DYNAMIC ANALYSIS OF HIGH-RISE TIMBER BUILDINGS

A factorial experiment

Victor Karlberg

Civil Engineering, master's level
2017

Luleå University of Technology
Department of Civil, Environmental and Natural Resources Engineering
DYNAMIC ANALYSIS OF HIGH-RISE TIMBER BUILDINGS

A factorial experiment

Department of Civil, Environmental and Natural Resources Engineering
Luleå University of Technology
Luleå
Dynamic analysis of high-rise timber buildings
A factorial experiment
*Master of Science Thesis in the Master’s Programme Structural Engineering*
Victor Karlberg
© Victor Karlberg, 2017

Examensarbete
Institutionen för samhällsbyggnad och naturresurser
Luleå Tekniska Universitet, 2017

Department of Civil, Environmental and Natural Resources Engineering
Division of Structural Engineering
Luleå University of Technology
971 87 Luleå
Sweden
Telephone: + 46 (0)920-491000

Cover:
A simple sine function overlapping an interaction plot.
PREFACE

This thesis concludes the Master’s Programme in Civil Engineering at Luleå University of Technology. It has been carried out at VBK Konsulterande ingenjörer AB in Gothenburg, in collaboration with the university during the second half of 2016.

I should like to express my particular gratitude to Martin Nilsson at VBK for his invaluable help. Without his expertise this thesis had not been possible.

I would also like to thank my supervisor Ida Edskär and my examiner Helena Lidelöw at the university for their advice.

The hardest part of this thesis has been the one concerning the factorial experiment and the underlying statistics. A special thanks to Adam Jonsson for the interpretation of factor effects, as well as Bjarne Bergquist and Erik Vanhatalo for their guidance.

Last but not least, I wish to thank Ulf Kjellberg and the co-workers at VBK for making me feel at home.

Göteborg, januari 2017

Victor Karlberg
ABSTRACT

Today high-rise timber buildings are more popular than ever and designers all over the world have discovered the beneficial material properties of timber. In the middle of the 1990’s cross-laminated timber (CLT), was developed in Austria. CLT consists of laminated timber panels that are glued together to form a strong and flexible timber element. In recent years CLT has been on the rise and today it is regarded as a good alternative to concrete and steel in the design of particularly tall buildings.

Compared to concrete and steel, timber has lower mass and stiffness. A high-rise building made out of timber is therefore more sensitive to vibration. The vibration of the building can cause the occupants discomfort and it is thus important to thoroughly analyze the building’s dynamic response to external excitation. The standard ISO 10137 provides guidelines for the assessment of habitability of buildings with respect to wind-induced vibration. The comfort criteria herein is based on the first natural frequency and the acceleration of the building, along with human perception of vibration.

The aim of this thesis is to identify the important structural properties affecting a dynamic analysis of a high-rise timber building. An important consequence of this study is hopefully a better understanding of the interactions between the structural properties in question.

To investigate these properties and any potential interactions a so-called factorial experiment is performed. A factorial experiment is an experiment where all factors are varied together, instead of one at a time, which makes it possible to study the effects of the factors as well as any interactions between these. The factors are varied between two levels, that is, a low level and a high level. The design of a factorial experiment includes all combinations of the levels of the factors.

The experiment is performed using the software FEM-Design, which is a modeling software for finite element analysis. A fictitious building is modelled using CLT as the structural system. The modeling and the subsequent dynamic analysis is repeated according to the design of the factorial experiment. The experiment is further analyzed using statistical methods and validated according to ISO 10137 in order to study performance and patterns between the different models.

The statistical analysis of the experiment shows that the height of the building, the thickness of the walls and the addition of mass are important in a dynamic analysis. It also shows that interaction is present between the height of the building and the thickness of the walls as well as between the height of the building and the addition of mass.

Most of the models of the building does not satisfy the comfort criteria according to ISO 10137. However, it still shows patterns that provides useful information about the dynamic properties of the building.

Lastly, based on the natural frequency of the building this study recognizes the stiffness as more relevant than the mass for a building with CLT as the structural system and with up to 16 floors in height.

Keywords: Acceleration, comfort criteria, cross-laminated timber, factorial design, finite element analysis, natural frequency, vibrations
SAMMANFATTNING

Idag är höga trähus mer populära än någonsin och konstruktörer runt om i världen har upptäckt de fördelaktiga materialegenskaperna hos trä. I mitten på 1990-talet utvecklades korslimmat trä (KL-trä) i Österrike. KL-trä består av hyvlade brädor som limmas ihop för att bilda en lätt och stark träskiva. På senare år har KL-trä varit på uppgång och idag anses materialet vara ett bra alternativ till betong och stål i framför allt höga byggnader.


Syftet med detta examensarbete är att identifiera de viktiga egenskaperna i en dynamisk analys av en hög träbyggnad. Förhoppningsvis leder det här examensarbetet till en ökad förståelse av samspelleffekterna mellan dessa egenskaper.


Den statistiska analysen av försöket visar att höjden på byggnaden, tjockleken på väggarna samt en ökad massa är viktiga i en dynamisk analys. Den visar också på en sambäddning mellan höjden på byggnaden och tjockleken på väggarna, samt mellan höjden på byggnaden och en ökad massa.

Merparten av modellerna av byggnaden uppfyller inte komfortkravet enligt ISO 10137. Däremot går det att urskönja mönster som bidrar med viktig information om byggnadens dynamiska egenskaper.

Avslutningsvis, baserat på byggnadens naturliga egenfrekvens framhåller den här studien byggnadens styvhet framför dess massa då byggnaden i fråga stabiliseras med KL-trä och har upp till 16 våningar.

Nyckelord: Acceleration, komfortkrav, korslimmat trä, faktorförsök, finita elementmetoden, naturlig egenfrekvens, vibrationer
NOTATIONS

Latin uppercase letters

\( A \) area
\( B \) background response factor
\( C_i \) constant, where \( i = 1, 2, 3 \ldots \)
\( D_{EI} \) stiffness against bending
\( E_m \) elastic modulus in bending
\( E_t \) elastic modulus in tension
\( E_c \) elastic modulus in compression
\( F \) point load
\( G_{ij} \) shear modulus for shear in plane \( ij \)
\( G_r \) rolling shear modulus
\( G_v \) panel shear modulus
\( I \) moment of inertia
\( I_v \) turbulence intensity
\( P \) peak magnitude
\( R_v \) resonance response factor
\( T_a \) number of years
\( U \) amplitude
\( U_0 \) static displacement
\( V \) volume
\( X_{\max} \) peak acceleration

Latin lowercase letters

\( a \) acceleration
\( b \) width of the building
\( b_f \) width of the floor element
\( c \) coefficient of viscous damping
\( c_{cr} \) critical damping coefficient
\( c_{dir} \) directional factor
\( c_e \) exposure factor
\( c_f \) force coefficient
\( c_{f,0} \) force coefficient of structural elements without free-end flow
\( c_o \) orography factor
\( c_r \) roughness factor
\( c_{season} \) season factor
\( d \) depth of the building
\( f \) frequency
\( f_i \) natural frequency corresponding to the \( i \)th root of the characteristic equation of a system, where \( i = 1, 2, 3 \ldots \)
\( f_n \) undamped natural frequency
\( g \) standard acceleration due to gravity
\( g_k \) dead load, characteristic value
\( h \) height of the building
\( k \) stiffness
\( k_{def} \) deformation factor
\( k_t \) turbulence factor
\( k_p \) peak factor
\( k_r \) terrain factor
\( l_f \) floor span
\( m \) mass
\( m_e \) equivalent mass per unit length
\( n \) number of replicates
\( n_1 \) first natural frequency of the building
\( n_{40} \) number of first-order modes with natural frequencies up to 40 Hz
\( p \) external force
\( p_0 \) excitation force amplitude
\( q_b \) basic velocity pressure
\( q_d \) combined load value
\( q_k \) imposed load, characteristic value
\( q_p \) peak velocity pressure
\( r \) frequency ratio
\( s \) number of runs
\( t \) time
\( t_i \) thickness of layer \( i \)
\( u \) displacement variable
\( u_p \) steady-state response
\( v_b \) basic wind velocity
\( v_{b,0} \) fundamental value of the basic wind velocity
\( v_m \) mean wind velocity
\( v_{Ta} \) characteristic wind velocity with a mean return period of \( T_a \) year(s)
\( w \) deflection
\( y_{ij} \) \( j \)th observation from factor level \( i \), where \( i = 1, 2 \) and \( j = 1, 2, 3 \ldots \)
\( z \) height above ground level
\( z_0 \) roughness length
\( z_e \) reference height
\( z_i \) distance in the \( z \)-direction to the center of gravity for bending around the \( i \)-axis
\( z_{max} \) maximum height
\( z_{min} \) minimum height
\( z_s \) reference height
Greek uppercase letters

\( \Omega \) forcing frequency

Greek lowercase letters

\( \beta_i \) natural mode for the \( i \)th natural frequency of a system, where \( i = 1, 2, 3 \ldots \)
\( \gamma \) density
\( \delta_a \) logarithmic decrement of aerodynamic damping
\( \delta_s \) logarithmic decrement of structural damping
\( \delta_{tot} \) total logarithmic decrement of damping
\( \epsilon_{ij} \) random error component of the model associated with the \( ij \)th observation, where \( i = 1, 2 \) and \( j = 1, 2, 3 \ldots \)
\( \zeta \) viscous damping factor
\( \lambda \) effective slenderness
\( \mu_e \) equivalent mass per unit area
\( \mu_i \) mean of the response at the \( i \)th factor level, where \( i = 1, 2 \)
\( \nu \) up-crossing frequency
\( \nu_i \) unit impulse velocity response of floor \( i \)
\( \rho \) density
\( \sigma_v \) standard deviation of the turbulence
\( \sigma_X \) standard deviation of the acceleration
\( \varphi \) solidity ratio
\( \phi_i \) \( i \)th mode shape for a system, where \( i = 1, 2, 3 \ldots \)
\( \psi_1 \) factor for frequent value of a variable action
\( \psi_2 \) factor for quasi-permanent value of a variable action
\( \psi_\lambda \) end-effect factor
\( \omega_d \) damped circular natural frequency
\( \omega_i \) circular natural frequency corresponding to the \( i \)th root of the characteristic equation of a system, where \( i = 1, 2, 3 \ldots \)
\( \omega_n \) undamped circular natural frequency
\( \zeta \) modal damping ratio

Abbreviations

CLT cross-laminated timber
DOF degree of freedom
EKS application of the European construction standards
FEM finite element method
ISO International Organization for Standardization
MDOFS multiple-degree-of-freedom system
RMS root-mean-square (also abbr. r.m.s)
SDOFS single-degree-of-freedom system
1 INTRODUCTION

1.1 Background

The popularity of high-rise timber buildings has absolutely soared the last years. In Sweden this is partly due to the fact that the regulations regarding multi-story timber buildings were released in the middle of the 1990s (Samuelson, 2011).

At the same time a new engineered timber product was developed in Austria. This product came to be known as cross-laminated timber (CLT). CLT consists of laminated timber panels. The panels are layered in sections and glued together to form a light and strong timber element with a high prefabrication ratio (Mohammad et al., 2012). The development of CLT is one reason for the success of timber as a building material today. Another reason is of course the fact that it is renewable, unlike steel and concrete (Frearson, 2015).

During the 19th and 20th century nearly all the high-rise structures worldwide were made of steel and concrete. Compared to these materials, timber is very light and has a much lower stiffness. Because of this, timber buildings are also more sensitive to vibration.

For a high-rise building in Northern Europe the vibration is mainly induced by the wind. The vibration of the building can cause discomfort for the occupants and, by extension, damage to the building (International Organization for Standardization, 2008). It is therefore important to perform a dynamic analysis of a high-rise building, especially one made out of timber. A dynamic analysis is usually performed in the serviceability limit state, where the comfort criteria is based on the dynamic response of the building and human perception of vibration.

Because of this, the key factors affecting a dynamic analysis needs to be thoroughly studied. Furthermore, these factors might affect one another and therefore their interactions need to be studied as well. The general method of studying interaction between factors is by using a so-called factorial experiment.

1.2 Aim

The aim of this thesis is to identify the important structural properties affecting a dynamic analysis of a high-rise timber building. An important consequence of this study is hopefully a better understanding of the interactions between the structural properties in question.

1.3 Method

This thesis is a continuation of a previously released article about the dynamic properties of high-rise timber buildings (Karlberg, 2016). The article consists of a literature study in which knowledge about structural dynamics, occupant comfort and factorial experiments was gathered. The rest of the necessary theory was gathered at the beginning of this study.

The literature study included in this thesis consists of wind loads on buildings, cross-laminated timber and vibrations. Also, further studies were made in the field of factorial experiments.
The study as a whole was divided into the following, chronological order

1. Gathering of information
2. Planning of the experiment
3. Modeling of the building
4. Analysis of the model
5. Evaluation of the model and the factors

The first step consists of the literature study that was mentioned previously. In the second step the factorial experiment was planned and the factors of interest were determined. To be able to investigate the interaction effects of the key factors in a high-rise timber building design a fictitious building was created. The building was modelled using a software for finite element analysis (step three) and then analyzed (step four). In order to perform a complete factorial experiment all combinations of the factors needs to be studied. Therefore, the magnitude of the experiment depends on the number of interesting factors.

Lastly, the results from the analysis were tested, validated and interpreted. Step three, four and five were repeated according to the design of the factorial experiment.

1.4 Limitations

This study is limited to a dynamic analysis, which was performed in the serviceability limit state. No seismic effects were considered since earthquakes are rare in Sweden.

The only parts of the design codes depicted are the ones associated with the dynamic properties of the building, the characteristics of the wind and human response to vibration in buildings.

The factors appear at only two levels in the factorial experiment, that is, they are varied between a minimum value and a maximum value only.

The structural system of the building is made out of cross-laminated timber and the building is intended for residential purposes. The individual parts of the building were chosen with the structural properties of the part in mind only. Such matters as fire resistance, thermal insulation and transmission of moisture were not included in the selection of the components of the building.

Generally, this study does not include any imposed loads since the building is most vulnerable when its empty. The wind load is only included in the calculations according to the design codes.

Lastly, the comfort criteria used to validate the models is the one given for wind-induced vibration in ISO 10137.
2 WIND LOADS ON STRUCTURES

The wind is an example of a dynamic load, that is, one whose magnitude and direction varies with time (Craig & Kurdila, 2006) (European Committee for Standardization, 2008). When the wind hits the external surface of a structure it generates a pressure on that surface, which leads to the development of forces normal to the surface.

Wind actions should be calculated according to SS-EN 1991-1-4 and are derived from the basic values of the wind velocity. These basic values are characteristic values with a mean return period of 50 years. The wind velocity and the velocity pressure consists of a mean and a fluctuating component, both of which are depending on the terrain roughness, terrain orography and the height above ground, amongst others (European Committee for Standardization, 2008).

2.1 Basic values

The fundamental value of the basic wind velocity \( v_{b,0} \) is the characteristic 10 minutes mean wind velocity, irrespective of wind direction and time of year, at 10 m above ground level in open country terrain with low vegetation. The basic wind velocity is defined as

\[
v_b = c_{dir} \cdot c_{season} \cdot v_{b,0}
\]

(2.1)

where

- \( v_b \) is the basic wind velocity, defined as a function of wind direction and time of year at 10 m above ground of terrain category II
- \( v_{b,0} \) is the fundamental value of the basic wind velocity
- \( c_{dir} \) is the directional factor
- \( c_{season} \) is the season factor

2.2 Wind velocity for determination of comfort criteria

The values of the wind velocity as defined above have a mean return period of 50 years. However, when determining the comfort criteria associated with human perception of motion, different mean return periods are required. The standards ISO 6897 and ISO 10137 uses mean return periods of 5 years and 1 year, respectively (International Organization for Standardization, 1984) (International Organization for Standardization, 2008). The application of the European construction standards, EKS 10 provides a formula for calculating the characteristic wind velocity \( v_{Ta} \) with a mean return period of 5 years (Boverket, 2015).
The wind velocity $v_{Ta}$ is defined as

$$v_{Ta} = 0.75 \cdot v_{50} \sqrt{\left\{1 - 0.2 \cdot \ln \left(-\ln \left(1 - \frac{1}{T_a}\right)\right)\right\}}$$

(2.2)

where

$$v_{50} = v_b$$

$T_a$ is the number of years.

When ISO 6897 is applicable $T_a = 5$ is used in accordance with the paragraph above. In some cases ISO 10137 is better suited and a mean return period of 1 year is thus required. In those cases equation (2.2) is no longer valid. ISO 6897 gives an alternative method: for storms with a one-year return period the suggested satisfactory acceleration magnitudes would be 0.72 times those for a five-year return period.

For the sake of simplicity the following notation is employed

$$v_{Ta} = v_{Ta,5}$$

(2.3)

Then

$$v_{Ta,1} = 0.72 \cdot v_{Ta,5}$$

(2.4)

where

$v_{Ta,1}$ is the characteristic wind velocity with a mean return period of 1 year.

From here on $v_{Ta,1}$ is used instead of the basic wind velocity $v_b$ in order to calculate the wind actions that are required for a satisfactory evaluation of the comfort criteria in question.

### 2.3 Mean wind

The mean wind velocity $v_m(z)$ is derived from the wind velocity. It also depends on the height $z$ above ground and the terrain.

The mean wind velocity is defined as

$$v_m(z) = c_r(z) \cdot c_o(z) \cdot v_{Ta,1}$$

(2.5)

where
\( c_r(z) \) is the roughness factor

\( c_o(z) \) is the orography factor

### 2.3.1 Terrain roughness

The mean wind velocity at the site of the structure varies depending on the height \( z \) above ground level and the ground roughness of the terrain. The roughness factor \( c_r(z) \) takes this into consideration.

The roughness factor is defined as

\[
c_r(z) = k_r \cdot \ln \left( \frac{z}{z_0} \right) \quad \text{for } z_{\text{min}} \leq z \leq z_{\text{max}}
\]

\[
c_r(z) = c_r(z_{\text{min}}) \quad \text{for } z \leq z_{\text{min}}
\]

where

\( z_0 \) is the roughness length (see Table 2.1)

\( k_r \) is a terrain factor depending on the roughness length \( z_0 \), defined as

\[
k_r = 0.19 \cdot \left( \frac{z_0}{z_{0,II}} \right)^{0.07}
\]

where

\( z_{0,II} = 0.05 \) (terrain category II, see Table 2.1)

\( z_{\text{min}} \) is the minimum height defined in Table 2.1

\( z_{\text{max}} = 200 \text{ m} \)

The terrain categories are illustrated in Figure 2.1.

### 2.3.2 Terrain orography

The orography factor \( c_o(z) \) is used where the orography (e.g. hills, cliffs etc.) increases wind velocities by more than 5 %. The effects of the orography may be neglected when the average slope of the upwind terrain is less than 3°.
### Table 2.1 Terrain categories and terrain parameters (European Committee for Standardization, 2008)

<table>
<thead>
<tr>
<th>Terrain Category</th>
<th>$z_0$ [m]</th>
<th>$z_{\text{min}}$ [m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Sea or coastal area exposed to the open sea</td>
<td>0.003</td>
</tr>
<tr>
<td>I</td>
<td>Lakes or flat and horizontal area with negligible vegetation and without obstacles</td>
<td>0.01</td>
</tr>
<tr>
<td>II</td>
<td>Area with low vegetation such as grass and isolated obstacles (trees, buildings) with separations of at least 20 obstacle heights</td>
<td>0.05</td>
</tr>
<tr>
<td>III</td>
<td>Area with regular cover of vegetation or buildings or with isolated obstacles with separations of maximum 20 obstacle heights (such as villages, suburban terrain, permanent forest)</td>
<td>0.3</td>
</tr>
<tr>
<td>IV</td>
<td>Area in which at least 15 % of the surface is covered with buildings and their average heights exceeds 15 m</td>
<td>1.0</td>
</tr>
</tbody>
</table>

### 2.4 Wind turbulence

The fluctuating component of the wind is represented by the turbulence intensity $I_v(z)$. The turbulent component of wind velocity has a mean value of 0 and a standard deviation $\sigma_v$.

