), which reduces equation (5.37) to

\[ \frac{1}{E_M} = \frac{1}{E_R} + \frac{f}{15} \left( \frac{3}{[k_{nn}]} + \frac{2}{[k_{ss}]} \right) \]  
(5.39)

At this point another relationship is needed to derive the other elastic constant. This can be found by applying a hydrostatic pressure \( p \) to the system

\[ \frac{p^2 V}{2K_M} = \frac{p^2 V}{2K_R} + \sum \frac{m}{i} (A_j \frac{1}{k_{nn}}) \]  
(5.40)

giving

\[ \frac{1}{K_M} = \frac{1}{K_R} + \frac{f}{[k_{nn}]} \]  
(5.41)

where \( K_M, K_R \) is the bulk modulus for the jointed rock and intact rock, respectively.

If the elastic and bulk modulus are known the Poisson’s ratio, \( \nu \), and the shear modulus, \( G \), can be determined using the relations

\[ \frac{1}{G} = \frac{E}{3} - \frac{1}{3K} \]

\[ \nu = \frac{1}{2} \left( 1 - \frac{E}{3K} \right) \]

giving the elastic properties for the jointed rock mass as

\[ E_M = \frac{1}{\left( \frac{1}{E_R} + \frac{f}{15} \left( \frac{3}{[k_{nn}]} + \frac{2}{[k_{ss}]} \right) \right)} \]

\[ K_M = \frac{1}{\left( \frac{3}{E_R} - \frac{6\nu_R}{E_R} + \frac{f}{[k_{nn}]} \right)} \]  
(5.43)

\[ \nu_M = \frac{1}{2} \left( 1 - \frac{E_M}{3K_M} \right) \]

\[ G_M = \frac{1}{\left( \frac{2}{E_M} - \frac{1}{3K_M} \right)} \]
5.4  Viscoplastic properties of the jointed rock mass

The viscoplastic strain rate for one set of joints can be calculated by the formula

$$\dot{\varepsilon}_{vp} = \sum_{i} \left( R^T \varepsilon_{vp} \right)_i = \sum_{i} \left( R^T \gamma \Phi(F) \frac{\partial \Phi}{\partial g} \right)_i$$  \hspace{1cm} (5.44)

where

$$\Phi(F) = \Phi(F) \quad \text{for } F \geq 0$$  \hspace{1cm} (5.45)

$$\Phi(F) = 0 \quad \text{for } F < 0$$

The tensor $R^T$ transforms the viscoplastic strain rate in the local coordinate system for joint set $i$ to the global coordinate system.

The factor $\gamma \Phi(F)$ determines the magnitude of the viscoplastic flow and the tensor $\frac{\partial \Phi}{\partial g}$ the direction. In order to evaluate the fluidity parameter $\gamma$ and the function $\Phi(F)$, the deformation rate behaviour of joints has to be experimentally determined. However, very few tests have so far been reported on joints including the deformation rate behaviour. Curran and Leong (1983) examined the shear strength of joints at different levels of normal stress using shear velocity rates from 2 to 256 mm/sec. They reported a decrease in shear strength with increasing shear velocity. However, experimental studies on time dependence are at present too few to be able to draw any conclusion concerning the $\gamma$-parameter and the $\Phi(F)$-function.

In this study we will restrict ourselves to the quasi-static solution, i.e. the plastic state, and use time only as a computational parameter. The choice of $\gamma$ and $\Phi(F)$ is then arbitrary. The only restriction on $\Phi(F)$ is that it should be a monotonic increasing function for $F > 0$. One fulfilment of these requirements is

$$\Phi(F) = F \quad \text{for } F > 0$$  \hspace{1cm} (5.46)
which reduces equation (5.44) to

$$
\dot{\xi}_l = \sum_{i=1}^{m} \left( R \gamma_{i} \alpha \right) \frac{\partial Q}{\partial g_i}
$$

(5.47)

Hence, the functions that have to be evaluated are the yield-function, \( F \), and the viscoplastic potential \( Q \).

5.4.1 Yield function, \( F \)

Different proposals on yield functions for joints have already been discussed in chapter 3. In this chapter the basic yield function in the \( \sigma_n, \sigma_s \) stress space will be extended to a three-dimensional state of stress, see Fig 5.12.

![Fig 5.12](image)

**Fig 5.12** Yield surface for a joint, \( F \), in the \( \sigma_{11}, \sigma_{12}, \sigma_{13} \) stress space. Note that the part of the yield cone that is shadowed is below the \( \sigma_{12}-\sigma_{13} \) plane.
The stress field, $Jg$, in the local coordinate system acts on a joint plane.

The basic yield function in the primed state of stress, $Jg'$, (i.e. when $\Phi_i = \lambda = 0$), can be written as

$$F_I' = J_{\sigma_{11}'}$$

$$F_{II}' = (J_{\sigma_{12}'})^2 + (J_{\sigma_{13}'})^2 - (c_0 - J_{\sigma_{11}'} \tan(\Phi_b + \Phi_s))^2$$  \(5.48\)

$$F_{III}' = N - J_{\sigma_{11}'}$$

where $N$ is the normal cap and $c_0$ the cohesion.

The general yield criterion when the dilation angle $\Phi_i$ and the shear direction angle $\lambda$ are $\neq 0$ can be evaluated by simple laws of coordinate transformation (no change in stress area, compare with the 2-D case in chapter 3).

Using

$$\begin{bmatrix} J_{\sigma_{11}'} \\ J_{\sigma_{12}'} \\ J_{\sigma_{13}'} \end{bmatrix} = \begin{bmatrix} \cos \Phi_i & \sin \Phi_i \cos \lambda & -\sin \Phi_i \sin \lambda \\ \sin \Phi_i & \cos \Phi_i \cos \lambda & \cos \Phi_i \sin \lambda \\ 0 & -\sin \lambda & \cos \lambda \end{bmatrix} \begin{bmatrix} J_{\sigma_{11}} \\ J_{\sigma_{12}} \\ J_{\sigma_{13}} \end{bmatrix}$$  \(5.49\)

equation (5.48) can be rewritten as

$$F_I' = J_{\sigma_{11}'} \cos \Phi_i - J_{\sigma_{12}'} \sin \Phi_i \cos \lambda - J_{\sigma_{13}'} \sin \Phi_i \sin \lambda$$

$$F_{II}' = (J_{\sigma_{11}'} \sin \Phi_i + J_{\sigma_{12}'} \cos \Phi_i \cos \lambda + J_{\sigma_{13}'} \cos \Phi_i \sin \lambda)^2 +$$

$$+ (- J_{\sigma_{12}'} \sin \lambda + J_{\sigma_{13}'} \cos \lambda)^2 - [c_0 - \tan(\Phi_b + \Phi_s) \cdot (J_{\sigma_{11}'} \cos \Phi_i - J_{\sigma_{12}'} \sin \Phi_i \cos \lambda - J_{\sigma_{13}'} \sin \Phi_i \sin \lambda)]^2$$  \(5.50\)

$$F_{III}' = N - J_{\sigma_{11}'} \cos \Phi_i + J_{\sigma_{12}'} \sin \Phi_i \cos \lambda + J_{\sigma_{13}'} \sin \Phi_i \sin \lambda$$
The shear direction angle $\lambda$ relates to the actual mating of the joint in the $x_2$-$x_3$ plane, i.e.

$$\lambda = \arctan\left(\frac{j_{c13}^{VP}}{j_{c12}^{VP}}\right)$$  \hspace{1cm} (5.51)$$

where $j_{c12}^{VP}$ and $j_{c13}^{VP}$ is the cumulated viscoplastic strain on the joint surface.