The standard deviation of the turbulence is defined as

$$\sigma_v = k_r \cdot v_{T,1} \cdot k_l$$

where

$k_l$ is the turbulence factor

The turbulence intensity at height $z$ is defined as

$$I_v(z) = \frac{\sigma_v}{v_m(z)} \quad \text{for} \quad z_{\text{min}} \leq z \leq z_{\text{max}}$$

$$I_v(z) = I_v(z_{\text{min}}) \quad \text{for} \quad z \leq z_{\text{min}}$$

### 2.5 Peak velocity pressure

The peak velocity pressure $q_p(z)$ includes mean and short-term velocity fluctuations and is, in EKS 10, defined as

$$q_p(z) = [1 + 6 \cdot I_v(z)] \cdot \left[k_r \cdot \ln \left(\frac{Z}{Z_0}\right) \cdot c_o(z)\right]^2 \cdot q_b$$
EKS 10 also states that equation (2.10) can be simplified if the effects of the orography can be neglected (see section 2.3.2).

In that case the peak velocity pressure is simplified to

\[ q_p(z) = c_e(z) \cdot q_b \]  

(2.11)

where
$c_e(z)$ is the exposure factor given in Figure 2.2

$q_b$ is the basic velocity pressure, defined as

$$q_b = \frac{1}{2} \cdot \rho_{air} \cdot v_{Ta,1}^2$$

where

$\rho_{air}$ is the air density

The reference height $z_e$ for windward walls of rectangular plan buildings depend on the ratio between the height $h$ and the width $b$ of the building and are always the upper height of the different parts of the walls. The following three cases for determining $z_e$ are visualized in Figure 2.3

- A building, whose height $h$ is less than its width $b$ should be considered to be one part.
- A building, whose height $h$ is greater than its width $b$, but less than $2b$, may be considered to be two parts: a lower part extending upwards from the ground by a height equal to $b$ and an upper part consisting of the remainder.
- A building, whose height $h$ is greater than $2b$ may be considered to be in multiple parts: a lower part extending upwards from the ground by a height equal to $b$; an upper part extending downwards from the top by a height equal to $b$ and a middle region, between the upper and lower parts, which may be divided into horizontal strips with a height $h_{strip}$.

![Figure 2.2 The exposure factor $c_e(z)$ for $c_o(z) = 1.0$ and $k_i = 1.0$ (Boverket, 2015)](image-url)
This chapter follows the calculations in SS-EN 1991-1-4, where the wind action is represented by a simplified set of pressures (or forces). The effects of those pressures (or forces) represents the extreme effects of the turbulent wind. The effect of the wind on the structure (i.e. the response of the structure) should be calculated from the peak velocity pressure $q_p$ at the reference height (European Committee for Standardization, 2008).
3 VIBRATION OF STRUCTURES

A dynamic analysis of a structure differs from a corresponding static analysis mainly in two ways. While a static analysis assumes that a structure is in complete equilibrium with its surroundings, a dynamic analysis takes into consideration the structure’s true response when subjected to a force. Furthermore, a static analysis treats the loads as static, which means that they vary sufficiently slowly with time. For some loads this is of course true. On the other hand, there are a lot of loads that in fact are dynamic, that is, they vary rather quickly with time.

A dynamic load results in time-varying displacements and stresses of the structure. These displacements and stresses are called the dynamical response of the structure (Craig & Kurdila, 2006). Depending on the magnitude of the structure’s response, the occupants may experience discomfort. In addition to this, human perception of motion depends on numerous factors and is thus very complicated to estimate (Griffin, 1996).

3.1 Structural dynamics

The dynamical response of the structure means that the structure starts to vibrate, more or less. When a mechanical system vibrates it simply means that it oscillates about a point of equilibrium. This motion of the mechanical system can be measured by its displacement, velocity or acceleration (Mansfield, 2005).

3.1.1 Analytical modeling

Structural dynamics problems can be analyzed using either continuous models or discrete-parameter models. The discrete-parameter models shown in Figure 3.2 and Figure 3.3 are called lumped-mass models, since the mass of the system is approximated by a certain number of point masses. Each of these point masses, along with its acceleration gives rise to an inertia force. For every one of these inertia forces there is a coupled displacement that needs to be considered for the model to be valid. The number of these displacements is usually called the system’s number of degrees of freedom (DOF). In the continuous model shown in Figure 3.1 the mass is distributed over the whole system. Thus, by the definition given above, this constitutes an infinite-DOF system.

Once the analytical model of the studied system is created, the physical laws can be applied to obtain the differential equation(s) of motion (this next step is also called the mathematical model). To derive the equations of motion of vibrating structures there are two ways to go: Vectorial Mechanics (also called Newtonian Mechanics) or Analytical Mechanics (also called Lagrangian Mechanics) (Craig & Kurdila, 2006). Here only Newtonian Mechanics are employed.

![Figure 3.1 A continuous model of a cantilever beam (Craig & Kurdila, 2006)](image-url)
3.1.2 The spring-mass model

The so-called spring-mass oscillator is a very simple structure undergoing simple, controlled vibration. It is depicted in Figure 3.4 and is a simplified interpretation of the mathematical model of a single-degree-of-freedom system (SDOFS). It has an elastic component (the spring), which can store and release potential energy. The structure has a mass, which can store and release kinetic energy. The mass \( m \) is a point mass, which can be displaced only in the \( x \)-direction. The variable \( u(t) \) is called the displacement variable, \( k \) is the stiffness of the spring and \( p(t) \) is an external force acting on the mass. The spring is assumed to be massless and linear (Craig & Kurdila, 2006).

3.1.3 The mathematical model

To describe the behavior of the spring-mass oscillator Newton’s Second Law is used along with the given forces shown in Figure 3.5 and the force-displacement relationship. This gives the equation of motion for the prototype undamped SDOFS as

\[
m \ddot{u} + ku = p(t)
\]  

(3.1)

This equation is a linear second-order ordinary differential equation, which constitutes the mathematical model of the simple system in Figure 3.4 (Craig & Kurdila, 2006).
3.1.4 Viscous-damping element

The spring-type member stores energy in a vibrating structure. At the same time, so-called damping mechanisms allow energy to dissipate from the structure. However, damping in a structure is usually impossible to determine. The simplest analytical model of damping uses a so-called damping force, which relates the relative velocity of the system in question to the coefficient of viscous damping $c$.

If this damping force is now included in the mathematical model, the equation of motion for the prototype damped SDOFS is

$$m\ddot{u} + cu + ku = p(t)$$  \hspace{1cm} (3.2)

This second-order differential equation is fundamental in structural dynamics and is applicable to both SDOFS and multiple-degree-of-freedom systems (MDOFS). This equation corresponds to the system in Figure 3.6 (Craig & Kurdila, 2006).

3.1.5 Free vibration, damping and natural frequencies

There are two types of dynamical behavior that are especially important in structural applications: free vibration and forced vibration (or forced response). Free vibration of a structure is the motion resulting from specified initial conditions; forced vibration is the motion resulting from external excitation of the system. A system is said to undergo free vibration if $p(t) = 0$. That is, for equation (3.1) and equation (3.2), respectively

$$m\ddot{u} + ku = 0$$  \hspace{1cm} (3.3)
\[ m\ddot{u} + c\dot{u} + ku = 0 \]  \hspace{1cm} (3.4)

The general solution to equation (3.3) is

\[ u = C_1\cos\omega_n t + C_2\sin\omega_n t \]  \hspace{1cm} (3.5)

where

\[ C_1 \text{ and } C_2 \text{ are constants chosen to satisfy the two initial conditions} \]

\[ \omega_n \text{ is the undamped circular natural frequency, defined as} \]

\[ \omega_n = \sqrt{\frac{k}{m}} \]  \hspace{1cm} (3.6)

in radians per second (rad/s). The undamped natural frequency can now be defined as

\[ f_n = \frac{\omega_n}{2\pi} \]  \hspace{1cm} (3.7)

in Hertz (Hz).

The problem with an undamped system is that once the system is set into motion with an initial displacement and/or an initial velocity that same motion continues indefinitely (at least theoretically). Fortunately, all systems have some damping that causes the motion to fade (and eventually stop) (Craig & Kurdila, 2006).

The viscous damping factor \( \zeta \) can be used to determine three different categories of damping for a free vibrating system: underdamped \((0 < \zeta < 1)\), critically damped \((\zeta = 1)\), and overdamped \((\zeta > 1)\). For structural dynamics, the underdamped case is the most common, and therefore this is the only one employed here.

For the underdamped case the damped circular natural frequency \( \omega_d \) can be determined as

\[ \omega_d = \omega_n\sqrt{1 - \zeta^2} \]  \hspace{1cm} (3.8)

in radians per second (rad/s).

The viscous damping factor \( \zeta \) is defined as

\[ \zeta = \frac{c}{c_{cr}} \]  \hspace{1cm} (3.9)

where

\[ c_{cr} \text{ is the critical damping coefficient, defined as} \]
\[ c_{cr} = 2\sqrt{km} \quad (3.10) \]

As mentioned before, the damping of a structure is very hard to determine and cannot be measured directly. The damping of a structure usually comes from such things as joints and internal damping in the material. However, if the amplitude of vibration decays exponentially, viscous damping can be used for the system.

Often in structural systems the damping is small. If \( \zeta < 0.2 \), \( \omega_d \approx \omega_n \), which means that even for a lightly damped system \( \omega_n \) is representative (Craig & Kurdila, 2006).

### 3.1.6 The steady-state response and the phenomenon of resonance

The total response of a linear system consists of a forced motion and a natural motion. When the excitation of the system can be described by a simple harmonic function, forced motion is referred to as the steady-state response.

If the excitation force amplitude \( p_0 \) and the forcing frequency \( \Omega \) (in rad/s) are constants, equation (3.1) can be written as

\[ m\ddot{u} + ku = p_0 \cos \Omega t \quad (3.11) \]

Then, the steady-state response is

\[ u_p = U \cos \Omega t \quad (3.12) \]

and the amplitude

\[ U = \frac{p_0}{k - m\Omega^2} \quad (3.13) \]

Let

\[ U_0 = \frac{p_0}{k} \quad (3.14) \]

which is the static displacement. This gives along with equation (3.13)

\[ \frac{U}{U_0} = \frac{1}{1 - r^2} \quad (3.15) \]

where

\[ r \] is the frequency ratio, defined as

\[ r = \frac{\Omega}{\omega_n} \quad (3.16) \]
Equation (3.15) is not valid when \( r = 1 \). Furthermore, equation (3.16) gives that \( \Omega = \omega_n \) if \( r = 1 \). This is called resonance. In Figure 3.7 the response of the system at resonance is visualized. Note that the response of the system becomes very large. Because of this it is of utmost importance to avoid the resonance condition (Craig & Kurdila, 2006).

### 3.1.7 Natural frequencies and mode shapes of 2-DOFS

Here the concepts that have been covered so far are extended to a 2-DOFS. For the case where the system undergoes undamped free-vibration, that is the 2-DOFS equivalent of equation (3.3), the differential equation would look like

\[
\begin{bmatrix}
  m_{11} & m_{12} \\
  m_{21} & m_{22}
\end{bmatrix}\begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\
  k_{21} & k_{22}\end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \{0\}
\]

(3.17)

The system is assumed to undergo harmonic motion. Thus, the solution to the corresponding algebraic eigenvalue problem consists of the two roots \( \omega_1^2 \) and \( \omega_2^2 \) (also called eigenvalues). These are the circular natural frequencies of the system, as has been noted before, and gives the respective natural frequencies

\[
f_1 = \frac{\omega_1}{2\pi}
\]

(3.18)

\[
f_2 = \frac{\omega_2}{2\pi}
\]

These can be used to calculate the so-called natural modes of the system. The natural mode for the first natural frequency is defined as

\[
\beta_1 = (U_2/U_1)^{(1)}
\]

(3.19)

where \( U_1 \) and \( U_2 \) is the amplitude of the respective motions of the 2-DOFS.

![Figure 3.7 Response \( u_p(t) \) at resonance: \( p(t) = p_0 \cos \omega_n t \) (Craig & Kurdila, 2006)](image-url)
The natural mode for the second natural frequency is then defined as

\[
\beta_2 = \left(\frac{U_2}{U_1}\right)^{(2)}
\]  

(3.20)

The natural mode of a system is, in mathematical terminology called an eigenvector. They are also called mode shapes, since they describe the deformation of the system. The most important topic in structural dynamic analysis is, by far, the determination of the natural frequencies and mode shapes of vibrating structures (Craig & Kurdila, 2006).

### 3.1.8 The concepts of MDOFS

A common notation that is used to characterize the mode shapes of MDOFS is a vector notation like the one in equation (3.21). As an example, the \( i \)th mode shape \( \phi_i \) for a 2-DOFS would look like

\[
\phi_i = \begin{bmatrix} \phi_1 \\ \phi_2 \end{bmatrix}_i = C_i \begin{bmatrix} 1 \\ \beta_i \end{bmatrix}
\]

(3.21)

where

\[
i = 1, 2
\]

\( C_i \) is a constant determined from initial conditions

It is also convenient to sketch the mode shapes of the system to get a better understanding of the deformation of the system.

By the concepts shown here, a MDOFS have multiple natural frequencies. At each natural frequency, resonance can develop. This means that for large structures resonance is a factor of big importance (Craig & Kurdila, 2006).

### 3.1.9 Finite element analysis

It is easy to see that for a system with many DOF the calculations are extensive. As mentioned earlier a continuous system possesses infinitely many DOF. A very powerful tool for generating a finite-DOF model of a continuous system is the Finite Element Method. Finite element analysis is almost always done with a computer software (Craig & Kurdila, 2006).

### 3.2 Human perception of vibration

Humans are frequently exposed to vibration through such things as machines, tools or music. It has been like this for a long time, but today it is even more so. Today more energy is used to build and create, which also means that more energy is transmitted to people in the form of vibration. Human response to vibration covers many disciplines, including engineering, psychology and the natural sciences (Mansfield, 2005).
3.2.1 Wave theory

Vibration is a form of mechanical wave that can only propagate through a mechanical structure (Mansfield, 2005). The magnitude of the vibration is determined by the extent of the wave and the frequency of the vibration is determined by the number of cycles of the wave per unit time (Griffin, 1996). Like all waves vibration carries energy but not matter. The sine wave is the simplest type of wave and is also referred to as simple harmonic motion (Mansfield, 2005) (Craig & Kurdila, 2006). It is defined as

\[ a(t) = U \sin(2\pi ft) \]  

(3.22)

where

\[ a(t) \] is the acceleration at time \( t \)

\( U \) is the amplitude

\( f \) is the frequency

These simple sine waves are seldom representative when humans are exposed to vibration; in everyday life, people encounters vibration that includes many different frequencies. Therefore, more complex descriptions of waves are necessary. Complex waves can be produced by adding sine waves with different amplitudes, frequencies and phases using the principle of superposition. Of course, complex waves can also be resolved into its simple components and analyzed one component at a time (Mansfield, 2005).

Vibration can be characterized by the waveform of the signal as can be seen in Figure 3.8. When the future waveform can be determined from knowledge of a previous waveform the motion is called deterministic. If the motion cannot be predicted from previous events the motion is called random. Deterministic motion can be of the repetitive kind, which means that each successive cycle is identical. This is called periodic motion. A sine wave is an example of a periodic motion. On the other hand, the (deterministic) motion can occur as only one cycle at a time and is then called nonperiodic. Transient or shock motion are examples of nonperiodic motion.

People are usually exposed to random motion. Random motion can be stationary or nonstationary. If the statistical properties of the (random) motion changes over time, it is referred to as nonstationary. If the motion’s statistical properties do not change over time the vibration is referred to as stationary. It is recommended to treat the motion as nonstationary when evaluating, for example, comfort criteria (Mansfield, 2005). Figure 3.9 gives some examples of waveforms of different types of oscillatory motion.

The magnitude of a vibration signal can be measured by its displacement, velocity or acceleration. When a system vibrates with a large amplitude and a low frequency it is possible to see the displacement of the system. The displacement between the maximum movement in one direction and the maximum movement in the opposite direction is called the peak-to-peak displacement. The displacement is not a very good measure of vibration though. It is hard to measure and often causes severe vibration before the eye can detect it (Griffin, 1996).
Instead, the magnitude of the vibration can be defined in terms of velocity. Consequently, the difference between the maximum velocity in one direction and the maximum velocity in the opposite direction is called the peak-to-peak velocity.

Although velocity could be used to measure the movement of the oscillating system, the acceleration is generally the most convenient way to do so. Therefore, many standards use the vibration acceleration as the main indicator of the severity of human vibration exposures.

As for the acceleration, the peak-to-peak acceleration or the peak acceleration could be used to quantify the magnitude of a vibration. Because of complex motions that fluctuate considerably with time these values may be unrepresentative. Often the root-mean-square (RMS) value is used instead. It is an average measure of the vibration or, more specifically the square root of the average value of the square of the acceleration record.

If the peak magnitude is $P$ for a simple harmonic motion the peak-to-peak magnitude is $2P$ and the RMS magnitude is $P/\sqrt{2}$ (Griffin, 1996). The difference between these measurements can be seen in Figure 3.10.

### 3.2.2 Classification of human vibration

People are exposed to either localized vibration or vibration that affects the whole body. Vibration can be divided into three distinct classes depending on how and where it affects the human body, as well as the characteristic frequency range. These are

- Hand-transmitted vibration
- Whole-body vibration
- Motion sickness

Localized vibration usually affects only the hand-arm system. It is then referred to as hand-transmitted vibration (or hand-arm vibration) and is caused by a person holding a vibrating object such as a tool or control device. The effects of hand-transmitted vibration are most prominent between frequencies of 8 to 1000 Hz.

Whole-body vibration affects all parts of the body and it is normally transmitted through seat surfaces and through the floor. It is noticeable for people in standing postures, seated postures as well as for people in recumbent postures. Whole-body vibration is most prominent between frequencies of 1 to 20 Hz.
Motion sickness is indeed notable for the whole body but the frequency range and the effects justifies the use of a separate class for this topic. A person can be subject to motion sickness when exposed to vibration with frequencies below 1 Hz (Mansfield, 2005). These three classes of human vibration are visualized in Figure 3.11.

![Figure 3.9 Examples of waveforms of different types of oscillatory motion (Griffin, 1996)]
Figure 3.10 Displacement, velocity and acceleration waveforms for a sinusoidal motion (Griffin, 1996)
Whole-body vibration is a common occurrence in everyday life. It is generated when a person is in contact with a shaking surface and is not limited to the contact site. Whole-body vibration is transmitted through the whole body and has numerous health effects.

The human body senses whole-body vibration through many different organs. It combines signals from the visual, vestibular, somatic and auditory systems. In turn, these systems can sense vibration in more than one way and are sensitive to vibration in different frequency ranges (Mansfield, 2005).

The sensitivity of the human body differs depending on the frequency of the induced vibration. The influence of this phenomenon is generally accounted for by the usage of frequency weightings. Frequency weightings adjusts or ‘weights’ the vibration magnitude at each frequency in accordance with its effect on the human body. That is to say, the weighting has a high value at frequencies where the effect is significant and a low value at frequencies with little effect (Griffin, 1996).

Vibration axes for whole-body vibration

Vibration can move along six different directions. These directions are represented as the axes of a coordinate system that is defined relative to the human body. Generally, the stimuli caused by the vibration is very complex and simultaneously move vertically, laterally and in the fore-and-aft directions. Rotation around these axes is also possible. The fore-and-aft direction is defined as the x-axis as shown in Figure 3.12. The lateral direction is parallel to the y-axis and the vertical direction goes along the z-axis. Roll is rotating around the x-axis, pitch is rotation around the y-axis, and jaw is rotation around the z-axis.

For a standing person, the coordinate system is defined as the one for the feet in Figure 3.12. For a recumbent person, the coordinate system, relative to the individual is the same as for a standing person (Mansfield, 2005).
3.3 **Occupant comfort in serviceability limit state according to Eurocode and ISO**

A vibrating structure can cause the occupants discomfort. The occupants experience the motion of the structure as whole-body vibration. There are many factors that influence whether the occupants experience the vibration as an inconvenience or not. Such factors as age, gender, body posture or type of activity to be performed are of significance (International Organization for Standardization, 1998).

ISO 4354 treats wind actions on structures and gives methods for calculating the dynamic response of structures (International Organization for Standardization, 2009). Additionally, there are a number of ISO-standards that concerns human exposure to whole-body vibration. According to ISO 2631-1 the vibration magnitude shall be quantified using the acceleration, which, in turn shall be frequency weighted in accordance with given methods (International Organization for Standardization, 1998) (International Organization for Standardization, 2003).

Furthermore, ISO 6897 and ISO 10137 present methods for evaluating the comfort criteria associated with human response to vibration in structures. Both standards uses the first natural frequency and the acceleration to assess the habitability of the structure (International Organization for Standardization, 1984) (International Organization for Standardization, 2008). The first natural frequency of the structure should be obtained using the methods described in section 3.1. The acceleration of the structure can be calculated using SS-EN 1991-1-4 and EKS 10 (European Committee for Standardization, 2008) (Boverket, 2015). These calculations are described in this section.
3.3.1 The comfort criteria

EKS 10 refers to ISO 6897 for determining the comfort criteria (Boverket, 2015). ISO 6897 provides guidelines for the evaluation of occupant comfort in fixed structures to low-frequency horizontal motion. Fixed structures in this sense means especially buildings and off-shore structures. ISO 6897 covers typical responses of a normal adult population to motion of buildings in the frequency range 0.063 to 1 Hz. In buildings used for general purposes, it is required that probably not more than 2 % of the occupants of the crucial parts of the building comment adversely about the motion caused by the peak 10 min of the worst wind storm with a return period of 5 years or more. This standard uses RMS acceleration for the comfort criteria (International Organization for Standardization, 1984).

ISO 4354 refers to ISO 10137 for the assessment of habitability of buildings with respect to wind-induced vibration (International Organization for Standardization, 2009). ISO 10137 evaluates human response to horizontal motions of buildings using winds with a one-year return period. Furthermore, it uses the peak acceleration and the first natural frequency of the building at the target floor, often the top of the building, in a structural direction of the building and in torsion to determine the habitability (International Organization for Standardization, 2008).

The comfort criteria according to ISO 6897 and ISO 10137 are visualized in Figure 3.13 and Figure 3.14, respectively.

3.3.2 The calculation of the acceleration

EKS 10 provides a method to calculate the peak acceleration of a cantilevered structure with constant mass along its main axis (Boverket, 2015). The peak acceleration $X_{\text{max}}(z)$ is defined as

$$X_{\text{max}}(z) = k_p \sigma_X(z)$$

(3.23)

where

$k_p$ is the peak factor

$\sigma_X(z)$ is the standard deviation of the acceleration

The peak factor $k_p$ in equation (3.23) is defined as

$$k_p = \sqrt{2 \ln(\nu t)} + \frac{0.6}{\sqrt{2 \ln(\nu t)}}$$

(3.24)

where

$\nu$ is the time duration of the peak wind
Figure 3.13  The comfort criteria according to ISO 6897
(International Organization for Standardization, 1984)

Figure 3.14  The comfort criteria according to ISO 10137
(International Organization for Standardization, 2008)
The up-crossing frequency $\nu$ and the rest of the factors that are needed to calculate the peak factor are given in equations (3.25)-(3.31)

$$\nu = n_1 \frac{R_{\nu}}{\sqrt{B^2 + R_{\nu}^2}}$$  \hspace{1cm} (3.25)

where

$n_1$ is the first natural frequency of the building

$$B = \sqrt{\exp \left[ -0.05 \left( \frac{z}{z_s} \right) + \left( 1 - \frac{b}{z} \right) \left( 0.04 + 0.01 \left( \frac{z}{z_s} \right) \right) \right]}$$  \hspace{1cm} (3.26)

where

$z$ is the height above ground as defined in section 2.3

$z_s$ is the reference height given in Figure 3.15

$b$ is the width of the building

$$R_{\nu} = \frac{2\pi F_C \phi_b \Phi_h}{\delta_s + \delta_a}$$  \hspace{1cm} (3.27)

where

$\delta_s$ is the logarithmic decrement of structural damping and is described later in this section

$\delta_a$ is the logarithmic decrement of aerodynamic damping and is defined in equation (3.34)

$$F_C = \frac{4y_c}{(1 + 70.8y_c^2)^{\frac{5}{6}}}$$  \hspace{1cm} (3.28)

$$y_c = \frac{150n_1}{\nu_m(z)}$$  \hspace{1cm} (3.29)

where

$\nu_m(z)$ is defined in section 2.3
\[ \phi_h = \frac{1}{1 + \frac{2n_1 h}{v_m(z)}} \]  

(3.30)

where

\( h \) is the height of the building

\[ \phi_b = \frac{1}{1 + \frac{3.2n_1 b}{v_m(z)}} \]  

(3.31)

The standard deviation of the acceleration \( \sigma_X(z) \) in equation (3.23) is defined as

\[ \sigma_X(z) = \frac{3l_v(z)R_vq_p(z)bc_f \phi_1(z)}{m_e} \]  

(3.32)

where

\( l_v(z) \) is defined in section 2.4

\( q_p(z) \) is defined in section 2.5
$c_f$ is the force coefficient of structural elements with sharp edged section defined in equation (3.36)

$\phi_1(z)$ is the fundamental flexural mode, defined as

$$\phi_1(z) = \left(\frac{z}{h}\right)^{1.5}$$  \hspace{1cm} (3.33)

$m_e$ is the equivalent mass per unit length of the building. This value may be approximated by the average value of $m$ over the upper third of the building.

Eurocode uses the Logarithmic Decrement Method to determine the damping of the building. However, it does not give any value for the logarithmic decrement of structural damping $\delta_s$ for timber buildings. The logarithmic decrement of aerodynamic damping is defined as

$$\delta_a = \frac{c_f \rho_{air} v_m(z_s)}{2n_1 \mu_e}$$  \hspace{1cm} (3.34)

where

- $\rho_{air}$ is the air density
- $v_m(z_s)$ is the mean wind velocity at the reference height $z_s$
- $\mu_e$ is the equivalent mass per unit area of the building

The total logarithmic decrement of damping for a building without special devices (tuned mass dampers for example) is simply

$$\delta_{tot} = \delta_s + \delta_a$$  \hspace{1cm} (3.35)

The force coefficient of structural elements with sharp edged section $c_f$ is defined as

$$c_f = c_{f,0} \cdot \psi_\lambda$$  \hspace{1cm} (3.36)

where

- $c_{f,0}$ is the force coefficient of structural elements without free-end flow as given in Figure 3.16
- $\psi_\lambda$ is the end-effect factor given in Figure 3.17 as a function of the solidity ratio $\varphi$ and the effective slenderness $\lambda$
The effective slenderness $\lambda$ is given in Table 3.1.

Figure 3.16  Force coefficient $c_{f,0}$ of rectangular sections without free-end flow against the ratio of the depth $d$ and the width $b$ of the building (European Committee for Standardization, 2008)

Figure 3.17  The end-effect factor $\psi_\lambda$ as a function of the solidity ratio $\varphi$ and the effective slenderness $\lambda$ (European Committee for Standardization, 2008)
Table 3.1 Values of the effective slenderness $\lambda$ for different structural sections and lattice structures (European Committee for Standardization, 2008)

<table>
<thead>
<tr>
<th>No.</th>
<th>Position of the structure, wind normal to the plane of the page</th>
<th>Effective slenderness $\lambda$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td><img src="image1.png" alt="Diagram 1" /> for $b \leq \ell$</td>
<td>For polygonal, rectangular and sharp edged sections and lattice structures: for $\ell \geq 50$, $\lambda = 1.4 \ell/b$ or $\lambda = 70$, whichever is smaller for $\ell &lt; 15$, $\lambda = 2 \ell/b$ or $\lambda = 70$, whichever is smaller</td>
</tr>
<tr>
<td>2</td>
<td><img src="image2.png" alt="Diagram 2" /> $b \leq \ell$, $b_0 \geq 2.5b$</td>
<td>For circular cylinders: for $\ell \geq 50$, $\lambda = 0.7 \ell/b$ or $\lambda = 70$, whichever is smaller for $\ell &lt; 15$, $\lambda = \ell/b$ or $\lambda = 70$, whichever is smaller</td>
</tr>
<tr>
<td>3</td>
<td><img src="image3.png" alt="Diagram 3" /></td>
<td>For intermediate values of $\ell$, linear interpolation should be used</td>
</tr>
<tr>
<td>4</td>
<td><img src="image4.png" alt="Diagram 4" /> for $z_3 \geq 2b$</td>
<td>For intermediate values of $\ell$, linear interpolation should be used for $\ell \geq 50$, $\lambda = 0.7 \ell/b$ or $\lambda = 70$, whichever is larger for $\ell &lt; 15$, $\lambda = \ell/b$ or $\lambda = 70$, whichever is larger</td>
</tr>
</tbody>
</table>
4 FACTORIAL EXPERIMENTS

Experiments are an essential part in almost all fields of study. Generally, an experiment is used to study the performance of a process or a system. A factorial experiment is an experiment in which all the relevant factors are varied together, instead of one at a time. The design of such an experiment is called a factorial design. Generally, factorial designs are the most efficient alternative when studying the effects of two or more factors (Montgomery, 2009).

4.1 Statistical models

Often when designing an experiment a statistical model is used. The statistical model describes the experimental data and gives a relationship between the response and the important design factors. A simple linear model is

\[ y_{ij} = \mu_i + \epsilon_{ij} \]

where

- \( y_{ij} \) is the \( j \)th observation from factor level \( i \)
- \( \mu_i \) is the mean of the response at the \( i \)th factor level
- \( \epsilon_{ij} \) is the random error component of the model associated with the \( ij \)th observation

Throughout the analysis of the statistical data the model is used to test for the significance of the effects of the design factors, as well as estimating these effects. Because of the importance of the statistical model, it is desirable that the model describes the data as accurately as possible. Therefore, the adequacy of the model needs to be checked before the results of the analysis can be interpreted.

The model in equation (4.1) includes a random error component denoted \( \epsilon_{ij} \). This term is of big importance when using statistical methods. It incorporates all other sources of variability in the experiment, including measurement, differences between the experimental units and general background noise in the process. This term can be described as the noise in the model as it represents the difference between the results of the model and observed results. If \( \epsilon_{ij} \neq 0 \) it means that there are other factors than those included in the model that influence the observed response. Therefore, the experimental error is fundamental when checking the assumptions and the adequacy of the model.

To be able to estimate the error the experiment must be replicated. If there is only one replication of an experiment it is impossible to validate the observed results. For example, the experimenter might be unlucky and gets an unusual observation. If the experiment is not replicated there is no way to check if this is the case. This can of course distort the results of the experiment.
In a factorial experiment, all possible combinations of the included factors are considered. This means that for even a moderate number of factors the design of the experiment will be large. This might make the experiment expensive and/or time-consuming. Sometimes the experimenter can afford only one replicate due to limited resources. There is an obvious risk with conducting an experiment with only one replicate. However, this design is often used and methods to overcome its shortcomings have been developed. Some of these methods are described in section 4.7.

Without an internal estimate of error the model fits the data perfectly and there is no way to check the assumptions or the adequacy of the model. This simplifies the statistical analysis considerably.

A single replicate of a factorial design is sometimes called an unreplicated factorial (Montgomery, 2009).

4.2 Basic definitions and principles of factorial designs

In a factorial design, all possible combinations of the levels of the factors in each complete replication of the experiment are studied. If there are \(a\) levels of factor \(A\) and \(b\) levels of factor \(B\), each replicate includes all \(ab\) treatment combinations. In these kinds of experiments, it is customary to have a so-called response variable to which the experimenter can relate the input variables (factors \(A\) and \(B\)). For example, factors \(A\) and \(B\) can be process variables in a manufacturing process where the response variable is a property of the manufactured product.

Here only two-level designs are considered, that is, designs where the factors (input variables) are represented at two levels. These levels are usually called low and high and denoted “−” and “+”, respectively. The effect of a factor is defined as the change in the response variable that comes directly from changing the factor itself (from the low level to the high level). This is usually called a main effect, since the effect refers to one of the primary factors in the experiment (Montgomery, 2009).

4.2.1 Main effects in a two-factor factorial

A two-factor factorial experiment can, for example, be visualized as in Figure 4.1. The experiment consists of two factors, both of which are represented at two levels. Figure 4.1 shows the factors \(A\) and \(B\) at the horizontal and vertical axis, respectively, against the response data. For example, when both factors are at the low level the value of the response variable is 20 (as shown in the lower left corner of the square in the figure).

The main effect of factor \(A\) is the difference between the average response at the low level of \(A\) and the average response at the high level of \(A\). Numerically, the main effect of factor \(A\) is defined as

\[
A = \frac{40 + 52}{2} - \frac{20 + 30}{2} = 21
\]  (4.2)
In the same manner, the main effect of factor $B$ is

$$B = \frac{30 + 52}{2} - \frac{20 + 40}{2} = 11$$

(4.3)

### 4.2.2 Interaction effects in a two-factor factorial

It is possible that the effect of one factor depends on the level chosen for the other factor. For the two-factor factorial experiment shown in Figure 4.2 this is the case. The effect of $A$ at the low level of factor $B$ is

$$A_{B^-} = 50 - 20 = 30$$

(4.4)

and at the high level of factor $B$

$$A_{B^+} = 12 - 40 = -28$$

(4.5)

The fact that the effect of $A$ depends on the level chosen for factor $B$ is called interaction. The magnitude of the interaction effect is the average difference between the $A$ effects shown in equation (4.5) and equation (4.4), respectively. Numerically, the interaction effect $AB$ is defined as

$$AB = \frac{(-28 - 30)}{2} = -29$$

(4.6)

To get a better understanding of how the factors interact in an experiment, it is practical to plot the interaction effects. Figure 4.3 shows the relationship between the response data in Figure 4.1 and factor $A$ for both levels of factor $B$. Since the $B^-$ and $B^+$ lines are approximately parallel there is no interaction between the factors. In fact, the slope of each line is the $A$ effect for that specific level of $B$.

---

**Figure 4.1** A two-factor factorial experiment (Montgomery, 2009)

**Figure 4.2** A two-factor factorial experiment with interaction (Montgomery, 2009)
The effect of \( A \) at the low level of factor \( B \) is
\[
A_{B^-} = 40 - 20 = 20 \tag{4.7}
\]
The effect of \( A \) at the high level of factor \( B \) is
\[
A_{B^+} = 52 - 30 = 22 \tag{4.8}
\]
Since the \( A \) effects at both levels of \( B \) are very similar, it can be concluded that the effect of factor \( A \) does not depend on the level chosen for factor \( B \). Similarly, Figure 4.4 plots the response data in Figure 4.2. The two lines in this plot are clearly not parallel, which indicates an interaction between the factors. The slope of the \( B^- \) line is given in equation (4.4) and the slope of the \( B^+ \) line is given in equation (4.5).

For the experiment in Figure 4.2 the interaction effect \( AB \) is large. In such a case, the main effect of, for example, \( A \) alone does not convey the necessary information for correctly interpreting the results of the experiment (Montgomery, 2009).

### 4.3 The advantage of factorial designs

Factorial designs have several advantages over one-factor-at-a-time designs. Two of the most important ones are described here. If factor \( A \) and factor \( B \), each at two levels, are varied one at a time three treatment combinations are possible. The levels of the factors are denoted \( A^-, A^+, B^- \) and \( B^+ \).

The three treatment combinations are
- \( A^+B^- \) (factor \( A \) at the high level and factor \( B \) at the low level)
- \( A^-B^- \) (factor \( A \) at the low level and factor \( B \) at the low level)
- \( A^-B^+ \) (factor \( A \) at the low level and factor \( B \) at the high level)

These treatment combinations are shown in Figure 4.5.

The effect of changing factor \( A \) can easily be computed
\[
A = A^+B^- - A^-B^- \tag{4.9}
\]
and the effect of changing factor \( B \) is
\[
B = A^-B^+ - A^-B^- \tag{4.10}
\]
Due to experimental error, each treatment combination should be observed twice to get an average response. This means that six observations are required.
If a factorial experiment had been used instead it would have required only four observations. The experimenter would have added a fourth treatment combination, namely $A^+B^+$ and already there he or she would have had the possibility to compute the average effects [as was done in, for example equation (4.2)]. The general efficiency of the factorial design to the one-factor-at-a-time design is in this case $6/4 = 1.5$. As the number of factors increase this efficiency increases as well. Furthermore, if interaction is present, the conclusions drawn from a one-factor-at-a-time design may be erroneous. In such a case, a factorial design is completely necessary (Montgomery, 2009).

4.4 The $2^2$ Design

The most important class of factorial designs is that of $k$ factors, each at only two levels. This is called a $2^k$ factorial design. It requires $2 \times 2 \times ... \times 2 = 2^k$ observations, hence the name. The levels of the factors may be quantitative, such as two values of temperature or concentration; or they may be qualitative, such as two types of clothing brand or the presence and absence of a factor. The simplest of these designs are the one with only two factors, like the one in section 4.2.
This design is called a $2^2$ factorial design. Since each factor has only two levels it is assumed that the response is approximately linear between the factor levels. The two factors $A$ and $B$ can be combined in four ways, yielding four runs. In turn, running the experiment once involves one treatment combination and yields one observation of the response. The levels and the corresponding treatment combinations can be seen in Table 4.1.

It is convenient to label the treatment combinations by using lowercase letters, as shown in Figure 4.6. The high level of any factor in the treatment combination is denoted by the corresponding lowercase letter and the low level of a factor in the treatment combination is denoted by the absence of the corresponding letter. For example, $a$ represents the treatment combination where $A$ is at the high level and $B$ is at the low level, $b$ represents $B$ at the high level and $A$ at the low level, and $ab$ represents both $A$ and $B$ at the high level. The label (1) is used to denote both factors at the low level. The letter $n$ denotes the number of replicates taken at all the treatment combinations (Montgomery, 2009).

The labels can now be used to define the effects of the factors. It is quite convenient to use the same examples as in section 4.2. Once again, the effect of factor $A$ is the difference in the average response of the two treatment combinations (on the right-hand side of the square) where $A$ is at the high level and the two treatment combinations (on the left-hand side) where $A$ is at the low level. Numerically, this is defined as

$$A = \bar{y}_{A^+} - \bar{y}_{A^-} = \frac{ab + a}{2n} - \frac{b + (1)}{2n} = \frac{1}{2n} [ab + a - b - (1)]$$ (4.11)

The effect of factor $B$ is the difference between the average of the two treatment combinations on the top of the square and the average of the two treatment combinations on the bottom, or

$$B = \bar{y}_{B^+} - \bar{y}_{B^-} = \frac{ab + b}{2n} - \frac{a + (1)}{2n} = \frac{1}{2n} [ab + b - a - (1)]$$ (4.12)

The interaction effect $AB$ is the average of the right-to-left diagonal treatment combinations in the square [$ab$ and (1)] minus the average of the left-to-right diagonal treatment combinations [$a$ and $b$], or

$$AB = \frac{ab + (1)}{2n} - \frac{a + b}{2n} = \frac{1}{2n} [ab + (1) - a - b]$$ (4.13)

Table 4.1 Design matrix for the $2^2$ design (Montgomery, 2009)

<table>
<thead>
<tr>
<th>Factor</th>
<th>Treatment Combination</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>$+$</td>
<td>$-$</td>
</tr>
<tr>
<td>$-$</td>
<td>$+$</td>
</tr>
<tr>
<td>$+$</td>
<td>$+$</td>
</tr>
</tbody>
</table>
Using the experiment in Figure 4.1 the average effect of factors $A$ and $B$ can be estimated as

$$A = \frac{1}{2} [52 + 40 - 30 - 20] = 21$$

(4.14)

$$B = \frac{1}{2} [52 + 30 - 40 - 20] = 11$$

(4.15)

where

$$n = 1.$$

Using the experiment in Figure 4.2 the average effect of factor $AB$ can be estimated as

$$AB = \frac{1}{2} [12 + 20 - 50 - 40] = -29$$

(4.16)

where

$$n = 1.$$

These effect estimates are the same as the ones given in equation (4.2), equation (4.3) and equation (4.6), respectively.