The parameters $\Phi_i$, $\Phi_s+\Phi_d$ and $N$ can be determined from equations (3.43), (3.45) and (3.50), see chapter 3.4.

5.4.2 Viscoplastic potential, $Q$

The viscoplastic potential, $Q$, controls the direction of the viscoplastic flow on the joint by the expression

$$\frac{j_{\varepsilon}^{VP}}{\varepsilon} = \gamma \Phi(F) \frac{\partial Q}{\partial \varepsilon}$$  \hspace{1cm} (5.52)$$

The viscoplastic potential can be represented by a surface in the stress space. The gradient, $\partial Q/\partial \varepsilon$, to the surface then represents the direction of the viscoplastic flow, $j_{\varepsilon}^{VP}$.

In Fig 5.13 the proposed viscoplastic function, $Q$, for a set of joints is shown: (A) in the local stress field $j_{\sigma_{11}}$, $j_{\sigma_{12}}$ and $j_{\sigma_{13}}$ and (B) in the normal stress, $\sigma_n$, and shear stress, $\sigma_s$, space.
Fig 5.13  Viscoplastic potential $Q$ for a set of joints
A. In the $j_{\sigma_{11}}$, $j_{\sigma_{12}}$ and $j_{\sigma_{13}}$ stress space
B. In the $\sigma_n$, $\sigma_s$ stress space
The viscoplastic potential is divided into three regions corresponding to the different failure modes of the joint, $Q_I - Q_{III}$:

- $Q_I =$ Separation of the two joint surfaces
- $Q_{II} =$ Sliding along the joint plane
- $Q_{III} =$ Wear of the joint surface due to compressive stresses acting normal to the joint surface

The direction of the viscoplastic flow in region I (the half-sphere on top of the cylinder) is in the same direction as the applied stress rate field.

Sliding along the plane of the joint in the almost mated and unmated region should follow the joint topography (the cylinder). Wear on the joint surface gives a viscoplastic flow directed normally to the joint surface (the cap).

In regions I and II of Fig 5.13 we have a non-associated flow rule and in the third region $Q \equiv F$ imposing an associated viscoplastic flow. In the primed state of stress, $\dot{\sigma'}$, the viscoplastic potential $Q$ can be written as

$$Q_I' = \left[ (\dot{\sigma}'_{11})^2 + (\dot{\sigma}'_{12})^2 + (\dot{\sigma}'_{13})^2 \right]^{1/2}$$

$$Q_{II}' = \left[ (\dot{\sigma}'_{12})^2 + (\dot{\sigma}'_{13})^2 \right]^{1/2}$$

$$Q_{III}' = -\dot{\sigma}'_{11} \quad (5.53)$$
However, when $\Phi_i$ and $\lambda$ are $\neq 0$ the viscoplastic potential $Q$ has to be transformed to the unprimed state of stress, $\dot{J}_g$, cf. equations (5.49) and (5.50), giving

\[
Q_I = [(\dot{J}_{\sigma_{11}})^2 + (\dot{J}_{\sigma_{12}})^2 + (\dot{J}_{\sigma_{13}})^2]^{1/2}
\]

\[
Q_{II} = [(\dot{J}_{\sigma_{11}}\sin\Phi_i + \cos\Phi_i(\dot{J}_{\sigma_{12}}\cos\lambda + \dot{J}_{\sigma_{13}}\sin\lambda))^2 + (\dot{J}_{\sigma_{13}}\cos\lambda - \dot{J}_{\sigma_{12}}\sin\lambda)^2]^{1/2}
\]

\[
Q_{III} = -\dot{J}_{\sigma_{11}}\cos\Phi_i + \dot{J}_{\sigma_{12}}\sin\Phi_i\cos\lambda + \dot{J}_{\sigma_{13}}\sin\Phi_i\sin\lambda
\]

Differentiating equation (5.54) gives the direction of the plastic flow, $\frac{\partial Q}{\partial g}$, i.e.

\[
\frac{\partial Q_I}{\partial g} = \frac{2}{Q_I} \begin{bmatrix} \dot{J}_{\sigma_{11}} \\ \dot{J}_{\sigma_{12}} \\ \dot{J}_{\sigma_{13}} \end{bmatrix}
\]

\[
\frac{\partial Q_{II}}{\partial g} = \frac{2}{Q_{II}} \begin{bmatrix} \sin\Phi_i A \\ \cos\Phi_i\cos\lambda A - \sin\lambda B \\ \cos\Phi_i\sin\lambda A + \cos\lambda B \end{bmatrix}
\]

\[
\frac{\partial Q_{III}}{\partial g} = \begin{bmatrix} -\cos\Phi_i \\ \sin\Phi_i\cos\lambda \\ \sin\Phi_i\sin\lambda \end{bmatrix}
\]
where

\[ A = j\sigma_{11}\sin\Phi_i + \cos\Phi_i (j\sigma_{12}\cos\lambda + j\sigma_{13}\sin\lambda) \]

\[ B = j\sigma_{13}\cos\lambda - j\sigma_{12}\sin\lambda \]

The dilation angle, \( \Phi_i \), and the shear direction angle \( \lambda \) are calculated using equations (3.42) and (5.51).
6 NUMERICAL TEST

The elasto-viscoplastic equivalent material model of jointed rock masses will be tested for different values on the material properties. The constitutive equations are solved numerically for a homogeneous state of stress and strain. The calculation will be restricted to:

- Two-dimensional, plane strain condition.
- The material properties of intact rock and joint are not rate dependent. Only the steady-state, plastic, solution will be studied.