The effect of $A$ is positive, which means that when factor $A$ is increased from the low level to the high level the response increases (by an average of 21 units). Similarly, increasing factor $B$ from the low level to the high level accounts for an average response increase of 11 units. If a main effect is negative it means that increasing this factor decreases the response (Montgomery, 2009). The interaction effect $AB$ is negative, which means that the effect of $A$ decreases as $B$ increases. If $AB$ instead is positive, it means that the effect of $A$ increases as $B$ increases$^1$.

---

$^1$ Jonsson, Adam; Senior Lecturer in Mathematical Statistics at Luleå University of Technology. 2016. E-mail correspondence regarding the interpretation of interaction effects.
The quantity inside the brackets on the right-hand side of, for example equation (4.11) is called a contrast. The contrast is usually referred to as the total effect of the factor. The sum of squares can easily be computed for any contrast. The sum of squares for a factor is a measure of the variation that is explained by the factor.

Equation (4.17) gives the contrast that is used to estimate A as

\[
Contrast_A = ab + a - b - (1)
\]

This corresponds to the sum of squares

\[
SS_A = \frac{[ab + a - b - (1)]^2}{4n}
\]

Consequently, the sum of squares for the two other contrasts are

\[
SS_B = \frac{[ab + b - a - (1)]^2}{4n}
\]

\[
SS_{AB} = \frac{[ab + (1) - a - b]^2}{4n}
\]

A simple table, like Table 4.2 can be used for determining the proper contrast coefficient for each treatment combination. The order in which the treatment combinations appear at the top of the table is called the standard order. The contrast coefficient is always either +1 or −1. The sign of the contrast coefficient for the main effects depends on the level of the factor for that specific treatment combination. For example, at treatment combination (1) factor A is at the low level and the main effect of A is thus associated with a minus. Factor B is also at the low level at treatment combination (1) and the main effect of B is thus associated with a minus as well. The contrast coefficients for estimating the interaction effects are just the product of the corresponding coefficients for the main effects. Thus, the interaction effect AB is associated with a plus. The remaining columns are obtained in the same manner (Montgomery, 2009).

<table>
<thead>
<tr>
<th>Effects</th>
<th>(1)</th>
<th>a</th>
<th>b</th>
<th>ab</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>−1</td>
<td>+1</td>
<td>−1</td>
<td>+1</td>
</tr>
<tr>
<td>B</td>
<td>−1</td>
<td>−1</td>
<td>+1</td>
<td>+1</td>
</tr>
<tr>
<td>AB</td>
<td>+1</td>
<td>−1</td>
<td>−1</td>
<td>+1</td>
</tr>
</tbody>
</table>

A table of plus and minus signs, such as in Table 4.3 is even more appealing to this strategy. Each column constitutes the effect given by the corresponding column heading and has an equal number of plus and minus signs. Column I represents the total or average of the entire experiment.
and has only plus signs. The contrast used for estimating any effect can be found by multiplying the signs in the appropriate column of the table by the corresponding treatment combination and add. For example, to estimate the effect of $A$, the contrast is $-(1) + a - b + ab$. This is the same result as in equation (4.17).

Table 4.3  Contrast coefficients for the $2^2$ design
(Montgomery, 2009)

<table>
<thead>
<tr>
<th>Treatment Combination</th>
<th>Factorial Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>+</td>
</tr>
<tr>
<td>$a$</td>
<td>+</td>
</tr>
<tr>
<td>$b$</td>
<td>+</td>
</tr>
<tr>
<td>$ab$</td>
<td>+</td>
</tr>
</tbody>
</table>

4.5  The $2^3$ Design

If three factors are included in the design it is called a $2^3$ factorial design. A complete replication of such a design requires eight runs since the three factors $A, B$ and $C$ can be combined in $2 \times 2 \times 2 = 8$ possible ways. These treatment combinations can be visualized geometrically using a cube. The cube shown in Figure 4.7 is the three-dimensional equivalent of Figure 4.6. The three factors and the eight runs are presented in the design matrix in Table 4.4. The labels used for the $2^2$ design in section 4.4 can be extended to the $2^3$ design. The labels for the treatment combinations written in standard order are $(1), a, b, ab, c, ac, bc$ and $abc$. Table 4.5 shows the design matrix including the labels for the treatment combinations (Montgomery, 2009).

To help understand how the factor effects for a $2^3$ design works it is of good practice to study the illustrations in Figure 4.8. The main effect of $A$ is defined as the difference between the average of the four runs where $A$ is at the high level and the average of the four runs where $A$ is at the low level. This is visualized in Figure 4.8 as the cube in the upper left corner. The four treatment combinations where $A$ is at the high level forms the rightmost plane in the cube and the four treatment combinations where $A$ is at the low level forms the leftmost plane in the cube. The main effect of $A$ is in other words the average of the plane marked with a plus sign minus the average of the plane marked with a minus sign.

Numerically, this is defined as

$$A = \bar{y}_A^+ - \bar{y}_A^- = \frac{a + ab + ac + abc}{4n} - \frac{1}{4n} \left( b + c + bc \right)$$  \hspace{1cm} (4.21)

Equation (4.21) can be rearranged as

$$A = \frac{1}{4n} \left[ a + ab + ac + abc - (1) - b - c - bc \right]$$  \hspace{1cm} (4.22)
The main effect of $B$ is then the difference in averages between the four treatment combinations in the plane at the front of the cube and the four treatment combinations in the plane at the back of the cube. Numerically

$$B = \bar{y}_B^+ - \bar{y}_B^- = \frac{1}{4n} [b + ab + bc + abc - (1) - a - c - ac] \quad (4.23)$$

The main effect of $C$ is the difference in averages between the four treatment combinations in the plane at the top of the cube and the four treatment combinations in the plane at the bottom of the cube. Numerically

$$C = \bar{y}_C^+ - \bar{y}_C^- = \frac{1}{4n} [c + ac + bc + abc - (1) - a - b - ab] \quad (4.24)$$

Table 4.4  Design matrix for the $2^3$ design  
(Montgomery, 2009)

<table>
<thead>
<tr>
<th>Run</th>
<th>Factor</th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>4</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>5</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>6</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>7</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>8</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
</tbody>
</table>
The $AB$ interaction is defined as one-half of the difference between the average $A$ effects at the two levels of $B$. The average $A$ effect at the high level of $B$ is

$$\bar{A}_B^+ = \frac{[(abc - bc) + (ab - b)]}{2n}$$ (4.25)

The average $A$ effect at the low level of $B$ is

$$\bar{A}_B^- = \frac{[(ac - c) + [a - (1)]]}{2n}$$ (4.26)

One-half of the difference of the two is then

$$AB = \frac{\bar{A}_B^+ - \bar{A}_B^-}{2} = \frac{[abc - bc + ab - b - ac + c - a + (1)]}{4n}$$ (4.27)

The $AB$ interaction is also defined as the difference in averages between runs on two diagonal planes in the corresponding cube in Figure 4.8. Numerically, this is defined as

$$AB = \frac{abc + ab + c + (1)}{4n} - \frac{bc + b + ac + a}{4n}$$ (4.28)

which is the same as in equation (4.27).

The two other two-factor interactions are computed in the same manner

$$AC = \frac{1}{4n}[(1) - a + b - ab - c + ac - bc + abc]$$ (4.29)

$$BC = \frac{1}{4n}[(1) + a - b - ab - c - ac + bc + abc]$$ (4.30)
The $ABC$ interaction is defined as the average difference between the $AB$ interaction at the two different levels of $C$. Numerically

\[
ABC = \frac{1}{4n}\{[abc - bc] - [ac - c] - [ab - b] + [a - (1)]\}
\]

\[
= \frac{1}{4n}[abc - bc - ac + c - ab + b + a - (1)]
\]

Figure 4.8  Illustrations of contrasts for the $2^3$ design (Montgomery, 2009)

As before the quantities inside the brackets in equations (4.22)-(4.24); equation (4.27) and equations (4.29)-(4.31) are the contrasts. For a $2^3$ design the table of plus and minus signs, that is the $2^3$ equivalent of Table 4.3 looks like the one in Table 4.6. For example, the contrast for the $A$ effect is

\[
Contrast_A = -(1) + a - b + ab - c + ac - bc + abc
\]

which is the same as in equation (4.22).
Table 4.6  Contrast coefficients for the $2^3$ design (Montgomery, 2009)

<table>
<thead>
<tr>
<th>Treatment Combination</th>
<th>Factorial Effect</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>I</td>
</tr>
<tr>
<td>(1)</td>
<td>+</td>
</tr>
<tr>
<td>a</td>
<td>+</td>
</tr>
<tr>
<td>b</td>
<td>+</td>
</tr>
<tr>
<td>ab</td>
<td>+</td>
</tr>
<tr>
<td>c</td>
<td>+</td>
</tr>
<tr>
<td>ac</td>
<td>+</td>
</tr>
<tr>
<td>bc</td>
<td>+</td>
</tr>
<tr>
<td>abc</td>
<td>+</td>
</tr>
</tbody>
</table>

Sums of squares for the effects can be computed using the corresponding contrast and equation (4.33)

$$SS_i = \frac{(\text{Contrast}_i)^2}{8n}$$  \hspace{1cm} (4.33)

where

- $i$ is the factor effect of interest
- $n$ is the number of replicates

For example, the sum of squares for the $A$ effect is (symbolically) defined as

$$SS_A = \frac{(\text{Contrast}_A)^2}{8n}$$  \hspace{1cm} (4.34)

4.6  The $2^k$ Design

The statistical model for the general $2^k$ design includes $k$ main effects, $\binom{k}{2}$ two-factor interactions, $\binom{k}{3}$ three-factor interactions, ..., and one $k$-factor interaction. The number of factor effects included in the model for a $2^k$ design is thus $2^k - 1$. The generally accepted procedure for statistical analysis of the $2^k$ design is summarized in Table 4.7. The first step is to estimate factor effects and examine their signs and magnitudes. The methodology for doing this has been presented in complete in section 4.4.

The full model initially includes all main effects and interactions. This subject has been briefly introduced in section 4.1. Step 2a requires that the experiment, or parts of it has been replicated. This text does not cover such an experiment. Step 2b is a modification to this step and the ideas behind this are presented in section 4.7. In the third step the model is formally tested for significant factor effects (Montgomery, 2009).
The nonsignificant factors are usually removed in the fourth step. In step five the adequacy of the model is checked and finally, in the last step, the results are interpreted. The results are often interpreted graphically, using for example a main effect or interaction plot. Like step 2a, steps three, four and five are left out of this text. Revisit section 4.1 for an explanation to this.

The contrast for an effect can be determined using a table of plus and minus signs, like the ones in Table 4.3 or Table 4.6. The contrast for the general factor effect $A_B...K$ can also be defined as

$$\text{Contrast}_{AB...K} = (a \pm 1)(b \pm 1)...(k \pm 1) \quad (4.35)$$

The right-hand side of equation (4.35) can easily be expanded. In the final expression, the number “1” is replaced by the treatment combination (1). The sign in each set of parenthesis is negative if the factor is included in the effect and positive if the factor is not included. For example, the contrast for $AB$ in a $2^3$ design is

$$\text{Contrast}_{AB} = (a - 1)(b - 1)(c + 1) \quad (4.36)$$

$$= abc + ab + c + (1) - ac - bc - a - b$$

which is the same as in equation (4.27).

The contrasts can then be used to estimate the effects and compute the sums of squares. These are defined as

$$AB ... K = \frac{2}{n2^k}(\text{Contrast}_{AB...K}) \quad (4.37)$$

and

$$SS_{AB...K} = \frac{1}{n2^k}(\text{Contrast}_{AB...K})^2 \quad (4.38)$$

where $n$ is the number of replicates.
4.7 An unreplicated $2^k$ Design

Sometimes the experimenter can only afford a single replicate of an experiment. As already pointed out in section 4.1, it is not possible to estimate the error when conducting an unreplicated experiment. If the response is highly variable, the results of the experiment may be erroneous.

However, there are methods to overcome this problem. One way is to spread out the factor levels widely. This is illustrated in Figure 4.10. The shaded band in Figure 4.9 and Figure 4.10 represents the random variability in the response variable. The dark dots are the two measured responses in the experiment and the straight line in the middle is the true factor effect. Increasing the distance between the factor levels results in a better estimate of the true factor effect, as can be seen by comparing the two figures.

Another way to overcome the problem is to arrange the factor effects in a normal probability plot. The negligible effects are normally distributed and tends to fall along a straight line on this plot, whereas significant effects do not lie along the straight line (Montgomery, 2009).

4.8 A numerical example of an unreplicated $2^4$ Design

This example is taken from Montgomery (2009). However, it is slightly more generalized to fit the scope of this chapter. The values of the response variables are just arbitrary numbers with the sole purpose of showing the principles of the design.

In a $2^4$ design there are four factors that are being varied. In the extended design matrix in Table 4.8 the four factors $A$, $B$, $C$ and $D$ are shown. The four factors can be combined in sixteen different ways, yielding sixteen runs and sixteen treatment combinations. The response variables for the sixteen runs can be seen in the table as well. They can also be visualized as in the cube in Figure 4.11, where the fourth factor, $D$ represents the “fourth dimension” of the figure. $D$ is at the high level for all the runs included in the rightmost cube and at the low level for all the runs included in the leftmost cube.

The contrast coefficients are shown in Table 4.9. From this table, it is easy to compute the contrasts by multiplying the signs in the appropriate column of the table by the corresponding treatment combination and add.
For example, the contrast for the effect $ABCD$ is

\[
Contrast_{ABCD} = (1) - a - b + ab - c + ac + bc - abc - d + ad + bd
- abd + cd - acd - bcd + abcd
= 45 - 71 - 48 + 65 - 68 + 60 + 80 - 65 - 43 + 100
+ 45 - 104 + 75 - 86 - 70 + 96 = 11
\]  

The effect of $ABCD$ is then estimated using equation (4.37)

\[
ABCD = \frac{2}{n2^k}(Contrast_{ABCD}) = \frac{2}{1 \cdot 2^4}(11) = 1.375
\]  

\[
\begin{array}{c}
\text{Figure 4.11 Treatment combinations in the } 2^4 \text{ design (Montgomery, 2009)}
\end{array}
\]
The sum of squares is computed using equation (4.38)

\[ SS_{ABCD} = \frac{1}{n^2} (Contrast_{ABCD})^2 = \frac{1}{16} \cdot 2^4 (11)^2 = 7.5625 \]  

(4.41)

The estimated effect in equation (4.40) and the sum of squares in equation (4.41) can be found in Table 4.10.

As this experiment has \(2^4 = 16\) runs, it has \(2^4 - 1 = 15\) effects. The effect of \(ABCD\) and the 14 other effects are summarized in Table 4.10. All the sums of squares are also there together with a column labeled *Percent contribution*. This is just a measure of the sum of squares that is associated with a specific effect relative to the total sum of squares. Numerically

\[ Percent\ \text{contribution} = 100 \frac{SS_{AB\ldots K}}{SS_{TOTAL}} \]  

(4.42)

By the definition given for the sum of squares, the percent contribution can be seen to, more specifically be a measure of how much of the change in the response that the factor in question is accountable for. For example, changing factor \(A\) (from the low level to the high level) accounts for nearly 33% of the change in the response variable.

Examination of the normal probability plot is a mandatory step in the analysis of an unreplicated factorial. The normal probability plot is constructed by arranging the effect estimates from the most negative to the most positive on the x-axis.
The y-axis consists of the so-called observed cumulative frequency, which is defined as

\[ y = 100 \left( \frac{j - 0.5}{s} \right) \]  

(4.43)

where

- \( j \) is the run number considered
- \( s \) is the number of runs

**Table 4.10  Summary of the results for the design**

(Montgomery, 2009)

<table>
<thead>
<tr>
<th>Model Term</th>
<th>Effect Estimate</th>
<th>Sum of Squares</th>
<th>Percent Contribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>21.625</td>
<td>1870.56</td>
<td>32.6397</td>
</tr>
<tr>
<td>B</td>
<td>3.125</td>
<td>39.0625</td>
<td>0.681608</td>
</tr>
<tr>
<td>C</td>
<td>9.875</td>
<td>390.062</td>
<td>6.80626</td>
</tr>
<tr>
<td>D</td>
<td>14.625</td>
<td>855.563</td>
<td>14.9288</td>
</tr>
<tr>
<td>AB</td>
<td>0.125</td>
<td>0.0625</td>
<td>0.00109057</td>
</tr>
<tr>
<td>AC</td>
<td>-18.125</td>
<td>1314.06</td>
<td>22.9293</td>
</tr>
<tr>
<td>AD</td>
<td>16.625</td>
<td>1105.56</td>
<td>19.2911</td>
</tr>
<tr>
<td>BC</td>
<td>2.375</td>
<td>22.5625</td>
<td>0.393696</td>
</tr>
<tr>
<td>BD</td>
<td>-0.375</td>
<td>0.5625</td>
<td>0.00981515</td>
</tr>
<tr>
<td>CD</td>
<td>-1.125</td>
<td>5.0625</td>
<td>0.0883363</td>
</tr>
<tr>
<td>ABC</td>
<td>1.875</td>
<td>14.0625</td>
<td>0.245379</td>
</tr>
<tr>
<td>ABD</td>
<td>4.125</td>
<td>68.0625</td>
<td>1.18763</td>
</tr>
<tr>
<td>ACD</td>
<td>-1.625</td>
<td>10.5625</td>
<td>0.184307</td>
</tr>
<tr>
<td>BCD</td>
<td>-2.625</td>
<td>27.5625</td>
<td>0.480942</td>
</tr>
<tr>
<td>ABCD</td>
<td>1.375</td>
<td>7.5625</td>
<td>0.131959</td>
</tr>
</tbody>
</table>

**Figure 4.12  Normal probability plot of the effects for the design**

(Montgomery, 2009)
In this case \( s = 15 \) since there are 15 effects. For example, if the leftmost value in the plot should be determined, that is \( AC \), the most negative effect estimate \((-18.125)\) is used as the x-value and

\[
y = 100 \left(1 - 0.5 \right) \frac{15}{15} = 3.33
\]

as the y-value.

In Figure 4.13 the main effects of factors \( A, C \) and \( D \) are plotted. These are the important main effects of this experiment, which Table 4.10 as well as the normal probability plot in Figure 4.12 reveals. The significant effects as shown in Figure 4.12 are the ones that does not lie on the straight line. In Table 4.10 the magnitude of the effect estimate and the percent contributed are the most substantial descriptors of the significance of the effect. Furthermore, the significant interaction effects \( AC \) and \( AD \) are plotted in Figure 4.14.

As for the main effect plots the slope of the curve is given by the corresponding effect estimate. The minimum of the curve is given by the average response at the low level of the factor. Take factor \( A \) as an example. There is one plane in each of the two cubes in Figure 4.11 that satisfies the condition above (that is, factor \( A \) at the low level). \( A \) is at the low level at the leftmost plane in each cube. As each plane incorporates four treatment combinations, there are eight treatment combinations that should be considered. The minimum of the curve is therefore the average of these eight treatment combinations. Consequently, the maximum of the curve is the average of the eight treatment combinations where \( A \) is at the high level.

In the interaction plots the interaction effects (as shown in Table 4.10) is one-half of the difference between the slope of the two lines in each of the plots. To get the minimum of, for example the \( C^- \) line the response when both \( C \) and \( A \) are at the low level needs to be studied. By studying Figure 4.11 it can be established that \( A \) is at the low level in the leftmost plane in each cube, while \( C \) is at the low level in the plane at the bottom of each cube. In fact, the solution to the problem above is given by the treatment combinations that lies on the mutual line of the two planes in each cube. So, the minimum of the \( C^- \) line is given by the average of the four treatment combinations that lies along the two lines where the two planes intersect in each cube.

![Figure 4.13 Main effect plots (Montgomery, 2009)](image-url)
Consequently, at the maximum of the $C^-$ line factor $A$ is at the high level (while $C$ is still at the low level). This is satisfied by the rightmost plane and the plane at the bottom of each cube. The maximum of the $C^-$ line is therefore given by the average of the four treatment combinations that lies along the two lines where these two planes intersect in each cube. Of course, the $C^+$ line is generated in the same manner.