6.1 Solution of constitutive equations

The constitutive equations of the model are solved incrementally for plane strain conditions, assuming a homogeneous state of stress and strain. The stress-strain relation, equation (5.15), can be written in incremental form as follows

\[
\Delta \sigma + \Delta \sigma^{vp} = S \Delta \varepsilon
\]  

(6.1)

where

\[
\Delta \sigma^{vp} = S \Delta \varepsilon^{vp} = S \sum_{i}^{m} (\Delta \varepsilon^{vp})_i
\]  

(6.2)

and

\[
S = [R + \sum_{i}^{m} (J C)_{ij}]^{-1}
\]  

(6.3)

The solution algorithm is based on the procedure proposed by Nilsson (1979) and Häggbland et al (1983). Writing equation (6.1) in component form (two-dimensional analysis) gives

\[
\begin{bmatrix}
\Delta \sigma_{11} + \Delta \sigma_{11}^{vp} \\
\Delta \sigma_{22} + \Delta \sigma_{22}^{vp} \\
\Delta \sigma_{12} + \Delta \sigma_{12}^{vp}
\end{bmatrix} =
\begin{bmatrix}
S_{11} & S_{12} & S_{13} \\
S_{21} & S_{22} & S_{23} \\
S_{31} & S_{32} & S_{33}
\end{bmatrix}
\begin{bmatrix}
\Delta \varepsilon_{11} \\
\Delta \varepsilon_{22} \\
\Delta \varepsilon_{12}
\end{bmatrix}
\]  

(6.4)
The strain increment vector, $\Delta \varepsilon$, is partitioned into the known, $\Delta \varepsilon_C$, and the unknown, $\Delta \varepsilon_U$, components. The stress increment vector is partitioned into an unknown, $\Delta \sigma_U$, and a constrained component, $\alpha^T \Delta \sigma_u$. The vector $\alpha$ contains the ratios of constrained stresses to the unknown stresses.

Rewriting equation (6.4) in unknown and prescribed components gives

$$
\begin{bmatrix}
\Delta \sigma_U \\
\alpha^T \Delta \sigma_u
\end{bmatrix}
+ 
\begin{bmatrix}
\Delta \sigma_{U}^{vp} \\
\Delta \sigma_{C}^{vp}
\end{bmatrix}
= 
\begin{bmatrix}
S_{11}^* & S_{12}^* \\
S_{22}^* & S_{22}^*
\end{bmatrix}
\begin{bmatrix}
\Delta \varepsilon_C \\
\Delta \varepsilon_U
\end{bmatrix}
$$

(6.5)

where stiffness components denoted by $*$ are the partitioned components of equation (6.4) according to the unknown and constrained components respectively. The viscoplastic stress increments $\Delta \sigma_{U}^{vp}$ and $\Delta \sigma_{C}^{vp}$ correspond to the unknown and constrained stress increments, respectively. The viscoplastic stress increments, $\Delta \sigma^{vp}$, are known for the current solution step.

Solving equation (6.5) gives

$$
\Delta \sigma_u = [I - S_{12}^* (S_{22}^*)^{-1} \alpha^T]^{-1} \left[ (S_{11}^* - S_{12}^* (S_{22}^*)^{-1} S_{21}^*) \Delta \varepsilon_C \\
+ S_{12}^* (S_{22}^*)^{-1} \Delta \sigma_{C}^{vp} - \Delta \sigma_{U}^{vp} \right]
$$

(6.6)

and

$$
\Delta \varepsilon_U = (S_{22}^*)^{-1} [\alpha^T \Delta \sigma_u + \Delta \sigma_{C}^{vp} - S_{21}^* \Delta \varepsilon_C]
$$

(6.7)

where $I$ is the unit matrix.
To exemplify equations (6.4)-(6.7) we will assume that strain increment $\Delta \varepsilon_{11}$ is prescribed and $\Delta \sigma_{22} = \Delta \sigma_{12} = 0$, corresponding to uniaxial testing. The terms in equations (6.6) and (6.7) are defined as

$$
\Delta \sigma_u = \Delta \sigma_{11} \\
\Delta \sigma_c = [a^T \Delta \sigma_{11}]^T \\
\mathbf{a}^T = [\Delta \sigma_{22}/\Delta \sigma_{11}, \Delta \sigma_{12}/\Delta \sigma_{11}] - [0, 0] \\
\Delta \varepsilon_c = \Delta \varepsilon_{11} \\
\Delta \varepsilon_u = [\Delta \varepsilon_{22}, \Delta \varepsilon_{12}]^T \\
\Delta \sigma_{u,VP} = \Delta \sigma_{11} \\
\Delta \sigma_{c,VP} = [\Delta \sigma_{22,VP}, \Delta \sigma_{12,VP}]^T \\
S_{11}^* = S_{11} \\
S_{12}^* = [S_{12}, S_{13}] \\
S_{21}^* = [S_{21}, S_{31}]^T \\
S_{22}^* = \begin{bmatrix} S_{22} & S_{23} \\ S_{32} & S_{33} \end{bmatrix} \\
I = 1
$$

The elasto-viscoplastic algorithm used here enables the solution of both rate dependent and quasi-static problems to be found.
In quasi-static calculations time is used as a computational parameter only. At each load step increment "trial" values of unknown stresses, $\Delta \sigma_u$, and strains, $\Delta \varepsilon_u$, are computed using equations (6.6) and (6.7). The prescribed strain, $\Delta \varepsilon_c$, is then set to zero and the viscoplastic algorithm is used to update unknown stresses and strains until steady-state conditions are obtained. The condition for steady-state convergence is expressed as

$$\frac{|\Delta \varepsilon_{vp}^{t=t}|}{|\Delta \varepsilon_{vp}^{t=1}|} \leq 0.001$$  \hspace{1cm} (6.8)

where $|\Delta \varepsilon_{vp}^{t=t}|$ is the length of the viscoplastic strain increment at the current time increment and $|\Delta \varepsilon_{vp}^{t=1}|$ is the length of the viscoplastic strain increment at time increment 1.

Fig 6.0 shows a flow chart of a computer program used to solve the constitutive equations for homogeneous state of stress and strain.
New load step

Calculate the elastic rock-joint stiffness

Solve the constitutive eq. at t=0, elastic response. Update stresses and strain.

Time iteration loop

Loop over joint sets

Calculate local stresses over joint set i.

Check yield condition

Compute viscoplastic strain increments and update local joint viscoplastic strains. Update the global viscoplastic strain increments.
Calculate the visco-plastic stress increment

Steady-state solution

Solve the constitutive equation

Update the total stresses and strains

Check for convergence steady-state solution

Fig 6.0  Flow chart of the viscoplastic algorithm used to solve the constitutive equations for the jointed rock mass model.
6.2 Comparison with shear box experiments

To check and verify the constitutive model, results from experimental shear box tests have been compared with results obtained with the constitutive model.

In most shear box experiments the normal load is applied first and then held constant during the application of the shear load. The two blocks are then displaced relatively to one another along the joint plane. In the theoretical model, on the other hand, the discontinuoues deformation is "smeared" over the considered volume of rock. The deformation pattern of the shear box experiments and the theoretical model are shown in Fig 6.1.

![Diagram of shear box experiment and theoretical model](image)

**Fig 6.1** Deformation pattern in shearing a block with a single joint
A. Shear box experiment
B. Numerical shear test

To compare the experimental and the theoretical results the following assumptions are made:

- The joint spacing $d$, is 1 m in the theoretical calculations.
- Intact rock deformations are ignored e.g. $E_R \to \infty$, $v_R = 0$. 
The normal strain $\varepsilon_2$ and the shear strain $\varepsilon_{12}$ in the theoretical model are defined by the equations

$$\varepsilon_2 = u_n/l$$
$$2\varepsilon_{12} = \gamma_{12} = u_s/l$$

The normal stress $\sigma_2$ and the shear stress $\sigma_{12}$ are calculated as

$$\sigma_2 = -F_n/A_j$$
$$\sigma_{12} = F_s/A_j$$

where $A_j$ is the area of the joint surface in the experiment.

Bandis et al. (1981) performed a series of shear tests on model joint replicas with different surface roughnesses. Figure 6.2 shows the calculated and measured shear stress - shear displacement behaviour on model joints with four different joint surface roughness, JRC, at two different normal loads. The model described in chapter 3.4 has been used to evaluate the viscoplastic properties from Barton's parameters JRC, JCS, $\Phi_r$, and $L$. Table 6.1 shows the input parameters for the four types of tested joints.

Table 6.1 Input parameters for the numerical calculations shown in Fig 6.2. The values for JRC, JCS, $\Phi_r$, and $L$ are adapted from Bandis et al. (1981).

<table>
<thead>
<tr>
<th>Joint no</th>
<th>$C_{nn}$</th>
<th>$C_{ss}$</th>
<th>JRC</th>
<th>JCS</th>
<th>$\Phi_r$</th>
<th>$L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>4</td>
<td>16.6</td>
<td>2</td>
<td>32</td>
<td>9</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>4</td>
<td>10.6</td>
<td>2</td>
<td>32</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>4</td>
<td>7.5</td>
<td>2</td>
<td>32</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>4</td>
<td>4</td>
<td>6.5</td>
<td>2</td>
<td>32</td>
<td>9</td>
</tr>
</tbody>
</table>

For all joints $C_{ns} = C_{sn} = 0$, $u_p = 0.01$ L, $u_r = 0.08$ L. Normal stress $-\sigma_2 = 34$ kPa, 90 kPa
Fig 6.2 Measured and predicted shear stress - shear displacement behaviour for joint replicas, with different surface roughness and normal stress. The calculated behaviour are marked with symbols. Experimental results after Bandis et al (1981).
The curves from the experimental test and the theoretical calculations are in good agreement. However, this is not surprising since equation (3.50) in chapter 3.4 have been fitted to the shear box experiments shown in Fig 6.2.

The next two figures 6.3 and 6.4 show the calculated experimental results obtained on model joints with identical surface roughness but with different joint length, \( L \).

The JCS and LRC values for the different joint lengths have been corrected in scale using Bandis experimentally determined scale correction chart, see Fig 3.14.

**Table 6.2** Input parameters for the numerical calculations shown in Fig 6.3. The values for \( \phi_r \), \( L \) and \( L_0 \) are adapted from Bandis et al (1981). The original values for JRC and JCS have been scale corrected using Fig 3.14.

<table>
<thead>
<tr>
<th>Joint no</th>
<th>( C_{nn} ) GPa(^{-1} )</th>
<th>( C_{ss} ) GPa(^{-1} )</th>
<th>JRC</th>
<th>JCS MPa</th>
<th>( \phi_r ) deg</th>
<th>( L ) cm</th>
<th>( L_0 ) cm</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>4</td>
<td>4</td>
<td>18.3</td>
<td>2.0</td>
<td>32</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>4</td>
<td>15.6</td>
<td>1.54</td>
<td>32</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>4</td>
<td>13.1</td>
<td>1.10</td>
<td>32</td>
<td>20</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>4</td>
<td>10.1</td>
<td>0.76</td>
<td>32</td>
<td>40</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>32</td>
<td>5.0</td>
<td>2.0</td>
<td>32</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>32</td>
<td>4.5</td>
<td>1.68</td>
<td>32</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>32</td>
<td>4.25</td>
<td>1.50</td>
<td>32</td>
<td>18</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>4</td>
<td>32</td>
<td>4.08</td>
<td>1.45</td>
<td>32</td>
<td>36</td>
<td>6</td>
</tr>
</tbody>
</table>

For all joints \( C_{ns} = C_{sn} = 0 \), \( u_p = 0.01 \) \( L \), \( u_r = 0.1 \) \( L \).

Normal stress \(-\sigma_2 = 24.5\) kPa
Fig 6.3 Measured and predicted shear stress - shear displacement behaviour for joint replicas with identical surfaces but with different joint length, L. The calculated values are shown with symbols. Experimental results after Bandis et al (1981).
Fig 6.4 Measured and predicted shear stress - shear displacement behaviour for joint replicas with identical surfaces but with different joint length, L. The calculated values are shown with symbols. Experimental results after Bandis et al (1981).
Bandis measured the peak dilation angle, $\phi_1^p$, from the normal displacement - shear displacement behaviour. The measured values of $\phi_1^p$ and the total peak friction angle, $\phi^p$, are compared with the calculated values for the case $-\sigma_2 = 24.5$ kPa in Table 6.3.

**Table 6.3** Measured and calculated values of peak dilation, $\phi_1^p$, and total peak friction angle, $\phi^p$. Measured values after Bandis et al. (1981).

<table>
<thead>
<tr>
<th>Joint no</th>
<th>L (cm)</th>
<th>$\phi_1^p$ (deg)</th>
<th>$\phi^p$ (deg)</th>
<th>Calculated $\phi_1^p$ (deg)</th>
<th>Calculated $\phi^p$ (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>16.7</td>
<td>67.3</td>
<td>22.5</td>
<td>65.9</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>12.0</td>
<td>61.1</td>
<td>13.8</td>
<td>60.1</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>10.0</td>
<td>52.7</td>
<td>8.4</td>
<td>53.6</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>7.8</td>
<td>47.4</td>
<td>4.4</td>
<td>45.9</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>2.3</td>
<td>42.4</td>
<td>2.5</td>
<td>41.6</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>2.0</td>
<td>40.3</td>
<td>1.9</td>
<td>40.2</td>
</tr>
<tr>
<td></td>
<td>18</td>
<td>1.8</td>
<td>39.2</td>
<td>1.7</td>
<td>39.6</td>
</tr>
<tr>
<td></td>
<td>36</td>
<td>1.7</td>
<td>38.1</td>
<td>1.6</td>
<td>39.2</td>
</tr>
</tbody>
</table>

The calculated total peak friction angles are in close agreement with the measured values. The predicted peak dilation angles diverge more from the measured values, especially for the joint with the higher surface roughness, joint no 1. Hence, the calculated normal displacement versus shear displacement curves for joint no 1 do not coincide with measured behaviour.