Keep in mind that the main effects are of little importance when significant interactions are present. In an experiment, it is important to plot the effects of the factors, mostly because the numerical values often can be hard to evaluate, especially in the case of interactions.

In this example, the important factors are $A$, $C$ and $D$ as well as the interactions $AC$ and $AD$. The effect of, for example factor $A$ is positive, which means that if $A$ is increased the response increases as well. The same goes for factors $C$ and $D$.

Interaction $AC$ indicates that factor $A$ has a larger effect on the response when $C$ is at the low level. At the same time interaction $AD$ indicates that $A$ has a larger effect on the response when $D$ is at the high level. These plots are of big importance when evaluation what actions might be necessary to improve the process or system that the experiment describes. Of equal importance is the knowledge of the response variable and what the satisfying values of it might be.

Figure 4.14 Interaction plots (Montgomery, 2009)
5 CROSS-LAMINATED TIMBER

Cross-laminated timber (CLT) was developed in Austria in the middle of the 1990s. CLT panels consist of several crosswise layers of timber planks glued together on their wide faces. Each layer is traditionally oriented in a right angle relative to the next layer. CLT can be used as the sole building material for the frame of a building or as a complement to other framing options. Furthermore, CLT is part of a new category of timber products called massive timber that has been produced to form a good alternative to concrete and steel in the design of particularly tall buildings (Mohammad et al., 2012).

CLT panels are easily prefabricated and efficiently assembled on the job site. They are relatively light weight, which makes them easy to handle and well suited for sites where the ground conditions are bad (Evans, 2013). The cross lamination in the panel provides relatively high strength and stiffness properties in both directions of the panel (Mohammad et al., 2012).

Lastly, wood is a natural and renewable material. As wood continues to store carbon absorbed by the trees while growing its carbon footprint is lighter than that for steel and concrete. At the same time, manufacturing of wood products requires less energy, which results in less greenhouse gas emissions (Evans, 2013).

5.1 CLT panels

In Sweden Martinsons is the only manufacturer of CLT. Their line of CLT products includes panels with thicknesses varying from 60 mm up to 300 mm. The panels can be used both as wall elements and floor element, with lengths up to 16 m. The width of the panels varies between 2.4 m and 3.0 m. The panels consist of either 3, 5 or 7 layers of timber planks. The outer layers of the panels can be oriented in either the longitudinal direction or the transverse direction, where the latter is recommended for wall elements. The layers consist of different types of construction wood and have different thicknesses, depending on the stiff direction and the weak direction of the panel. The density of the panels should be taken as 400 kg/m³ (Martinsons, 2016b). In Figure 5.1 the dimensions and the different layers of the panel are illustrated.

![Figure 5.1 Illustration of the dimensions and the different layers of a CLT panel (Martinsons, 2016b)]
5.2 Connections

Figure 5.2 shows examples of typical connections that are being used when building with CLT. The leftmost figure shows a typical jointing of two floor elements. The elements are jointed with a plywood spline that is attached to the panels with screws. The connection of the floor element to the wall element is usually done with metal brackets. Either there is one metal bracket present, like in Figure 5.2 or there is one metal bracket attached to each wall element in the assembly. The connection of the wall element to the concrete foundation can be done in several ways as well. For example, a sill plate can be attached to the slab and then used to mount the wall element. The simplest wall to wall connection looks like the one in the rightmost figure in Figure 5.2. For this connection self-tapping screws are driven perpendicular to the panel. Traditional fasteners between elements are screws or nails (Mohammad, 2011).

![Figure 5.2 Examples of different connections for CLT panels (Martinsons, 2016c) (Martinsons, 2016d)](image-url)
6 FINITE ELEMENT MODELING

The fictitious building used in this study is located in a suburb of Gothenburg where the terrain is more or less flat. The building was modelled using the software FEM-Design 3D Structure, which is an advanced modeling software for finite element analysis and design of load-bearing concrete, steel, and timber structures according to Eurocode (StruSoft, 2016). A dynamic analysis was performed using the software in order to get the natural frequency of the building. FEM-Design does not consider any damping in the dynamic (eigenfrequency) analysis.

Before the building could be modeled, the experiment was carefully planned. These two steps are elaborated in this chapter.

6.1 The planning of the experiment

This section follows part of the guidelines for designing an experiment as given by Montgomery (2009). Many of the choices that are being described in this section are based on what was presented in the article referred to in section 1.3. These choices are therefore explained only briefly here. For a more elaborate explanation the reader is referred to the article in question (Karlberg, 2016).

6.1.1 Selection of the response variable

The response variable for this experiment is the natural frequency of the building. This quantity was selected as the response variable based on its utmost importance in a dynamic analysis as well as in the evaluation of the comfort criteria. A building has many natural frequencies but the highest magnitudes of acceleration is generally found at the first natural frequency (International Organization for Standardization, 1984). Thus, the response variable for this experiment is, more precisely found to be, the first natural frequency of the building.

6.1.2 Choice of factors

The response variable is selected based on the information it provides about the process or system under study. The factors, on the other hand, are chosen in order to illustrate their influence on the performance of the process or system (Montgomery, 2009).

According to equation (3.6) and equation (3.7) the (undamped) natural frequency depends on the stiffness $k$ and the mass $m$ of the system. The stiffness of a building depends mainly on the type of frame chosen, the layout, and the geometry as well as the stiffness of the individual materials. The mass of the building is of course highly dependent on the dimensions and mass density of the materials, and the loads included.

For the purpose of this experiment, two classes of potential design factors should be defined: design factors and held-constant factors. The design factors are the factors that are actually

---

2 Alaoui, Youssef; Coach Engineering and Sales at StruSoft AB. 2016. E-mail correspondence regarding the integration of damping in the eigenfrequency analysis in FEM-Design.
studied in the experiment. The held-constant factors are factors that are not of interest for the experiment but still may affect the response (Montgomery, 2009).

For this experiment, only one type of frame was investigated. Also, the layout of the building was chosen in an arbitrary manner and remained the same during the whole experiment. Thus, these two factors can be regarded as held-constant factors; they do indeed affect the response but they were not of interest for this experiment. These decisions were mostly due to the time-consuming aspect of changing the frame of the building during the experiment and the fact that the layout of the building lies beyond the scope of this thesis.

The design factors of this experiment are

- The shape of the building
- The number of floors
- The thickness of the walls
- The addition of mass

The shape of the building is of interest since it affects both the mass and the stiffness of the building. The number of floors are interesting, not only for the same reason as the shape of the building, but also because of the quest to build taller buildings. The thickness of an individual wall element of the building has a direct influence on both the mass and the stiffness of that element. Of course, the properties of each individual element influence the global properties of the building as well. The addition of mass, without affecting the stiffness of the building, is highly effective in a dynamic analysis.

Lastly, the ambition to concretize the stiffness and the mass of the building was highly significant when choosing the design factors.

### 6.1.3 Choice of factor levels

These four design factors are all present at two levels: the low level and the high level. The levels were chosen so that the spacing between them would be as large as possible, according to Figure 4.10.

Two different shapes of the building were studied: a square shaped building and a rectangular shaped building. As for the dimensions of the building the depth $d$ is held constant at 20 m, while the width $b$ varies. The width of the building is 20 m at the low level (for the square shape) and 40 m at the high level (for the rectangular shape). These dimensions were chosen so that the floors of the square shaped building would be of reasonable size to actually house one or more apartments. The width of the rectangular shaped building, that is the width at the high level was chosen so that the rectangular shaped building would be twice as wide as the square shaped one.

The highest number of floors were chosen so that the building would reach a considerable height when at its highest. The high level was chosen to be a little bit higher than the current record-holder Treet, which has fourteen floors (Hixson, 2016). The highest number of floors was
therefore chosen to be 16 and the lowest number of floors was chosen to be one half of that, which is 8.

The thickness of the walls was chosen in accordance with the range of CLT panels provided by Martinsons. Their assortment of CLT panels has been briefly introduced in section 5.1. Thus, the low level of the thickness of the walls for this experiment is 60 mm. Consequently, the high level of the thickness of the walls is 300 mm. This variation of the thickness of the wall elements changes both the mass and the stiffness of the walls and thereby also of the building.

As for the addition of mass, the low level represents a building where the additional mass is absent, while the high level of the factor represents a building where the additional mass is present.

### 6.1.4 Choice of experimental design

The design of an experiment with four design factors, each at two levels is called a $2^4$ factorial design. This is an unreplicated design with the run order as shown in the design matrix in Table 6.1.

<table>
<thead>
<tr>
<th>Model Number</th>
<th>Building Shape</th>
<th>Number of Floors</th>
<th>Thickness Walls</th>
<th>Additional Mass</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>−</td>
<td>−</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td>2</td>
<td>−</td>
<td>+</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td>3</td>
<td>−</td>
<td>+</td>
<td>+</td>
<td>−</td>
</tr>
<tr>
<td>4</td>
<td>−</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>5</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>6</td>
<td>+</td>
<td>−</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>7</td>
<td>+</td>
<td>−</td>
<td>−</td>
<td>+</td>
</tr>
<tr>
<td>8</td>
<td>+</td>
<td>−</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td>9</td>
<td>−</td>
<td>−</td>
<td>+</td>
<td>−</td>
</tr>
<tr>
<td>10</td>
<td>−</td>
<td>−</td>
<td>−</td>
<td>+</td>
</tr>
<tr>
<td>11</td>
<td>−</td>
<td>−</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>12</td>
<td>+</td>
<td>+</td>
<td>−</td>
<td>−</td>
</tr>
<tr>
<td>13</td>
<td>−</td>
<td>+</td>
<td>−</td>
<td>+</td>
</tr>
<tr>
<td>14</td>
<td>+</td>
<td>−</td>
<td>+</td>
<td>−</td>
</tr>
<tr>
<td>15</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>−</td>
</tr>
<tr>
<td>16</td>
<td>+</td>
<td>+</td>
<td>−</td>
<td>+</td>
</tr>
</tbody>
</table>
### 6.2 The modeling of the building

The design of the experiment requires sixteen runs (and sixteen treatment combinations). This yields sixteen unique alterations of the building. In accordance with the factor levels chosen, the following can be concluded:

- one half of these models are square shaped (while the other half are rectangular shaped)
- one half of these models has 8 floors (while the other half has 16 floors)
- one half of these models has walls that are 60 mm thick (while the other half has walls that are 300 mm thick)
- one half of these models has extra mass (while the other half has not)

This can be seen in Table 6.2, which is the numeric equivalent of the design matrix in Table 6.1.

**Table 6.2 The design matrix, with the actual factor levels**

<table>
<thead>
<tr>
<th>Model Number</th>
<th>Building Shape</th>
<th>Number of Floors</th>
<th>Thickness Walls [mm]</th>
<th>Additional Mass</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>square</td>
<td>8</td>
<td>60</td>
<td>off</td>
</tr>
<tr>
<td>2</td>
<td>square</td>
<td>16</td>
<td>60</td>
<td>off</td>
</tr>
<tr>
<td>3</td>
<td>square</td>
<td>16</td>
<td>300</td>
<td>off</td>
</tr>
<tr>
<td>4</td>
<td>square</td>
<td>16</td>
<td>300</td>
<td>on</td>
</tr>
<tr>
<td>5</td>
<td>rectangular</td>
<td>16</td>
<td>300</td>
<td>on</td>
</tr>
<tr>
<td>6</td>
<td>rectangular</td>
<td>8</td>
<td>300</td>
<td>on</td>
</tr>
<tr>
<td>7</td>
<td>rectangular</td>
<td>8</td>
<td>60</td>
<td>on</td>
</tr>
<tr>
<td>8</td>
<td>rectangular</td>
<td>8</td>
<td>60</td>
<td>off</td>
</tr>
<tr>
<td>9</td>
<td>square</td>
<td>8</td>
<td>300</td>
<td>off</td>
</tr>
<tr>
<td>10</td>
<td>square</td>
<td>8</td>
<td>60</td>
<td>on</td>
</tr>
<tr>
<td>11</td>
<td>square</td>
<td>8</td>
<td>300</td>
<td>on</td>
</tr>
<tr>
<td>12</td>
<td>rectangular</td>
<td>16</td>
<td>60</td>
<td>off</td>
</tr>
<tr>
<td>13</td>
<td>square</td>
<td>16</td>
<td>60</td>
<td>on</td>
</tr>
<tr>
<td>14</td>
<td>rectangular</td>
<td>8</td>
<td>300</td>
<td>off</td>
</tr>
<tr>
<td>15</td>
<td>rectangular</td>
<td>16</td>
<td>300</td>
<td>off</td>
</tr>
<tr>
<td>16</td>
<td>rectangular</td>
<td>16</td>
<td>60</td>
<td>on</td>
</tr>
</tbody>
</table>
The two building shapes can be seen in Figure 6.1 and Figure 6.2, where the square shaped building is $20 \times 20 \text{ m}^2$ and the rectangular shaped building is $40 \times 20 \text{ m}^2$. Notice that the rectangular shaped building is just two duplicates of the square shaped building put together. In Figure 6.3 and Figure 6.4 the building can be seen at its lowest and its highest, respectively. Figure 6.3 shows model 1, which is one of the models with only 8 floors. As each floor is 3 m in height this alteration of the building has a total height of 24 m. Consequently, the model in Figure 6.4, model 12, is one of the models with 16 floors and a total building height of 48 m. All 16 models can be seen in Appendix B.
Figure 6.3  One of the models that has 8 floors

Figure 6.4  One of the models that has 16 floors
6.2.1 Concrete slab

The building’s foundation consists of a concrete slab. It was made 250 mm thick using C25/30 concrete. The slab is not reinforced since the mass added by the reinforcement would have been negligible compared to the mass of the slab itself. Because of this, the reinforcement would have affected the dynamic analysis only slightly. Thus, neglecting the reinforcement in the slab is a valid simplification.

The slab is supported by line supports that are placed along the location of the walls of the building (see section 6.2.2 for the location of the walls). Figure 6.5 shows the set-up of the connection rigidity for the line support group.

6.2.2 Walls

The walls of the building are all made of CLT panels from Martinsons’ product line-up. There are three types of walls in the building: inner walls, outer walls and walls included in the core of the building. All walls are load-bearing and all walls are varied according to the factorial design. This means that all walls are either 60 mm thick or 300 mm thick. The location of the different types of walls in the square shaped building is visualized in the floor plan in Figure 6.6.

The inner walls are placed symmetrically around the core to function as support for the floors as well as for adding stiffness to the building. The walls that are included in the core of the building forms the elevator shaft/staircase that runs through the center of each square shaped building. Consequently, in the rectangular shaped building two elevator shafts/staircases are present. Here the right faced outer wall in Figure 6.6 becomes an inner wall as the hypothetical merging of the two square shaped buildings takes place.

FEM-Design 3D Structure provides CLT as a predefined material. The CLT panels can be used both as floor elements and wall elements. In line with the factor levels two new CLT panels were created: CLT 60 and CLT 300. The stiffness properties of these are shown in Figure 6.7. These properties have been taken from Table 6.3. Figure 6.8 shows the local coordinate system from which these properties are defined. The CLT panels used as walls were modeled as 3 m wide, except for a few that automatically were set as 1 m or 2 m wide by the software. The height of the panels was naturally set as the height of each floor, which is 3 m.

6.2.3 Floors

The floors of the building consist of CLT panels that are 300 mm thick and are included in Martinsons’ product line-up. The floors are validated according to Appendix A. The CLT panels used as floors were modeled as 3 m wide, except for a few that automatically were set as 1 m wide by the software.

The stiff direction of the panel is the same as the longitudinal direction of the panel. As the CLT panels can be used both as floor elements and wall elements the same properties are valid for the floor panels as the ones given for the thickest walls in Figure 6.7.
Figure 6.9 shows the floor plan of the square shaped building. Remember that the rectangular shaped building is just two duplicates of the square shaped building put together. Each square shaped building has five continuous floor panels. In turn, each continuous panel is made up of many individual panels. This is of course a simplification, which relies on satisfactory jointing of the individual panels. The same simplification goes for the wall elements. Here one continuous panel goes along each of the four faces of each floor of the building.

One of the continuous floor panels spans the elevator shaft/staircase. This is due to the fact that most of this area is occupied by the staircase rather than the elevator shaft. Notice in Figure
6.9 that each of the other four continuous panels are supported by the outer walls at two sides, two of the inner walls and one of the walls that makes up the core of the building.

All the individual panels have a span length of 7 m, except the ones that span the shaft; the span length of these are 6 m.

![Timber panel library](image)

**Figure 6.7** The stiffness properties of the two created CLT panels

**Table 6.3** The stiffness properties of the two CLT panels as given by Martinsons (2016a)

<table>
<thead>
<tr>
<th>Thickness [mm]</th>
<th>$E_{mz,50}$</th>
<th>$E_{mx,50}$</th>
<th>$E_{tx,50}$</th>
<th>$E_{tz,50}$</th>
<th>$E_{cx,50}$</th>
<th>$E_{cz,50}$</th>
<th>$G_{xy,50}$</th>
<th>$G_{yz,50}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>10606</td>
<td>764</td>
<td>7457</td>
<td>3913</td>
<td>7457</td>
<td>3913</td>
<td>117</td>
<td>96</td>
</tr>
<tr>
<td>300</td>
<td>8157</td>
<td>2120</td>
<td>6692</td>
<td>3022</td>
<td>6692</td>
<td>3022</td>
<td>100</td>
<td>104</td>
</tr>
</tbody>
</table>

![Local coordinate system of a CLT panel](image)

**Figure 6.8** Local coordinate system of a CLT panel (Martinsons, 2016a)
6.2.4 Additional concrete coating

Since concrete is heavier than timber it is quite convenient to add concrete to a timber building in order to get a heavier building. In this experiment this was done by the addition of a concrete coating on each of the timber floors of the building. Remember: one half of the models has these additional concrete coatings, while the other half of the models has not. The concrete coating was determined to be 80 mm thick, which equals an additional load of 2 kN/m². The coating was made of C25/30 concrete. It was modeled as a uniformly distributed load on each timber floor of the building. In this way, the coatings provide additional mass but no additional stiffness.

6.2.5 Connections

The connections associated with the concrete slab and the line supports are given in section 6.2.1. All the connections between the timber elements as well as between the timber elements and the concrete slab are assumed to be hinged.

6.2.6 Loads

As for the loads in the models only the self-weight of the structure and the additional concrete coating (when present) was included. Both the self-weight of the structure and the additional concrete coating was set as permanent loads. Before the dynamic analysis was performed the loads were converted to mass. Load combinations were not utilized since the loads were only used to provide mass to the building.

6.2.7 Mesh

The mesh that was used for the models consisted of fine, rectangular elements with nine nodes at each face of the element. The average surface element size was set as 0.4 m.
Figure 6.9 The floor plan of the square shaped building with the five continuous panels as well as the individual panels visualized.
7 RESULTS

The results of this study can be divided into three parts, as described in this chapter.

7.1 The response data from the experiment

The experiment, as described in chapter 6, was performed in order to study the response and the influence of the given factors. If the design matrix in Table 6.1 once again is presented, this time with the labels for the treatment combinations and the response data, it would look like the one in Table 7.1. The treatment combinations are also visualized in Figure 7.1 and Figure 7.2.

Table 7.1 The design matrix, including labels for the treatment combinations and the response data

<table>
<thead>
<tr>
<th>Model Number</th>
<th>Factor</th>
<th>Run Label</th>
<th>First Natural Frequency [Hz]</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Building Shape</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Number of Floors</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Thickness Walls</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>Additional Mass</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>(1)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>b</td>
<td></td>
<td>0.673</td>
</tr>
<tr>
<td>3</td>
<td>bc</td>
<td></td>
<td>1.188</td>
</tr>
<tr>
<td>4</td>
<td>bcd</td>
<td></td>
<td>0.866</td>
</tr>
<tr>
<td>5</td>
<td>abcd</td>
<td></td>
<td>0.809</td>
</tr>
<tr>
<td>6</td>
<td>acd</td>
<td></td>
<td>1.630</td>
</tr>
<tr>
<td>7</td>
<td>ad</td>
<td></td>
<td>0.799</td>
</tr>
<tr>
<td>8</td>
<td>a</td>
<td></td>
<td>1.254</td>
</tr>
<tr>
<td>9</td>
<td>c</td>
<td></td>
<td>2.421</td>
</tr>
<tr>
<td>10</td>
<td>d</td>
<td></td>
<td>0.864</td>
</tr>
<tr>
<td>11</td>
<td>cd</td>
<td></td>
<td>1.750</td>
</tr>
<tr>
<td>12</td>
<td>ab</td>
<td></td>
<td>0.626</td>
</tr>
<tr>
<td>13</td>
<td>bd</td>
<td></td>
<td>0.431</td>
</tr>
<tr>
<td>14</td>
<td>ac</td>
<td></td>
<td>2.274</td>
</tr>
<tr>
<td>15</td>
<td>abc</td>
<td></td>
<td>1.120</td>
</tr>
<tr>
<td>16</td>
<td>abd</td>
<td></td>
<td>0.400</td>
</tr>
</tbody>
</table>

Figure 7.1 Treatment combinations in the design using labels
7.2  Statistical analysis of the data

To be able to analyze the response data given in section 7.1 the contrasts of each of the factors need to be computed. Remember: this experiment has four design factors. These four design factors can be combined in $2^4 = 16$ ways. Thus, there are $2^4 - 1 = 15$ effects to be considered. The contrasts needed to estimate these effects can be found by examining Table 7.2. From these contrasts the effects can be estimated and the sums of squares can be computed. These are shown in Table 7.3.

Table 7.2  Contrast coefficients for the design

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>AB</th>
<th>AC</th>
<th>AD</th>
<th>BC</th>
<th>BD</th>
<th>CD</th>
<th>ABC</th>
<th>ABD</th>
<th>ACD</th>
<th>BCD</th>
<th>ABCD</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td></td>
</tr>
<tr>
<td>b</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td></td>
</tr>
<tr>
<td>bc</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td></td>
</tr>
<tr>
<td>bcd</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>abcd</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td></td>
</tr>
<tr>
<td>acd</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>ad</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>a</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>c</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>d</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td></td>
</tr>
<tr>
<td>cd</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>ab</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td></td>
</tr>
<tr>
<td>bd</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>ac</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td></td>
</tr>
<tr>
<td>abc</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>abd</td>
<td>+</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
</tbody>
</table>
Table 7.3  Summary of the results for the design

<table>
<thead>
<tr>
<th>Model Term</th>
<th>Effect Estimate</th>
<th>Sum of Squares</th>
<th>Percent Contribution</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>-0.08</td>
<td>0.03</td>
<td>0.45</td>
</tr>
<tr>
<td>B</td>
<td>-0.78</td>
<td>2.43</td>
<td>44.07</td>
</tr>
<tr>
<td>C</td>
<td>0.71</td>
<td>2.00</td>
<td>36.35</td>
</tr>
<tr>
<td>D</td>
<td>-0.42</td>
<td>0.71</td>
<td>12.81</td>
</tr>
<tr>
<td>AB</td>
<td>0.03</td>
<td>0.00</td>
<td>0.06</td>
</tr>
<tr>
<td>AC</td>
<td>-0.02</td>
<td>0.00</td>
<td>0.03</td>
</tr>
<tr>
<td>AD</td>
<td>0.01</td>
<td>0.00</td>
<td>0.01</td>
</tr>
<tr>
<td>BC</td>
<td>-0.24</td>
<td>0.24</td>
<td>4.33</td>
</tr>
<tr>
<td>BD</td>
<td>0.14</td>
<td>0.08</td>
<td>1.52</td>
</tr>
<tr>
<td>CD</td>
<td>-0.07</td>
<td>0.02</td>
<td>0.33</td>
</tr>
<tr>
<td>ABC</td>
<td>0.01</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>ABD</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>ACD</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>BCD</td>
<td>0.03</td>
<td>0.00</td>
<td>0.05</td>
</tr>
<tr>
<td>ABCD</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
</tbody>
</table>

The normal probability plot of the factor effects is shown in Figure 7.3. It reveals, along with Table 7.3 that the important factor effects from this experiment are the main effects of $B$, $C$ and $D$ and the $BC$ and $BD$ interactions. With the help of Table 7.1 the factors that have the greatest influence on the response can be determined as

- the number of floors ($B$)
- the thickness of the walls ($C$)
- the addition of mass ($D$)
- the interaction between the number of floors and the thickness of the walls ($BC$)
- the interaction between the number of floors and the addition of mass ($BD$)

The effects of these factors are plotted in Figure 7.4-Figure 7.8.
Figure 7.3  Normal probability plot of the effects for the design

Figure 7.4  Main effect plot of factor B

Figure 7.5  Main effect plot of factor C
Figure 7.6  Main effect plot of factor D

Figure 7.7  Interaction plot of the BC interaction

Figure 7.8  Interaction plot of the BD interaction
7.3 Evaluation of the comfort criteria

To validate the models of the building the comfort criteria for residential buildings according to ISO 10137 was evaluated. ISO 10137 uses the peak acceleration of the horizontal motion of the building together with the first natural frequency to determine the comfort criteria associated with human response to motions of buildings caused by the wind.

The peak acceleration was calculated according to section 3.3.2 for all 16 models. These calculations can be found in Appendix D. The first natural frequency for the models are given in Table 7.1. Also, Appendix C summarizes all the necessary information from the models for the calculations in Appendix D. Table 7.4 gives the first natural frequency and the peak acceleration for all 16 models of the building.

<table>
<thead>
<tr>
<th>Model Number</th>
<th>First Natural Frequency [Hz]</th>
<th>Peak Acceleration [m/s²]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.35</td>
<td>0.09</td>
</tr>
<tr>
<td>2</td>
<td>0.67</td>
<td>0.22</td>
</tr>
<tr>
<td>3</td>
<td>1.19</td>
<td>0.09</td>
</tr>
<tr>
<td>4</td>
<td>0.87</td>
<td>0.07</td>
</tr>
<tr>
<td>5</td>
<td>0.81</td>
<td>0.05</td>
</tr>
<tr>
<td>6</td>
<td>1.63</td>
<td>0.02</td>
</tr>
<tr>
<td>7</td>
<td>0.80</td>
<td>0.05</td>
</tr>
<tr>
<td>8</td>
<td>1.25</td>
<td>0.06</td>
</tr>
<tr>
<td>9</td>
<td>2.42</td>
<td>0.04</td>
</tr>
<tr>
<td>10</td>
<td>0.86</td>
<td>0.08</td>
</tr>
<tr>
<td>11</td>
<td>1.75</td>
<td>0.03</td>
</tr>
<tr>
<td>12</td>
<td>0.63</td>
<td>0.15</td>
</tr>
<tr>
<td>13</td>
<td>0.43</td>
<td>0.17</td>
</tr>
<tr>
<td>14</td>
<td>2.27</td>
<td>0.02</td>
</tr>
<tr>
<td>15</td>
<td>1.12</td>
<td>0.06</td>
</tr>
<tr>
<td>16</td>
<td>0.40</td>
<td>0.12</td>
</tr>
</tbody>
</table>

In Figure 7.9 the comfort criteria as given in Figure 3.14 is recreated using the values given in Table 7.4. In accordance with the comfort criteria given in ISO 10137 it can be concluded that only four models of the building are habitable: model 6, model 9, model 11 and model 14.
Figure 7.9  Validation of all 16 models in accordance with the comfort criteria for residential buildings as given in ISO 10137
8 DISCUSSION

8.1 Interpretation of results

The statistical analysis of the response data shows that the number of floors of the building influences the frequency the most. Changing the height of the building is accountable for about 44% of the change in the frequency. The thickness of the wall elements of the building is also important in a dynamic analysis. Changing the thickness of the walls is accountable for a little over 36% of the change in the frequency of the building. The addition of mass through the additional concrete coatings on the floors of the building yields a change in the frequency of about 13%. Furthermore, according to this study and in line with the factor levels chosen the shape of the building does not have a significant effect on the frequency.

The analysis also shows that interaction is present. There are two interactions that stand out: the one between the number of floors and the thickness of the walls, and the one between the number of floors and the addition of mass. These are accountable for 4.3% and 1.5% of the change in the frequency of the building, respectively.

The main effect plot in Figure 7.4 shows that increasing the number of floors from 8 to 16 decreases the frequency of the building. In turn, the main effect plot shown in Figure 7.5 reveals that increasing the thickness of the wall elements from 60 mm to 300 mm leads to a higher frequency of the building. Consequently, the main effect plot in Figure 7.6 shows that adding mass to the building lowers the frequency.

It is already known that the taller the building, the lower the frequency. This is for example studied by Hesselmar (2016). Figure 7.4 confirms this. It is also widely known that if mass is added to a system, the natural frequency of that system decreases. This can be seen by studying equation (3.6). Figure 7.6 confirms this. It should be noted that the thickness of the walls affects both the mass and the stiffness of the building.

The interaction plot in Figure 7.7 shows the interaction between the number of floors and the thickness of the walls. This plot indicates that the effect of the number of floors indeed depends on the thickness of the walls. Consequently, the interaction plot in Figure 7.8 shows that the number of floors also depends on the additional mass. However, the two lines in Figure 7.8 is close to being parallel, which indicates only a small interaction effect between the factors in question.

Figure 7.7 shows that as the building gets taller (that is, as factor B increases from the low level to the high level) the first natural frequency decreases depending on how thick the walls are (that is, the level chosen for factor C). If the two lines in the plot are compared it can be concluded that when the walls are thick (when factor C is at the high level) the first natural frequency decreases more than when the walls are thin (when factor C is at the low level). At the same time the first natural frequency is higher overall for a building with thick walls, according to Figure 7.7. Of course, Figure 7.8 can be subject to the same kind of interpretation. This plot shows that when the additional mass is present the first natural frequency of the rising building decreases less than when the additional mass is absent.

Clearly, the interpretations above can only be validated within the limits of the chosen factor levels. Furthermore, they are valid when the other factors are being held constant and they can only be confirmed for the first natural frequency of the building.
In addition to this, it can be concluded that only four models satisfies the comfort criteria according to ISO 10137. In other words, there are twelve models that can not be deemed habitable according to this criteria and under the given circumstances. The four models that does make it have two things in common: they all have 8 floors and 300 mm walls. Consequently, the four models that has 16 floors and only 60 mm walls are the ones that are the least favourable when evaluating the habitability.

The four most favourable models also has the four highest values of the natural frequency as well as the four lowest values of the acceleration. For the four least favourable models the situation is reversed. As given by the appearance of Figure 7.9, the other eight models do seem to place themselves in the region right in between the models that form the extremes of this evaluation, which ought to be reasonable.

Alas, the relationship between the natural frequency and the acceleration is complex. But there are patterns: lower buildings tend to have higher natural frequencies and thus, lower magnitudes of acceleration and better chances of making the comfort criteria, while the situation is reversed for taller buildings. These patterns are confirmed by Hesselmar (2016).

Based on these patterns, a comparison of the two interaction plots in Figure 7.7 and Figure 7.8 shows that the combination of 16 floors and 300 mm walls is to be preferred, rather than the combination of 16 floors and the addition of mass. This conclusion is drawn from the fact that thicker walls yields higher values of the natural frequency of the building than the addition of mass does. Once again, this conclusion is valid under the given circumstances and in line with the factor levels chosen for the study.

Additionally, the comfort criteria shows that both the addition of mass and the use of thicker walls decreases the acceleration of the building.

The thickness of the walls affects both the mass and the stiffness of the building, compared to the concrete coatings that only affects the mass. This might suggest that the stiffness is more relevant than the mass for the building with 16 floors.

**8.2 Further limitations and errors**

The general idea at the beginning of this study was that the factorial design should include two response variables: the natural frequency and the acceleration of the building. However, this was soon determined to be too comprehensive to fit the scope of this thesis, mostly due to the complexity of the calculation of the acceleration (see section 3.3.2). Another reason for not including the acceleration as a response variable is the strong interplay between the acceleration and the natural frequency of the building. As the natural frequency already is included as a response variable this would have made it complex as well.

Since there can be no replication in an experiment like the one that has been presented in this thesis, no error can be estimated. In accordance with section 4.1 this implies certain limitations. As the factors are present at only two levels in the experiment it is assumed that the response variable is approximately linear between the levels. The phenomenon of curvature of the response variable can indeed be checked using statistical methods; unfortunately, they do require that the experiment is replicated.

Finite element modeling provides a powerful tool for testing a building like this. Yet, it does not allow the experimenter to validate the outcome of the experiment. One way of estimating the
error might be to alter the mesh of the building in order to study the change in the calculations and thus be able to validate the experiment.

The design factors of this experiment were chosen based on their influence on the natural frequency of the building. The statistical analysis of the response data does not recognize the shape of the building as influential when it comes to the natural frequency of the building. However, the general conclusion that the shape of the building does not influence the natural frequency might be erroneous, since there are many kinds of building shapes. The conclusion that can be drawn from this result is therefore highly influenced by the factor levels in question. Of course, this applies to all the factors.

By comparing the models and their respective values of the acceleration it can be noted that the building shape, as given by this study, do affect the acceleration of the building. Edskär and Lidelöw (2016) performs a parametric study where they vary the shape of the building in a different manner. They vary both the width and the depth of the building so that the footprint of the building at all times is quadratic. This would have been more convenient since tall buildings tend to have quadratic footprints. Since they also vary the height of the building, they manage to vary the slenderness of the building quite effectively in this way. Unfortunately, this was not done in this study. Nonetheless, they also show that the building footprint do affect the acceleration of the building, but not its natural frequency.

Edskär and Lidelöw (2016) also vary the number of floors of the building. They show that the comfort criteria are breached at twelve floors, which should motivate the use of different factor levels for the number of floors of the building than in this study. Also, this study shows that a building with eight floors are far from being problematic when it comes to vibration, at least according to the given comfort criteria, which further confirms this conclusion.

One might argue against the chosen lower level of the thickness of the walls. In practice, it might not be that realistic to use 60 mm load bearing walls to support even a medium-rise building. Therefore, the lower level of the thickness of the walls probably should have been determined by some sort of buckling check.

Damping is another important property in a dynamic analysis. It affects the natural frequency of the building only slightly, which is why it is excluded as a design factor in this study. The fact that the undamped natural frequency and the damped natural frequency of a building are very similar can be seen by studying equation (3.8). However, damping is still very important when studying the acceleration of a building and is thus prominent when evaluating the comfort criteria.
This thesis presents a way of combining a statistical method with a structural problem. This approach was taken in order to further analyze the dynamic properties of a high-rise timber building. By using the given methodology these dynamic properties were studied in depth and any potential interactions between them were elucidated.

The study shows that the first natural frequency of the building varies with the number of floors in the span 8 floors-16 floors. It also shows that the frequency varies with the thickness of the walls in the span 60 mm-300 mm. Furthermore, the study shows that the addition of mass through an extra concrete coating on each of the floors affects the frequency of the building. However, when varying the shape of the building between a square one and a rectangular one, the frequency does not vary.

The study also recognizes two quite significant interactions: the one between the number of floors and the thickness of the walls, and the one between the number of floors and the addition of mass.

Since taller buildings tend to have lower natural frequencies this study recognizes the solution where the natural frequency has a relatively high value for the building with 16 floors. According to this study and under the given circumstances it is advantageous to increase the thickness of the walls in the span 60 mm-300 mm instead of adding mass to the building through an extra concrete coating on each floor. Of course, this conclusion is based only on the difference in the natural frequency between the two alternatives. Such things as cost and workability also need to be considered when choosing between the two.

The study of interaction effects are important in order to correctly evaluate the experiment. The interaction effects of this experiment may not be that large, but they still provide useful information about the building.

A crucial step in a factorial experiment is the selection of the factor levels. The span in which the factors are varied determines the validity and the variety of the experiment. It would be interesting to examine an even taller building than in this study, as well as one where both the width and the depth of the footprint are varied.

Lastly, since the damping of the building affects the natural frequency only slightly it is excluded from the factorial design used in this study. It is, however highly significant for the acceleration-levels of the building, which should motivate the inclusion of the damping in the factorial design in future studies.
10 REFERENCES


Available at: http://www.svenskttra.se/siteassets/6-om-oss/publikationer/pdfer/limtrahandbok-del-2-svenska.pdf [Accessed 15 November 2016].


Frearson, A., 2015. Architects embrace "the beginning of the timber age". Dezeen magazine. 9 november. [Online]


International Organization for Standardization, 1984. Guidelines for the evaluation of the response of occupants of fixed structures, especially buildings and off-shore structures, to low-
frequency horizontal motion (0.063 to 1 Hz) (ISO 6897:1984). Switzerland: International Organization for Standardization.


APPENDIX A – VALIDATION OF FLOORS

All figures and equations are taken from SS-EN 1990, SS-EN 1995-1-1 and Limträhandboken, except where noted.

All calculations are valid for a 1 meter wide floor element

\( b_f := 1 \text{m} \)

**Dead loads**

The additional concrete coating on the floor

\[ \gamma_c := 25 \frac{\text{kN}}{\text{m}^3} \]  
\[ t_c := 0.08 \text{m} \]  
\[ g_{k,c} := \gamma_c \cdot t_c = 2 \frac{\text{kN}}{\text{m}^2} \]

dead load concrete coating

The self-weight of the CLT panel

\[ \gamma_{\text{CLT}} := 4 \frac{\text{kN}}{\text{m}^3} \]  
\[ t_{\text{CLT}} := 0.300 \text{m} \]  
\[ g_{k,\text{CLT}} := \gamma_{\text{CLT}} \cdot t_{\text{CLT}} = 1.2 \frac{\text{kN}}{\text{m}^2} \]

dead load CLT panel

Additional loads

\[ g_{k,\text{walls}} := 0.5 \frac{\text{kN}}{\text{m}^2} \]
\[ g_{k,\text{installations}} := 0.3 \frac{\text{kN}}{\text{m}^2} \]

dead load walls

dead load installations

**Imposed load**

\[ q_k := 2 \frac{\text{kN}}{\text{m}^2} \]

imposed load due to residents

**Load combinations**

The deflection of the floor when it consists only of a CLT panel will be checked using the weight of the additional concrete coating, the self-weight of the panel, the weight of the walls and the weight of installations as permanent loads. The imposed load from residents is also included.
The characteristic combination

\[ q_{d,ch,G} := (g_{k,c} + g_{k,CLT} + g_{k,walls} + g_{k,installations}) \cdot b_f = 4 \frac{kN}{m} \] (for the permanent loads)

\[ q_{d,ch,Q} := q_k \cdot b_f = 2 \frac{kN}{m} \] (for the variable load)

The frequent combination

\[ \psi_1 := 0.5 \] combination factor for variable actions

\[ q_{d,freq} := (g_{k,c} + g_{k,CLT} + g_{k,walls} + g_{k,installations} + \psi_1 \cdot q_k) \cdot b_f = 5 \frac{kN}{m} \]

The stiffness of the CLT panel

\[ l_f := 7m \] the floor span

The CLT panel consists of seven layers that needs to be considered when calculating the stiffness of the panel. Only the layers that lies in the longitudinal direction of the panel are considered when the panel is subject to bending around the stiff axis. Consequently, only the panels that lies in the transverse direction of the panel are considered when the panel is subject to bending around the weak axis.