In Fig 6.5 measured values on peak friction and peak dilation angle have been used to calculate the shear behaviour for joint no 1, L = 5 cm. A much better fit is obtained for both the shear stress - shear displacement and normal displacement - shear displacement behaviour.
Fig 6.5  Calculated and measured shear stress and normal displacement behaviour versus shear displacement for joint no 1, $L = 5 \text{ cm}$. Measured values after Bandis et al (1981).

Also the residual shear displacement, $u_r$, will effect the calculated normal displacement behaviour. Fig 6.5 shows the calculated result for joint no 2, $L = 36 \text{ cm}$, using measured values on $\phi_i^D$, $\phi^D$ and $u_r = 0.05 \text{ L}$. 
Fig 6.6  Calculated and measured shear stress and normal displacement behaviour versus shear displacement for joint no 2, $L = 36$ cm. Measured values after Bandis et al (1981).
Sun (1983) carried out shear tests in a large shear rig. He measured the normal and shear displacements with high accuracy with a specially designed device mounted in a hole close to the joint surface.

Fig 6.7 shows the results from a shear test on a granite specimen with artificially constructed joint surface with relatively low roughness. The nominal joint area was 731 cm². Since the two surfaces had different roughnesses the joint was tested in an unmated position. First a normal stress of 8.7 MPa was applied and then held constant during the application of shear load. After a shear displacement of about 3 mm both the shear and normal loads were relaxed. Profile measurements were conducted and the experiment was repeated with the same blocks at the same level of normal stress.

The two tests on the same joint have completely different deformation behaviour. The first test shows an almost ideal-plastic behaviour. After the maximum shear stress levels have been reached the joint starts to slide at a constant shear stress level. The joint has no "off-diagonal" behaviour in the elastic region, i.e. no shear displacements during the application of normal stress. However some joint contraction can be noted after the joint has started to slide. In the second test the joint starts to move in the shear direction when the normal stress is applied. When the shear stress is applied the joint almost immediately starts to slide. The shear resistance gradually increases reaching almost the same level as in the first test at the end of the experiment. The joint in the second test has a clear "off-diagonal behaviour". Also the normal contraction during shearing of the joint is much more pronounced.
Fig 6.7 Shear testing of granite specimen, normal stress, 8.7 MPa, after Sun, 1983
a) dilatancy curve
b) shear load versus shear displacement.
According to the proposed model in this thesis an unmated joint, $\Phi_1 = 0$, has no diagonal behaviour. Secondly, the shear resistance is constant with respect to shear displacement, $\sigma_s = \sigma_n \tan \Phi_{bs}$. This is all in agreement with the first test by Sun. However, it seems as if the first test damaged the joint surface and changed the joint behaviour. If the contacting asperities were sheared off, in the first test the second test would start with a negative dilation angle, i.e. downhill sliding.

To model the behaviour in the second test the following function on the dilation angle has been used.

$$\Phi_i = 9.29 \log_{10} u_s - 3.165 \quad -18^0 \leq \Phi_i \leq 0$$

where $u_s$ is in mm.

The yield shear stress is as usual calculated from $\sigma_s = \sigma_n \tan(\Phi_i + \Phi_{bs})$. $\Phi_{bs}$ have been calculated from the maximum shear stress obtained in the first test assuming $\Phi_i = 0$. At starting condition, $u_s = 0$, the dilation angle is $\Phi_i = -18^0$ giving the off-diagonal compliance according to equation (6.2), as

$$c_{ns} = c_{sn} = (c_{ss} - c_{nn}) \cos(-18^0) \sin(-18^0)$$

Table 6.4 summarizes the input parameters for the calculated second test.

Table 6.4  Data for shear test 2 on a joint in granite, normal stress $\sigma_n = 8.7$ MPa.

<table>
<thead>
<tr>
<th>$\Phi_{bs}$</th>
<th>$\Phi_i$</th>
<th>$c_{nn}$</th>
<th>$c_{ss}$</th>
<th>$c_{ns}$</th>
<th>$c_{sn}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>deg</td>
<td>deg</td>
<td>1/GPa</td>
<td>1/GPa</td>
<td>1/GPa</td>
<td>1/GPa</td>
</tr>
<tr>
<td>27$^0$</td>
<td>eq (6.2)</td>
<td>0.03</td>
<td>0.057</td>
<td>0.008</td>
<td>0.008</td>
</tr>
</tbody>
</table>
Fig 6.8 shows the measured and calculated shear stress – shear displacement, shear stress – normal displacement for the second test. Note that also the normal and shear displacement due to the normal pressure are included.

Fig 6.8  Calculated and measured shear stress and normal displacement versus shear displacement for a joint in a granite block. Experimental result after Sun (1983), calculated values are shown with triangles.
The calculated values show good agreement with measured values both on the off-diagonal and the shear stress versus shear displacement behaviour. The predicted normal contraction is somewhat smaller compared with the measured result.

6.3 Examples of stress-strain behaviour for jointed rock masses

To demonstrate the possibilities of the combined jointed rock mass model some examples will be given for a rock mass intersected with two major sets of joints. The evaluation of input parameters to the proposed rock mass model will also be demonstrated.

A rock mass intersected by two major sets of joints is used as a demonstration example, see Fig 6.9.

![Diagram](image)

**Fig 6.9** Demonstration example of a rock mass with two sets of joints.
The first set of joints has an inclination of $\alpha = 30^\circ$ with respect to the horizontal axes, $x_1$. The average joint distance $d$ is 0.8 m. The second sets has a direction of $\alpha = -45^\circ$ and an average joint distance of $d = 0.4$ m. Small samples from both sets of joints and the intact rock was collected from site and the following parameters was evaluated in the laboratory.

**Rock:**

$E_R = 40$ GPa, $\nu_R = 0.2$

**Joint set 1:** $c_{nn} = 0.1$ GPa$^{-1}$, $c_{ss} = 0.2$ GPa$^{-1}$, JRC = 5, JCS = 50 MPa, $\phi_r = 20^\circ$, $L_0 = 0.14$ m

**Joint set 2:** $c_{nn} = 0.1$ GPa$^{-1}$, $c_{ss} = 0.2$ GPa$^{-1}$, JRC = 10, JCS = 80 MPa, $\phi_r = 20^\circ$, $L_0 = 0.14$ m.

The actual joint length in the site can be calculated from

- **Set 1:** $L_1 = d_2 / \sin(\alpha_1 - \alpha_2) = 0.43$, $L_1 / L_0 = 3$
- **Set 2:** $L_2 = d_1 / \sin(\alpha_1 - \alpha_2) = 0.83$, $L_2 / L_0 = 6$

The scale reduction factors is determined from Fig 3.14. Table 6.5 shows the input parameters for the intact rock, joint set 1 and 2.