Figure A-1  Cross-section of the CLT panel

- \( t_{L1} := 45mm \) the thickness of the first longitudinal layer of the panel
- \( t_{T1} := 40mm \) the thickness of the first transverse layer of the panel
- \( t_{L2} := 45mm \) the thickness of the second longitudinal layer of the panel
- \( t_{T2} := 40mm \) the thickness of the second transverse layer of the panel
- \( t_{L3} := 45mm \) the thickness of the third longitudinal layer of the panel
the thickness of the third transverse layer of the panel
the thickness of the fourth longitudinal layer of the panel
the area of the first longitudinal layer of the panel
the area of the first transverse layer of the panel
the area of the second longitudinal layer of the panel
the area of the second transverse layer of the panel
the area of the third longitudinal layer of the panel
the area of the third transverse layer of the panel
the area of the fourth longitudinal layer of the panel

The layers consist of construction wood of different quality

the modulus of elasticity of the first longitudinal layer of the panel, C24
the modulus of elasticity of the first transverse layer of the panel, C14
the modulus of elasticity of the second longitudinal layer of the panel, C24
the modulus of elasticity of the second transverse layer of the panel, C14
the modulus of elasticity of the third longitudinal layer of the panel, C24
the modulus of elasticity of the third transverse layer of the panel, C14
the modulus of elasticity of the fourth longitudinal layer of the panel, C24

the moment of inertia of the first longitudinal layer of the panel
the moment of inertia of the first transverse layer of the panel
the moment of inertia of the second longitudinal layer of the panel

A - 3
\[ I_{T2} := \frac{b \cdot t_{T2}^3}{12} = 5.333 \times 10^{-6} \text{ m}^4 \]

\[ I_{L3} := \frac{b \cdot t_{L3}^3}{12} = 7.594 \times 10^{-6} \text{ m}^4 \]

\[ I_{T3} := \frac{b \cdot t_{T3}^3}{12} = 5.333 \times 10^{-6} \text{ m}^4 \]

\[ I_{L4} := \frac{b \cdot t_{L4}^3}{12} = 7.594 \times 10^{-6} \text{ m}^4 \]

The moment of inertia of the second transverse layer of the panel.

The moment of inertia of the third longitudinal layer of the panel.

The moment of inertia of the third transverse layer of the panel.

The moment of inertia of the fourth longitudinal layer of the panel.

\[ z_{\text{CLT,y}} = \frac{E_{L1} \cdot A_{L1} \left( I_{\text{CLT}} - \frac{t_{L1}}{2} \right) + E_{L2} \cdot A_{L2} \left( I_{\text{CLT}} - t_{L1} - t_{T1} - \frac{t_{L2}}{2} \right)}{E_{L1} \cdot A_{L1} + E_{L2} \cdot A_{L2} + E_{L3} \cdot A_{L3} + E_{L4} \cdot A_{L4}} \quad \text{...} = 0.15 \text{ m} \]

\[ z_{\text{CLT,y}} \] is the distance in the \( z \)-direction to the center of gravity for bending around the stiff axis of the panel.

\[ D_{E_{\text{CLT,y}}} := E_{L1} \left[ I_{L1} + A_{L1} \left( I_{\text{CLT}} - \frac{t_{L1}}{2} - z_{\text{CLT,y}} \right) \right]^2 \quad \text{...} = 1.822 \times 10^4 \text{ kN m}^2 \]

\[ + E_{L2} \left[ I_{L2} + A_{L2} \left( I_{\text{CLT}} - t_{L1} - t_{T1} - \frac{t_{L2}}{2} - z_{\text{CLT,y}} \right) \right]^2 \]

\[ + E_{L3} \left[ I_{L3} + A_{L3} \left( I_{L4} + t_{T3} + \frac{t_{L3}}{2} - z_{\text{CLT,y}} \right) \right]^2 \]

\[ + E_{L4} \left[ I_{L4} + A_{L4} \left( \frac{t_{L4}}{2} - z_{\text{CLT,y}} \right) \right]^2 \]

\( D_{E_{\text{CLT,y}}} \) is the stiffness against bending around the stiff axis of the panel.

\[ z_{\text{CLT,x}} = \frac{E_{T1} \cdot A_{T1} \left( I_{\text{CLT}} - t_{L1} - \frac{t_{T1}}{2} \right) + E_{T2} \cdot A_{T2} \left( I_{T4} + t_{T3} + t_{L3} + \frac{t_{T2}}{2} \right)}{E_{T1} \cdot A_{T1} + E_{T2} \cdot A_{T2} + E_{T3} \cdot A_{T3}} \quad \text{...} = 0.15 \text{ m} \]

\[ z_{\text{CLT,x}} \] is the distance in the \( z \)-direction to the center of gravity for bending around the weak axis of the panel.

A-4
\[ D_{EI,CLT,x} = E_T \left[ I_{T1} + A_{T1} \left( t_{CLT} - \frac{t_{T1}}{2} - z_{CLT,x} \right) \right]^2 + E_{T2} \left[ I_{T2} + A_{T2} \left( t_{L4} + t_{T3} + \frac{t_{T2}}{2} - z_{CLT,x} \right) \right]^2 + E_{T3} \left[ I_{T3} + A_{T3} \left( t_{L4} + \frac{t_{T3}}{2} - z_{CLT,x} \right) \right]^2 \]

\[ D_{EI,CLT,x} = 4.158 \times 10^3 \text{kN}\cdot\text{m}^2 \]

\( D_{EI,CLT,x} \) is the stiffness against bending around the weak axis of the panel

**The instantaneous deflection**

\[ w_{inst,G} = \frac{5q_{d,ch.G} l_f^4}{384D_{EI,CLT,y}} = 6.865 \times 10^{-3} \text{ m} \]

the instantaneous deflection due to the permanent loads

\[ w_{inst,Q} = \frac{5q_{d,ch.Q} l_f^4}{384D_{EI,CLT,y}} = 3.432 \times 10^{-3} \text{ m} \]

the instantaneous deflection due to the variable load

\[ w_{inst} := w_{inst,G} + w_{inst,Q} = 0.01 \text{ m} \]

the instantaneous deflection due to both permanent and variable loads

**The creep deflection**

\[ k_{def} := 0.6 \]

the deformation factor for solid timber and service class 1

\[ \psi_2 := 0.3 \]

combination factor for variable actions

\[ w_{creep,G} := k_{def} w_{inst,G} = 4.119 \times 10^{-3} \text{ m} \]

the creep deflection due to the permanent loads

\[ w_{creep,Q} := \psi_2 k_{def} w_{inst,Q} = 6.178 \times 10^{-4} \text{ m} \]

the creep deflection due to the variable load

**The final deflection**

\[ w_{fin,G} := w_{inst,G} + w_{creep,G} = 0.011 \text{ m} \]

the final deflection due to the permanent loads

\[ w_{fin,Q} := w_{inst,Q} + w_{creep,Q} = 4.05 \times 10^{-3} \text{ m} \]

the final deflection due to the variable load

\[ w_{fin} := w_{fin,G} + w_{fin,Q} = 0.015 \text{ m} \]

the final deflection due to both permanent and variable loads

**The frequent deflection**

\[ w_{freq} = \frac{5q_{d,freq} l_f^4}{384D_{EI,CLT,y}} = 8.581 \times 10^{-3} \text{ m} \]

the deflection due to frequent loading, i.e. people walking on the floor
Limiting values for deflections

\[ w_{\text{inst}} = \frac{1}{679} l_f \]

\[ w_{\text{fin}} = \frac{1}{465} l_f \sqrt{\frac{D_{\text{EI,CLT,}y}}{m_{\text{CLT}}}} \]

\[ w_{\text{freq}} = \frac{1}{815} l_f \sqrt{\frac{D_{\text{EI,CLT,}y}}{m_{\text{CLT}}}} \]

<table>
<thead>
<tr>
<th>Floors</th>
<th>( w_{\text{max,inst}} )</th>
<th>( w_{\text{max,fin}} )</th>
<th>( w_{\text{max,freq}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>L/500</td>
<td>L/300</td>
<td>L/375</td>
<td></td>
</tr>
</tbody>
</table>

Figure A-2  Limiting values for deflections

The fundamental frequency of the CLT panel

The fundamental frequency of the CLT panel will be checked when there is no additional concrete coating present. The only mass regarded here is the self-weight of the CLT panel.

\[ m_{\text{CLT}} := \frac{g_{\text{k,CLT}}}{g} = 122.366 \frac{\text{kg}}{\text{m}^2} \]

the mass of timber per unit area

\[ f_{1,\text{CLT}} := \frac{\pi}{2 l_f^2} \sqrt{\frac{D_{\text{EI,CLT,}y}}{m_{\text{CLT}}}} l_m = 12.369 \text{ Hz} \]

the fundamental frequency of the floor

The fundamental frequency of a floor should be higher than 8 Hz.

The unit impulse velocity response

\[ n_{40,\text{CLT}} := \left[ \left( \frac{40 \text{ Hz}}{f_{1,\text{CLT}}} \right)^2 - 1 \right] \left( \frac{b_l}{l_f} \right)^4 \frac{D_{\text{EI,CLT,}y}}{D_{\text{EI,CLT,x}}} \right]^{-0.25} \]

the number of first-order modes with natural frequencies up to 40 Hz

\[ v_{\text{CLT}} := \left[ \frac{4 \left( 0.4 + 0.6 n_{40,\text{CLT}} \right)}{m_{\text{CLT}} b_l l_f^2 + 200} \right] \left( \frac{m}{\text{N} \cdot \text{s}^2} \right) = 2.338 \times 10^{-3} \frac{m}{\text{N} \cdot \text{s}^2} \]

the unit impulse velocity response of the floor

The vertical deflection of the CLT panel

\[ F_{\text{unit}} := 1 \text{kN} \]

a unit point load applied at any point on the floor

\[ w_{\text{CLT}} := \frac{F_{\text{unit}} l_f^3}{48 D_{\text{EI,CLT,}y}} = 3.923 \times 10^{-4} \text{ m} \]

the maximum instantaneous vertical deflection of the floor caused by a vertical concentrated static force
Requirements for residential floors with respect to vibrations

For residential floors with a fundamental frequency greater than 8 Hz the following requirements should be satisfied

\[ \frac{w}{F_{\text{unit}}} \leq a \]

and

\[ v \leq b \cdot \zeta - 1 \]

where

\[ a := 1.5 \text{ mm kN} \]

\[ b := 100 \frac{\text{m}}{\text{Ns}^2} \]

(values for a and b are taken from EKS 10)

The unit impulse velocity response as calculated above

\[ v_{\text{CLT}} = 0.00234 \frac{m}{\text{Ns}^2} \]

Limiting values for vibrations

\[ a_{\text{CLT}} := \frac{w_{\text{CLT}}}{F_{\text{unit}}} = 0.392 \frac{\text{mm}}{\text{kN}} \]

\[ \zeta_t := 0.01 \]

the modal damping ratio for timber floors

\[ v_{\text{CLT, max}} = b \cdot \frac{f_{1, \text{CLT}}}{\text{Hz}} \cdot \zeta_t^{-1} \cdot \frac{m}{\text{Ns}^2} = 0.018 \frac{m}{\text{Ns}^2} \]

\[ a_{\text{CLT}} \leq a = 1 \]

\[ v_{\text{CLT}} \leq v_{\text{CLT, max}} = 1 \]

Thus, the requirements for the CLT panel are satisfied.
The fundamental frequency of the composite structure

The fundamental frequency when the CLT panel and the concrete coating acts as a composite structure will be checked as well. The only masses regarded here are the self-weight of the CLT panel and the self-weight of the concrete. The CLT panel and the concrete coating are assumed to fully function as a composite.

\[ E_c := 31000 \text{MPa} \]

\[ A_c := b_f \cdot t_c = 0.08 \text{m}^2 \]

\[ I_c := \frac{b_f \cdot t_c^3}{12} = 4.267 \times 10^{-5} \text{ m}^4 \]

The modulus of elasticity for the concrete, C25/30

\[ \text{the area of the concrete} \]

\[ \text{the moment of inertia for the concrete, with no reinforcement and under the assumption of no cracks} \]

\[
\chi_{\text{comp.y}} = \frac{E_{L1} \cdot A_{L1} \left( t_{CLT} - \frac{t_{L1}}{2} \right) + E_{L2} \cdot A_{L2} \left( t_{CLT} - t_{L1} - \frac{t_{L2}}{2} \right)}{E_{L1} \cdot A_{L1} + E_{L2} \cdot A_{L2} + E_{L3} \cdot A_{L3} + E_{L4} \cdot A_{L4} + E_c \cdot A_c} \]

\[
+ \frac{E_{L3} \cdot A_{L3} \left( t_{L4} + t_{T3} + \frac{t_{L3}}{2} \right) + E_{L4} \cdot A_{L4} \frac{t_{L4}}{2} + E_c \cdot A_c \left( t_{CLT} + \frac{t_c}{2} \right)}{E_{L1} \cdot A_{L1} + E_{L2} \cdot A_{L2} + E_{L3} \cdot A_{L3} + E_{L4} \cdot A_{L4} + E_c \cdot A_c} \]

\[ \chi_{\text{comp.y}} \] is the distance in the z-direction to the center of gravity of the composite structure for bending around the stiff axis.

Figure A-3  Cross-section of the composite structure
\[ D_{\text{EI,comp.y}} := E_L \left[ l_{L1} + A_{L1} \left( t_{\text{CLT}} - \frac{t_{L1}}{2} - z_{\text{comp.y}} \right)^2 \right] + E_{L2} \left[ l_{L2} + A_{L2} \left( t_{\text{CLT}} - t_{L1} - \frac{t_{T1}}{2} - z_{\text{comp.y}} \right)^2 \right] + E_{L3} \left[ l_{L3} + A_{L3} \left( t_{L4} + t_{T3} + \frac{t_{L3}}{2} - z_{\text{comp.y}} \right)^2 \right] + E_{L4} \left[ l_{L4} + A_{L4} \left( \frac{t_{L4}}{2} - z_{\text{comp.y}} \right)^2 \right] + E_c \left[ l_c + A_c \left( t_{\text{CLT}} + \frac{t_c}{2} - z_{\text{comp.y}} \right)^2 \right] \]

\[ D_{\text{EI,comp.y}} = 5.928 \times 10^4 \text{kN} \cdot \text{m}^2 \]

\[ m_c := \frac{g_{k.c}}{g} = \frac{203.943}{\text{kg}} \text{m}^2 \]

\[ m_{\text{comp}} := m_{\text{CLT}} + m_c = \frac{326.309}{\text{kg}} \text{m}^2 \]

\[ f_{1,\text{comp}} := \frac{\pi}{21f} \sqrt{\frac{D_{\text{EI,comp.y}}}{m_{\text{comp}}}} = 13.664 \text{ Hz} \]

The fundamental frequency of a floor should be higher than 8 Hz.

The unit impulse velocity response

\[ z_{\text{comp.x}} := \frac{E_{T1} \cdot A_{T1} \left( t_{\text{CLT}} - t_{L1} - \frac{t_{T1}}{2} \right) + E_{T2} \cdot A_{T2} \left( t_{L4} + t_{T3} + \frac{t_{L3}}{2} + \frac{t_{T2}}{2} \right)}{E_{T1} \cdot A_{T1} + E_{T2} \cdot A_{T2} + E_{T3} \cdot A_{T3} + E_c \cdot A_c} \]

\[ z_{\text{comp.x}} = 0.292 \text{ m} \]

\[ z_{\text{comp.y}} = \text{the distance in the z-direction to the center of gravity of the composite structure for bending around the weak axis} \]
\[ D_{\text{EI.comp.x}} := E_{T1} \left[ t_{T1} + A_{T1} \left( t_{\text{CLT}} - t_{L1} + \frac{t_{T1}}{2} - z_{\text{comp.x}} \right) \right]^2 \ldots = 2.813 \times 10^4 \text{kN} \cdot \text{m}^2 \]

\[ + E_{T2} \left[ t_{T2} + A_{T2} \left( t_{L4} + t_{T3} + t_{L3} + \frac{t_{T2}}{2} - z_{\text{comp.x}} \right) \right]^2 \ldots \]

\[ + E_{T3} \left[ t_{T3} + A_{T3} \left( t_{L4} + \frac{t_{T3}}{2} - z_{\text{comp.x}} \right) \right]^2 \ldots \]

\[ + F_c \left[ I_c + A_c \left( t_{\text{CLT}} + \frac{t_c}{2} - z_{\text{comp.x}} \right) \right] \]

\( D_{\text{EI.comp.x}} \) is the stiffness against bending around the weak axis of the composite structure

\[ n_{40,\text{comp}} := \left[ \left( \frac{\text{40Hz}}{t_{\text{comp}}} \right)^2 - 1 \right] \left( \frac{b_f}{l_f} \right)^4 \left( \frac{D_{\text{EI.comp.y}}}{D_{\text{EI.comp.x}}} \right)^{0.25} = 0.285 \]

the number of first-order modes with natural frequencies up to 40 Hz

\[ v_{\text{comp}} := \left[ \frac{4 \left( 0.4 + 0.6n_{40,\text{comp}} \right)}{n_{\text{comp},b_f}l_f + 200} \right] \left( \frac{m}{\text{N} \cdot \text{s}^2} \right) = 9.199 \times 10^{-4} \left( \frac{m}{\text{N} \cdot \text{s}^2} \right) \]

the unit impulse velocity response of the floor

The vertical deflection of the composite structure

\[ F_{\text{unit}} = 1 \text{kN} \]

a unit point load applied at any point on the floor

\[ w_{\text{comp}} := \frac{F_{\text{unit}} l_f^3}{48 D_{\text{EI.comp.y}}} = 1.205 \times 10^{-4} \text{m} \]

the maximum instantaneous vertical deflection of the floor caused by a vertical concentrated static force

Requirements for residential floors with respect to vibrations

For residential floors with a fundamental frequency greater than 8 Hz the following requirements should be satisfied

\[ \frac{w}{F_{\text{unit}}} \leq a \]

and

\[ v \leq b f_1 \zeta^{-1} \]

where

\[ a = 1.5 \left( \frac{\text{mm}}{\text{kN}} \right) \]

\[ b = 100 \left( \frac{\text{m}}{\text{N} \cdot \text{s}^2} \right) \]

(values for a and b are taken from EKS 10)
The unit impulse velocity response as calculated above

\[ \nu_{\text{comp}} = 0.00092 \frac{m}{N \cdot s^2} \]

Limiting values for vibrations

\[ a_{\text{comp}} = \frac{w_{\text{comp}}}{F_{\text{unit}}} = 0.121 \frac{\text{mm}}{\text{kN}} \]

\[ \zeta_{t.c} := 0.02 \]

Thus, the requirements for the composite structure are satisfied.
APPENDIX B – PICTURES OF MODELS