**Table 6.5** Input parameters for numerical calculations on a rock mass with two sets of joint, test 1-3.

<table>
<thead>
<tr>
<th>Intact rock: $E_R = 40$ GPa, $\nu_R = 0.2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Joint set</td>
</tr>
<tr>
<td>no</td>
</tr>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
</tbody>
</table>

$c_{nn} = 0.1$ GPa$^{-1}$, $c_{ss} = 0.2$ GPa$^{-1}$, $c_{ns} = c_{sn} = 0$ for both sets of joints.

1) Scale reduced values.
In the first two tests a uniform stress of -1 MPa is first applied. In test 1 the stress in the 1-direction is then increased while $\sigma_2$ is held constant. In the second test the stress in the 2-direction is increased while $\sigma_1$ is held constant. Fig 6.10 shows the loading path for test 1 and 2.

![Graph showing stress-strain relationship for tests 1 and 2.](image)

**Fig 6.10** Loading path for numerical calculations, test 1 and 2.

Fig 6.11 and 6.12 shows the stress-strain behaviour for the two loading paths. The stress in the respective loading direction, $\sigma_1/\sigma_2$, is shown as a function of the strain $\varepsilon_1$, $\varepsilon_2$ and $\gamma_{12}$. 
Fig 6.11  Calculated result for test 1, $\sigma_1$ versus $\varepsilon_1$, $\varepsilon_2$, and $\gamma_{12}$.

Fig 6.12  Calculated results for test 2, $\sigma_2$ versus $\varepsilon_1$, $\varepsilon_2$, and $\gamma_{12}$. 
For the first test, failure in rock mass is initiated by plastic deformation in joint set no 1 when $\sigma_1 = -2.5$ MPa and $\sigma_2 = -1$ MPa. In the post-failure region the rock mass deforms along joint planes in set no 1. Since JRC and JCS are low for joint set no 1 almost no peak-stress behaviour can be recognized.

For the second loading path, test 2, the failure is initiated in joint no 2. Here a clear peak-stress behaviour is found. The maximum stress in the loading direction is $\sigma_2 = -4$ MPa ($\sigma_1 = -1$ MPa). Since the joint set is oriented at a $-45^\circ$ angle to the 1-direction the shear deformation along the plane of the joint results only in a negative component added to $\varepsilon_2$ and a positive component added to $\varepsilon_1$, i.e. no shear strain component, $\gamma_{12}$, except for a small component due to joint dilation. That is, the 1- and 2-direction nearly coincide with the principal strain direction.

The third test shows the behaviour of the jointed rock mass in pure shear. Fig 6.13 shows the strain in direction 2, $\varepsilon_2$, as a function of the strain in direction 1, $\varepsilon_1$. Since the joint has no tensile strength, set no 2 will open normal to the plane of the joint for tensile normal stresses. The shear strength of the jointed rock mass shown in Fig 6.9 will be zero if $\sigma_1 = \sigma_2 = 0$.

![Graph showing strain $\varepsilon_2$ vs strain $\varepsilon_1$](image)

Fig 6.13 Calculated result for test 3, shear loading. $\varepsilon_1$ versus $\varepsilon_2$. 
In test 4, joint set no 1 is orientated perpendicular to the horizontal axes, $\alpha = 90$. Joint set no 2 is parallel with the horizontal axes. Except for the joint orientation all the other parameters, joint length, JRC, JCS etc are the same as for example 1-3, see Table 6.5. Fig 6.14 shows the orientations of the joint sets.

![Diagram with labels Set no 2 and Set no 1]

**Fig 6.14** Joint orientations used in test 4.

First an uniform stress of 1 MPa is applied to the model in Fig 6.14. The rock mass is then sheared until one of the sets begins to slide and becomes unmated. The shear stress is then reversed and the joint set is sheared back and beyond its initial mated position.
Fig 6.15  Load path for test 4. Shear stress $\sigma_{12}$ as a function of normal stresses $\sigma_1$ and $\sigma_2$. The numbers 1-4 show the loading sequence.

Fig 6.16 shows the shear stress, $\sigma_{12}$, versus shear strain, $\gamma_{12}$. Since the stress conditions are the same for both sets of joints, set no 1 will start to fail due to its lower shear strength.

Fig 6.16  Calculated shear stress, $\sigma_{12}$, versus shear strain, $\gamma_{12}$, for test 4. The numbers 1-4 correspond to the loading sequence.
When the shear stress is reversed the joint will start to deform from an unmated position, i.e. $\Phi_j \neq 0$. This will decrease the shear strength since the dilation will be negative. That is, the dilation angle will be added to the shear strength when shearing in one direction and subtracted when the direction of shear stress reverses. However, when the joint has been moved back to its mated position the dilation will again be positive irrespective of shearing direction and increase the shear resistance. This can be observed in Fig 6.17 where the variations in shear stress is magnified after sliding have been initiated. The lower figure in Fig 6.17 shows the strain $\varepsilon_1$, normal to joint set no 1, as a function of shear strain $\gamma_{12}$. Note the difference in scale between the $\varepsilon_1$ and $\gamma_{12}$. 
Fig 6.17  Calculated stress – strain behaviour for test 4.
A. Shear stress, $\sigma_{12}$, versus shear strain, $\gamma_{12}$.
B. Normal strain $\varepsilon_1$ versus shear strain $\gamma_{12}$. 
A 2-D plane strain version of the equivalent material model described in chapter 5 has been implemented in a Finite element program, FEMP, developed by Nilsson and Oldenburg (1983).

A circular opening in a jointed rock mass will be used as a demonstration example of the capabilities of the model, see Fig 7.1.

Fig 7.1  A circular opening in a jointed rock mass.

Due to symmetry in loading, geometry and joint orientations only a quarter of the geometry will be analyzed. Fig 7.2 show the finite element mesh and the boundary condition. Eight-node iso-parametric elements are used. No gravity forces are applied to the model.
Fig 7.2  Finite element mesh

An initial stress field of $\sigma_x = \sigma_y = -1 \text{ MPa}$ is applied before the removal of elements 17-20 in Fig 7.2. The calculated displacements refer to the excavation of the circular opening.

Three different cases of joint orientation will be studied. The influence of joint roughness, JRC, is analyzed for the case of two orthogonal sets of joints.

Table 7.1 shows the input parameters for all tested cases.
Table 7.1  Input parameters for test 1-3.

<table>
<thead>
<tr>
<th>Test</th>
<th>Joint set no</th>
<th>Joint orientation $\alpha$</th>
<th>Joint roughness JRC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1:1</td>
<td>1</td>
<td>$90^\circ$</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>$0^\circ$</td>
<td>0</td>
</tr>
<tr>
<td>1:2</td>
<td>1</td>
<td>$90^\circ$</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>$0^\circ$</td>
<td>10</td>
</tr>
<tr>
<td>1:3</td>
<td>1</td>
<td>$90^\circ$</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>$0^\circ$</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>$+45^\circ$</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>$-45^\circ$</td>
<td>10</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>$0^\circ$</td>
<td>10</td>
</tr>
</tbody>
</table>

Input parameters for all tests:

Rock: $E = 70$ GPa, $\nu = 0.15$

Joint: $c_{nn} = c_{ss} = 2 \times 10^{-10}$ m$^2$/N, $c_{ns} = c_{sn} = 0$.

JCS = 100 MPa, $\phi_r = 20^\circ$, joint spacing = 1 m,

joint length = 1 m, cohesion and tensile strength = 10 kPa

Note that test 1:2 and 2 are equal due to the symmetry in load. The result from test 2 can be extracted from the output of test 1:2 by rotating it $\pm 45$ degrees. Test 2 is used to check the result of the calculations.

First an elastic calculation with no joints will be conducted. In Fig 7.3 the principal stresses and the displacement field are shown for an elastic solution with $E = 70$ GPa and $\nu = 0.15$.

The calculated maximum tangential stresses in the vicinity of the circular opening is approximately 2 MPa, which is in good agreement with the analytical solution.
Fig 7.3  Principal stresses and displacements for a circular opening in an elastic rock without joints and with a hydrostatic in-situ stress field of 1 MPa. A) principal stresses and B) displacements.
To obtain a "steady state" solution for the elasto-viscoplastic calculations the sum of viscoplastic yield functions in each Gauss point in the current time step, \( t \), is compared with the sum at the first time step \( t=1 \). When the ratio is less than a preset tolerance the iteration is stopped, i.e.

\[
\text{tol} < \frac{\sum_{i}(F_i)^2}{\sum_{i}(F_i)_{t=1}^2}
\]

The tolerance is set to 1%. However the tolerance requirements were only achieved for test 1:3 and 3 within reasonable numbers of iterations, (< 100). For test 1:1, 1:2 and 2 the tolerance first decreased rapidly to 10%. Then the tolerance slowly decreased to an almost constant value of approximately 6-7% which indicate instability in the vicinity of the opening. Over 90% of the sum of \( \sum (F_i)^2 \) was due to joints opening up in 2-3 Gauss points close to the opening. Therefore the results for test 1:1, 1:2 and 2 gives the viscoplastic solution with a higher tolerance, 6-7%. The difference in tolerance mainly effects the magnitude of viscoplastic deformations, as can be seen from Fig 7.8 where the result for test 3 is calculated using two different tolerances, 1% and 4%.

In Fig 7.4-7.6 the result from case 1:1-1:3 are shown. Fig 7.4 shows the principal stresses and displacements around the opening. Fig 7.5 shows the total viscoplastic deformation of the joints, \( (u_n^2 + u_s^2)^{1/2} \), in the Gauss points of each element. The length of the scale corresponds to the magnitude of the viscoplastic deformation. The direction shows the set of joints where the displacement occurred, not the direction of viscoplastic flow! Fig 7.6 shows the magnitude and relative shear movement for closed joints, \( \leftarrow \). Open joints are shown with the symbol \( = \).

With a larger joint roughness the joint displacement decreases as expected. The stress field around the opening for the case of smooth joints shows an almost stress free area located \( \pm 45^0 \) from the x-y axis, Fig 7.4 A. As the roughness increases the joints are able to transmit the stresses and form a continuous arch around the opening.

Note that the maximum shear displacements around the opening occurs in 45 degrees from the x-y axis and that joints located at 0 and 90 degrees from the axis opens.
The same failure mechanism was obtained in a study of joint movement around circular openings used for radioactive waste repositories in jointed rock by Stephansson et al (1981). The intact rock was intersected with two sets of joints with the same orientation as in test 1:1-1:3. Stephansson et al used discrete joint elements based on the Goodman joint element, Goodman (1976).
Fig 7.4 Principal stresses and displacements for a circular opening in a jointed rock mass with two perpendicular sets of joints, $\alpha_1 = 0^\circ$, $\alpha_2 = 90^\circ$ and $\sigma_x = \sigma_y = \frac{1}{2}$ MPa, A) Test 1:1, JRC = 0, B) Test 1:2, JRC = 10 and C) Test 1:3, JRC = 20
Fig 7.5  Total viscoplastic deformation occurring in the two sets of joint. Only the magnitude are shown. A) Test 1:1, JRC = 0, B) Test 1:2, JRC = 10 and C) Test 1:3, JRC = 20
Fig 7.6 Shear displacement along closed joints, ←→, and normal displacement in open joints, =, in a jointed rock mass with two perpendicular sets of joints. Only the magnitudes are shown. A) Test 1:1, JRC = 0, B) Test 1:2, JRC = 10 and C) Test 1:3, JRC = 20
The symmetry of the applied stress field for test 2 with two sets of joints oriented in $\pm 45^\circ$ to the coordinate system and JRC = 10 should give the same results as test 1:2, if the finite element net is rotated $45^\circ$ around the center of the opening. The result of the calculations shown in Fig 7.7, shows that the direction of the stress field, displacements and joint viscoplastic deformations are essentially the same as for case 1:2 if the two cases are compared rotated with an angle of $45^\circ$. The somewhat higher magnitudes of joint viscoplastic deformations in case 2 is due to the incompleteness in calculating the "steady-state" solutions for the two cases.

The third case with one horizontal set of joints have been calculated with two tolerances, A) $\text{tol} = 4\%$ and B) $\text{tol} = 1\%$. As can be seen from Fig 7.8 only the viscoplastic deformations are affected, the stress distributions in Fig 7.8 A and Fig 7.8 B are the same.
Fig 7.7  Principal stresses, displacements and viscoplastic deformations for two sets of joints oriented ± 45° to the coordinate system. A) Principal stresses. B) Displacements. C) Total viscoplastic deformations. D) Viscoplastic shear displacements. E) Open joints.
Fig 7.8  Principal stresses, displacements and joint viscoplastic opening deformations for test 3 with one set of joints horizontally oriented around a circular opening.
A) Tolerance, tol = 4 %. B) Tolerance, tol = 1 %.
8 DISCUSSION

The discussion is divided into two parts. First the discrete joint model described in chapter 3 will be discussed and evaluated. The second part treats the proposed constitutive model for jointed rock masses, the so called equivalent material approach.

8.1 The proposed joint model

The proposed joint model presented in chapter 3 has the following characteristics:

- Constant normal and shear compliance.

- Off-diagonal compliances are related to the difference between shear compliance and normal compliance and the initial dilation angle.

- Shear strength is divided into four components; cohesion, basic friction angle, shear friction angle and dilation angle. The cohesion and basic friction angle are assumed to be constant properties independent of stress and displacements.

- The normal strength is zero in tension and N in compression. N is assumed to be a function of the viscoplastic displacements only.

- The viscoplastic joint displacements are assumed to be directed with the angle of dilation from the joint, plane in shear. Normal to the joint surface in compression and in the direction of the applied stress field in tension.

The model is formulated as a stress - displacement law. With this formulation the model can be directly used to model discrete joint deformations in rock masses, e.g. using joint elements in FE-codes. However, in this thesis a continuum approach has been used where the discrete displacements are distributed over a certain volume of rock, see chapter 5.
There are two reasons for using a constant elastic compliance matrix. Firstly to facilitate the numerical calculations, secondly to reduce the number of input parameters. However, variable elastic joint properties are probably needed for a more sophisticated analysis, especially in the normal direction which show a highly non-linear elastic behaviour in the lower stress region, see Swan (1983), Bandis et al (1983).

The comparative studies between numerical calculations and experimental result from shear box test show that a good fit can be obtained provided the proper input parameters are used. It is also noted that the peak stress behaviour is mainly caused by dilation in the tested range of normal stress. The comparison with experimental test on an unmated joint specimen performed by Sun (1983) shows that the initial dilation angle can explain the measured off-diagonal compliances. Further, that a negative dilation angle in the shear direction will decrease the shear resistance.

Larsson (1984) conducted a series of calculations in his licentiate thesis on a reinforced jointed rock with different joint spacing, normal stress, reinforcement system, etc. He concluded that the rock mass properties dominate the behaviour of the bolt system. The frictional resistance of the joints was in many cases larger than the supporting forces given by the reinforcement system. The normal stress acting on the joint surfaces was the most significant factor for the supporting effect on the rock mass. Since the dilation of the joint system in locked rock masses increases the normal stress considerably, joint dilation seems to be one of the most important properties to model accurately.

However, since very few experiments on joints have been conducted under different loading conditions and paths, the proposed joint model cannot be properly tested. Repeated loading and unloading would be necessary to resolve the elastic and non-recoverable parts from the total joint displacements. It is also essential to establish under what condition the joint was tested, e.g. mated, almost mated or unmated and how the measurements of the joint deformation were conducted, etc.
8.2 The continuum approach

The proposed constitutive model for a jointed rock mass has a number advantages over the traditional discrete formulation.

The equivalent material model simplifies the analyses e.g. for a finite element procedure. It reduces the degrees of freedom for a numerical calculation considerably. The construction of a FE-mesh is simplified since the orientation and location of each joint need not to be considered. Zienkiewicz and Pande (1977) showed that a rock mass consisting of two sets of joints easily can be analysed with this type of model. In conclusion, reduced degrees of freedom and simple finite element mesh construction implies reduced computational cost for analysing jointed rock masses.

The continuum approach is only applicable for evenly spaced joints where the spacing are much smaller than the critical dimensions of the underground construction studied. A rock mass with irregular joint orientation or with individual major discontinuities or faults makes the equivalent material model less suitable, especially when large discrete displacements along discontinuity planes must be considered.

Also, the model assumes that the rock mass is intersected by sets of joint with continuous joint planes. In order to analyse rock masses intersected by discontinuous joints, i.e. sets of joints with bridges of intact rock, the material properties of such joints has to be evaluated.

The equivalent material model will probably not predict the "discontinuous beam" action that is considered to occur in roofs and hanging walls in layered rocks, Gerrard (1983).

At present the equivalent material model is unable to simulate the interlocking phenomena from intersecting joint sets in a rock mass. Gerrard (1983) gave an example of a rock mass with two orthogonal sets of joints. If there is relative shear displacements in the first set the surfaces of the second become stepped. Subsequent shearing along the second set in one direction causes interlooking of the corners of
the blocks. However, there is a possibility to introduce "inter-
locking" in a continuum model by introducing a parameter which
remembers the direction of previous joint deformation.

The model developed in this thesis is particularly relevant for hard
rock masses, where the inelastic joint deformation dominates the mech-
anics of the rock mass. The intact rock can thus be modelled as a linear
elastic continuum. In soft rock or in areas with extreme rock stresses
inelastic and viscous deformations occur in the intact rock due to

crack propagation, creep, rock burst etc. Viscoplastic properties of
the intact rock must then be incorporated in the model. Nilsson and
Oldenburg (1983) studied crack propagation in concrete structures with
a model with "smeared" properties of the fractures. The development of
irregularly orientated fractures was predicted in finite element calcu-
lations.

An elasto-visco plastic formulation of the model with an explicit
solution procedure has been used to obtain the steady-state plastic
solution. Since all elastic properties are assumed to be constant the
stiffness matrix for the jointed rock mass must be calculated only
once. This is even more advantageous in FE-calculation where a large
number of transformation between local and global coordinate systems
of the composite elements in the structure are required to set up the
structural stiffness matrix. However, effective convergence criteria
for the iterative time-stepping procedure is an absolute requirement
to obtain an effective solution procedure, especially for large Fe-
calculations. Criteria of this type have been proposed by Owen and
Hinton (1980).
9. CONCLUSIONS

The constitutive model presented in this thesis includes both elastic and viscoplastic behaviour of a jointed rock mass. The capabilities and main advantages of the model presented here are as follows:

- The model is characterized by flexible formulations of the relations governing the mechanical behaviour of jointed rock. Other functions for the dilation angle, cohesion etc than those that are proposed in this thesis can easily be implemented. Input parameters are directly taken from physical, measurable quantities.

- The constitutive equations can be used to study the influence of joint parameters as well as joint orientation and joint spacing. Arbitrary stress and strain paths can be studied. The model can be implemented in existing finite element codes as a discrete joint element or as an equivalent material model taken into account the elastic deformations in the intact rock.

- From the numerical calculations conducted, we can conclude that satisfactory agreement is obtained in comparison with laboratory shear test on joints. It was also showed that peak shear strength behaviour of joints was mainly a function of dilation rate. The elastic off-diagonal behaviour measured in some test by Sun (1983) was explained by a dilation angle $\Phi_i \neq 0$ at the contacting asperities.

- The joint orientation and joint spacing need not to be considered in the construction of a finite element mesh.

- The equivalent material approach reduces the degrees of freedom in FE-analysis, especially in heavily jointed rock.
The main limitations of the equivalent material model are:

- The model is only applicable for regularly jointed and evenly spaced continuous joints. The spacing of joints must be smaller than the critical dimensions of the underground opening.

- At present phenomena like beam effect in stratified roof, interlocking structures in jointed rock cannot be simulated by the model.

- To obtain an effective solution procedure for the explicit viscoplastic formulation used in the FE-analysis, a powerful convergence criteria is an absolute requirement.

Considering the present state of knowledge the following generalizations in the model are suggested for further research:

- Modelling of interlocking of intersecting joints and beam building in layered roofs and hanging walls of underground openings.

- Incorporation of viscoplastic behaviour of the intact rock. Modelling of fracture initiation and propagation.

- Systematic laboratory testing of single joint rock systems with continuous and discontinuous joint planes for various loading path, including loading and unloading, for proper modelling of joint behaviour.

- Comparative numerical studies between the equivalent material approach and discrete formulation.
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