Figure B-1  Model 1

Figure B-2  Model 1, vibration shape 1
Figure B-3  Model 2

Figure B-4  Model 2, vibration shape 1
Figure B-5  Model 3

Figure B-6  Model 3, vibration shape 1
Figure B-9  Model 5

Figure B-10  Model 5, vibration shape 1
Figure B-11  Model 6

Figure B-12  Model 6, vibration shape 1
Figure B-13  Model 7

Figure B-14  Model 7, vibration shape 1
Figure B-15  Model 8

Figure B-16  Model 8, vibration shape 1
Figure B-17  Model 9

Figure B-18  Model 9, vibration shape 1
Figure B-19  Model 10

Figure B-20  Model 10, vibration shape 1
Figure B-21  Model 11

Figure B-22  Model 11, vibration shape 1
Figure B-23  Model 12

Figure B-24  Model 12, vibration shape 1
Figure B-25  Model 13

Figure B-26  Model 13, vibration shape 1
Figure B-27  Model 14

Figure B-28  Model 14, vibration shape 1
Figure B-29  Model 15

Figure B-30  Model 15, vibration shape 1
Figure B-31  Model 16

Figure B-32  Model 16, vibration shape 1
| Model Number | Shape of Building | Number of Floors | Thickness Walls [mm] | Additional Mass | Specifications | Geomerty | Mass | Additional Mass | Elecnfreuqencies | Vibration Shape |
|--------------|------------------|------------------|----------------------|----------------|---------------|-----------|------|----------------|----------------|----------------|---------|
| 1            | square           | 6                | 60                   | off            |               | 24        | 20   | 20             | 2.55           | 100            | 2.55    |
|              |                  |                  |                      |                |               | 1.055     | 100  | 450            | 1.95           | 0.643          | 0.8055  |
| 2            | square           | 15               | 60                   | off            |               | 48        | 20   | 20             | 2.55           | 100            | 2.55    |
|              |                  |                  |                      |                |               | 1.055     | 100  | 163.1          | 1.961          | 0.643          | 0.8055  |
| 3            | square           | 15               | 300                  | off            |               | 48        | 20   | 20             | 2.55           | 100            | 2.55    |
|              |                  |                  |                      |                |               | 1.055     | 100  | 183.8          | 1.961          | 0.643          | 0.8055  |
| 4            | square           | 15               | 300                  | on             |               | 48        | 20   | 20             | 2.55           | 100            | 2.55    |
|              |                  |                  |                      |                |               | 1.055     | 100  | 162.8          | 1.961          | 0.643          | 0.8055  |
| 5            | rectangular      | 15               | 300                  | on             |               | 48        | 40   | 20             | 2.55           | 200            | 2.55    |
|              |                  |                  |                      |                |               | 1.055     | 200  | 235.4          | 1.961          | 0.643          | 0.8055  |
| 6            | rectangular      | 8                | 300                  | on             |               | 24        | 40   | 20             | 2.55           | 200            | 2.55    |
|              |                  |                  |                      |                |               | 1.055     | 200  | 147.1          | 1.961          | 0.643          | 0.8055  |
| 7            | rectangular      | 8                | 300                  | on             |               | 24        | 40   | 20             | 2.55           | 200            | 2.55    |
|              |                  |                  |                      |                |               | 1.055     | 200  | 306.5          | 1.961          | 0.643          | 0.8055  |
| 8            | rectangular      | 6                | 300                  | off            |               | 24        | 40   | 20             | 2.55           | 200            | 2.55    |
|              |                  |                  |                      |                |               | 1.055     | 200  | 306.5          | 1.961          | 0.643          | 0.8055  |
| 9            | square           | 6                | 300                  | off            |               | 24        | 20   | 20             | 2.55           | 100            | 264.2   |
|              |                  |                  |                      |                |               | 1.055     | 100  | 11.07          | 0.546          | 0.546          | 0.546   |
| 10           | square           | 6                | 300                  | on             |               | 24        | 20   | 20             | 2.55           | 100            | 264.2   |
|              |                  |                  |                      |                |               | 1.055     | 100  | 332.0          | 1.961          | 0.643          | 0.8055  |
| 11           | square           | 6                | 300                  | on             |               | 24        | 20   | 20             | 2.55           | 100            | 764.2   |
|              |                  |                  |                      |                |               | 1.055     | 100  | 1.17           | 0.546          | 0.546          | 0.546   |
| 12           | rectangular      | 15               | 60                   | off            |               | 48        | 40   | 20             | 2.55           | 200            | 1.977   |
|              |                  |                  |                      |                |               | 1.055     | 200  | 38.9           | 1.961          | 0.643          | 0.8055  |
| 13           | square           | 15               | 60                   | off            |               | 48        | 20   | 20             | 2.55           | 100            | 320.1   |
|              |                  |                  |                      |                |               | 1.055     | 100  | 183.8          | 1.961          | 0.643          | 0.8055  |
| 14           | rectangular      | 6                | 300                  | off            |               | 24        | 40   | 20             | 2.55           | 200            | 1.977   |
|              |                  |                  |                      |                |               | 1.055     | 200  | 117.1          | 0.943          | 0.943          | 0.943   |
| 15           | rectangular      | 15               | 300                  | on             |               | 48        | 40   | 20             | 2.55           | 200            | 254.1   |
|              |                  |                  |                      |                |               | 1.055     | 200  | 143.0          | 1.961          | 0.643          | 0.8055  |
| 16           | rectangular      | 15               | 60                   | on             |               | 48        | 40   | 20             | 2.55           | 200            | 310.0   |
|              |                  |                  |                      |                |               | 1.055     | 200  | 1.42           | 0.546          | 0.546          | 0.546   |

<table>
<thead>
<tr>
<th>Vibration Shape</th>
<th>n₁</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>3</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX D – DYNAMIC ANALYSIS

All figures and equations are taken from SS-EN 1991-1-4 and EKS 10, except where noted. The calculations are done for all 16 models of the building.

Properties of the building

The properties of the building are taken from the FEM-models

The height of the building ($h_{\text{building}}$) in meters:

\[
\begin{array}{ccccccc}
24 & 48 & 48 & 48 & 48 & 24 & 24 \\
24 & 24 & 48 & 48 & 48 & 48 & 48 \\
\end{array}
\]

The width of the building ($b_{\text{building}}$) in meters:

\[
\begin{array}{ccccccc}
20 & 20 & 20 & 20 & 40 & 40 & 40 \\
20 & 20 & 20 & 20 & 40 & 40 & 40 \\
\end{array}
\]

The depth of the building ($d_{\text{building}}$) in meters:

\[
\begin{array}{ccccccc}
20 & 20 & 20 & 20 & 20 & 20 & 20 \\
20 & 20 & 20 & 20 & 20 & 20 & 20 \\
\end{array}
\]

The mass of the building ($m_{\text{building}}$) in tonne:

\[
\begin{array}{cccc}
715.03 \\
1175.06 \\
1783.32 \\
3088.92 \\
6062.64 \\
3286.32 \\
2724.14 \\
1418.54 \\
1019.16 \\
1367.83 \\
1671.96 \\
2327.09 \\
2480.66 \\
1980.72 \\
3451.44 \\
4938.29 \\
\end{array}
\]
\[
\begin{pmatrix}
1.352 \\
0.673 \\
1.188 \\
0.866 \\
0.809 \\
1.630 \\
0.799 \\
1.254 \\
2.421 \\
0.864 \\
1.750 \\
0.626 \\
0.431 \\
2.274 \\
1.120 \\
0.400 \\
\end{pmatrix}
\text{Hz}
\]

\(v_{b,0} := 25 \frac{m}{s}\)

The first natural frequency of the building

Wind velocity

\(c_{\text{dir}} := 1\)  
\(c_{\text{season}} := 1\)

(recommended value)

\(v_{b} := c_{\text{dir}} \cdot c_{\text{season}} \cdot v_{b,0} = 25 \frac{m}{s}\)

the fundamental value of the basic wind velocity in Gothenburg

The basic wind velocity

The wind velocity for a 5 year return period in Gothenburg

\(T_a := 5\text{ years}\)

\[v_{T_a.5} := 0.75 \cdot v_b \cdot \sqrt{1 - 0.2 \cdot \ln\left(-\ln\left(1 - \frac{1}{T_a}\right)\right)} = 21.378 \frac{m}{s}\]

The wind velocity for a 1 year return period in Gothenburg

\[v_{T_a.1} := 0.72 \cdot v_{T_a.5} = 15.392 \frac{m}{s}\]

Terrain roughness

\(z_{\text{min}} := 5\text{ m}\)  
\(z_{\text{max}} := 200\text{ m}\)

(for terrain category III)

\(z := h_{\text{building}}\)

(the height \(z\) equals the height of the building)
\[ z_0 = 0.3 \text{m} \quad \text{(for terrain category III)} \]
\[ z_{0, \text{II}} = 0.05 \text{m} \quad \text{(for terrain category II)} \]

\[ k_r = 0.19 \left( \frac{z_0}{z_{0, \text{II}}} \right)^{0.07} = 0.215 \]

The terrain factor

The roughness factor at height \( z \)

\[
c_r := k_r \ln \left( \frac{z}{z_0} \right)
\]

\[
\begin{pmatrix}
0.944 \\
1.093 \\
1.093 \\
1.093 \\
1.093 \\
0.944 \\
0.944 \\
0.944 \\
0.944 \\
1.093 \\
0.944 \\
1.093 \\
0.944 \\
1.093
\end{pmatrix}
\]

Mean wind velocity

The orography factor

\[ c_o := 1 \quad \text{(the terrain can be regarded as flat)} \]
The mean wind velocity at height \( z \) above the terrain

\[
\]

Wind turbulence

The turbulence factor

\( k_l = 1 \) \hspace{1cm} \text{(recommended value)}

The standard deviation of the turbulent component of wind velocity

\[
\sigma_v := k_r v_{Ta.1} k_l = 3.315 \text{ m/s}
\]

The turbulence intensity at height \( z \)

\[
I_v := \frac{\sigma_v}{v_m} = \begin{pmatrix} 0.228 \\ 0.197 \\ 0.197 \\ 0.197 \\ 0.228 \\ 0.228 \\ 0.228 \\ 0.228 \\ 0.228 \\ 0.197 \\ 0.197 \\ 0.197 \\ 0.197 \\ 0.228 \\ 0.197 \\ 0.197 \\ 0.197 \end{pmatrix}
\]
Peak velocity pressure

\[ \rho_{\text{air}} = 1.25 \text{ kg/m}^3 \quad \text{the air density} \]

The basic velocity pressure

\[ q_b := \frac{1}{2} \rho_{\text{air}} \cdot v \cdot Ta.1^2 = 148.077 \text{ Pa} \]

The exposure factor at height \( z \)

\[
\begin{pmatrix}
2.10 \\
2.65 \\
2.65 \\
2.65 \\
2.65 \\
2.10 \\
2.10 \\
2.10 \\
2.10 \\
2.10 \\
2.10 \\
2.10 \\
2.65 \\
2.65 \\
2.10 \\
2.65 \\
2.65 \\
2.65 \\
2.65 \\
2.65 \\
2.65 \\
2.65 \\
2.65 \\
2.65
\end{pmatrix}
\]

(taken from Figure C-5 in EKS 10)

The peak velocity pressure at height \( z \)

\[
q_p := c_e \cdot q_b = \begin{pmatrix}
310.961 \\
392.403 \\
392.403 \\
392.403 \\
392.403 \\
310.961 \\
310.961 \\
310.961 \\
310.961 \\
392.403 \\
392.403 \\
392.403 \\
310.961 \\
392.403 \\
392.403 \\
310.961 \\
392.403 \\
392.403 \\
392.403 \\
392.403 \\
392.403 \\
392.403 \\
392.403
\end{pmatrix} \text{ Pa}
\]
Logarithmic decrement of damping

\[ d_{f,0} \geq b_{\text{building}} \]

\[ b_{f,0} \geq d_{\text{building}} \]

The force coefficient of structural elements without free-end flow

\[
\begin{array}{c}
2.1 \\
2.1 \\
2.1 \\
1.65 \\
1.65 \\
1.65 \\
1.65 \\
2.1 \\
2.1 \\
2.1 \\
2.1 \\
1.65 \\
2.1 \\
1.65 \\
1.65 \\
1.65 \\
1.65 \\
\end{array}
\]

\[ c_{f,0} \geq \] (taken from Figure 7.23 in Eurocode)

For the effective slenderness

\[ b_{\lambda} := h_{\text{building}} \]

\[ b_{\lambda} := d_{\text{building}} \] (the measurements of the building for the effective slenderness are given in Table 7.16 in Eurocode)

Linear interpolation according to Table 7.16 in Eurocode

\[ \lambda_{\text{high}} = \min \left( 1.4 \frac{b_{\lambda}}{b_{\lambda}}, 70 \right) = 1.68 \]

\[ \lambda_{\text{low}} = \min \left( 2 \frac{b_{\lambda}}{b_{\lambda}}, 70 \right) = 2.4 \]
\[ \lambda := \lambda_{\text{low}} + (\lambda_{\text{high}} - \lambda_{\text{low}}) \frac{(b_\lambda - 14.99m)}{(50m - 14.99m)} = \begin{pmatrix} 2.215 \\ 1.721 \\ 1.721 \\ 1.721 \\ 1.721 \\ 2.215 \\ 2.215 \\ 2.215 \\ 2.215 \\ 2.215 \\ 1.721 \\ 1.721 \\ 1.721 \\ 2.215 \\ 2.215 \\ 2.215 \\ 1.721 \\ 1.721 \end{pmatrix} \]  

the effective slenderness

The solidity ratio

\[ \varphi := \begin{pmatrix} 1 \end{pmatrix} \]  

(solid members)

The end-effect factor

\[ \psi_\lambda := \begin{pmatrix} 0.63 \\ 0.62 \\ 0.62 \\ 0.62 \\ 0.62 \\ 0.62 \\ 0.63 \\ 0.63 \\ 0.63 \\ 0.63 \\ 0.63 \\ 0.63 \\ 0.62 \\ 0.62 \\ 0.63 \\ 0.63 \\ 0.62 \\ 0.62 \\ 0.62 \\ 0.62 \end{pmatrix} \]  

(taken from Figure 7.36 in Eurocode)
The force coefficient of structural elements with sharp edged section

\[
\begin{pmatrix}
1.323 \\
1.302 \\
1.302 \\
1.302 \\
1.302 \\
1.023 \\
1.039 \\
1.039 \\
1.039 \\
1.323 \\
1.323 \\
1.323 \\
1.023 \\
1.302 \\
1.039 \\
1.023 \\
1.023
\end{pmatrix}
\]

\[
f_c := (e_0 \cdot \psi_\lambda) = \begin{pmatrix}
1.323 \\
1.302 \\
1.302 \\
1.302 \\
1.302 \\
1.023 \\
1.039 \\
1.039 \\
1.039 \\
1.323 \\
1.323 \\
1.323 \\
1.023 \\
1.302 \\
1.039 \\
1.023 \\
1.023
\end{pmatrix}
\]

The logarithmic decrement of structural damping

\[\delta_s := 0.015\] [according to Utne (2012)]

The mean wind velocity at height \(z_s\) above the terrain

\[z_s := 0.6 \cdot h_{\text{building}} = \begin{pmatrix}
14.4 \\
28.8 \\
28.8 \\
28.8 \\
28.8 \\
14.4 \\
14.4 \\
14.4 \\
14.4 \\
28.8 \\
28.8 \\
28.8 \\
14.4 \\
28.8 \\
28.8 \\
28.8
\end{pmatrix}\] m (taken from Figure 6.1a in Eurocode)
The equivalent mass per unit area of the building

$$c_{r, zs} := k_r \ln \left( \frac{z}{z_0} \right) = \begin{pmatrix} 0.834 \\ 0.983 \\ 0.983 \\ 0.983 \\ 0.983 \\ 0.834 \\ 0.834 \\ 0.834 \\ 0.834 \\ 0.834 \\ 0.983 \\ 0.983 \\ 0.983 \\ 0.983 \\ 0.983 \end{pmatrix}$$

$$v_{m, zs} := c_{r, zs} \cdot c_0 \cdot v_{T_a, 1} = \begin{pmatrix} 12.834 \\ 15.132 \\ 15.132 \\ 15.132 \\ 15.132 \\ 12.834 \\ 12.834 \\ 12.834 \\ 12.834 \\ 15.132 \\ 15.132 \\ 15.132 \\ 12.834 \\ 15.132 \\ 15.132 \end{pmatrix} \text{ m/s}$$

$$\mu_c = \left( \frac{m_{\text{building}}}{b_{\text{building}} \cdot d_{\text{building}}} \right) = \begin{pmatrix} 1.773 \times 10^3 \\ 2.548 \times 10^3 \\ 3.42 \times 10^3 \\ 4.18 \times 10^3 \\ 2.909 \times 10^3 \\ 4.18 \times 10^3 \\ 6.20 \times 10^3 \\ 2.476 \times 10^3 \\ 4.314 \times 10^3 \\ 6.173 \times 10^3 \end{pmatrix} \text{ kg/m}^2$$
The logarithmic decrement of aerodynamic damping

\[ \delta_a = \left( \frac{c_f \rho_{\text{air}} v_{m,z}}{2 a_1 \mu_c} \right) = \begin{bmatrix} 4.391 \times 10^{-3} \\ 6.228 \times 10^{-3} \\ 2.325 \times 10^{-3} \\ 1.841 \times 10^{-3} \\ 1.578 \times 10^{-3} \\ 1.245 \times 10^{-3} \\ 3.065 \times 10^{-3} \\ 3.75 \times 10^{-3} \\ 1.72 \times 10^{-3} \\ 3.592 \times 10^{-3} \\ 1.451 \times 10^{-3} \\ 5.313 \times 10^{-3} \\ 4.607 \times 10^{-3} \\ 1.481 \times 10^{-3} \\ 2.002 \times 10^{-3} \\ 3.918 \times 10^{-3} \end{bmatrix} \]

Structural factors

\[ B := \sqrt{\exp \left[ -0.05 \left( \frac{z}{z_s} \right) + \left( 1 - \frac{b_{\text{building}}}{z} \right) \left[ 0.04 + 0.01 \left( \frac{z}{z_s} \right) \right] \right]} = \begin{bmatrix} 0.964 \\ 0.975 \\ 0.975 \\ 0.975 \\ 0.964 \\ 0.941 \\ 0.941 \\ 0.964 \\ 0.964 \\ 0.964 \\ 0.964 \\ 0.964 \\ 0.964 \\ 0.964 \\ 0.975 \\ 0.964 \end{bmatrix} \]

the background response factor
\[ y_C := \left( \frac{150 n_1 m}{v_m} \right) = \begin{pmatrix} 13.959 \\ 6 \\ 10.591 \\ 7.72 \\ 7.212 \\ 16.83 \\ 8.25 \\ 12.948 \\ 24.997 \\ 8.921 \\ 18.069 \\ 5.581 \\ 3.842 \\ 23.479 \\ 9.985 \\ 24.997 \end{pmatrix} \]

\[ F_C := \left( \frac{4 y_C}{5 \left(1 + 70.8 y_C^2\right)^6} \right) = \begin{pmatrix} 0.02 \\ 0.035 \\ 0.024 \\ 0.029 \\ 0.031 \\ 0.017 \\ 0.028 \\ 0.021 \\ 0.013 \\ 0.027 \\ 0.017 \\ 0.037 \\ 0.047 \\ 0.014 \\ 0.025 \\ 0.049 \end{pmatrix} \]

\[ \phi_h := \left( \frac{1}{2 n_1 b_{\text{building}} v_m} \right) = \begin{pmatrix} 0.183 \\ 0.207 \\ 0.129 \\ 0.168 \\ 0.178 \\ 0.157 \\ 0.275 \\ 0.194 \\ 0.111 \\ 0.259 \\ 0.147 \\ 0.219 \\ 0.289 \\ 0.117 \\ 0.135 \\ 0.305 \end{pmatrix} \]

\[ \phi_b := \left( \frac{1}{3.2 n_1 b_{\text{building}} v_m} \right) = \begin{pmatrix} 0.144 \\ 0.281 \\ 0.181 \\ 0.233 \\ 0.14 \\ 0.065 \\ 0.124 \\ 0.083 \\ 0.086 \\ 0.208 \\ 0.115 \\ 0.174 \\ 0.379 \\ 0.048 \\ 0.105 \\ 0.247 \end{pmatrix} \]
\[ R_v := \frac{2\pi F C \phi_b \phi_h}{\sqrt{\delta_s + \delta_a}} \]

\( = \begin{pmatrix} 0.411 \\ 0.773 \\ 0.449 \\ 0.656 \\ 0.539 \\ 0.263 \\ 0.578 \\ 0.336 \\ 0.219 \\ 0.698 \\ 0.328 \\ 0.655 \\ 1.282 \\ 0.173 \\ 0.361 \\ 1.11 \end{pmatrix} \]

the resonance response factor

For the acceleration at the top of the building

\[ v := n_1 \frac{R_v}{\sqrt{B^2 + R_v^2}} \]

\( = \begin{pmatrix} 0.53 \\ 0.418 \\ 0.497 \\ 0.483 \\ 0.395 \\ 0.438 \\ 0.418 \\ 0.418 \\ 0.421 \\ 0.537 \\ 0.507 \\ 0.565 \\ 0.352 \\ 0.343 \\ 0.41 \\ 0.393 \end{pmatrix} \text{Hz} \]

the up-crossing frequency

\( t := 600 \text{s} \)

10 min peak wind
\[ k_p := \sqrt{2 \ln(\nu + 1)} + \frac{0.6}{\sqrt{2 \ln(\nu + 1)}} = \begin{pmatrix} 3.572 \\ 3.505 \\ 3.553 \\ 3.546 \\ 3.488 \\ 3.518 \\ 3.505 \\ 3.507 \\ 3.575 \\ 3.559 \\ 3.589 \\ 3.455 \\ 3.448 \\ 3.499 \\ 3.487 \\ 3.411 \end{pmatrix} \]

the peak factor

\[ \phi_1 := \begin{pmatrix} z \\ \frac{z}{h_{\text{building}}} \end{pmatrix}^{1.5} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \]

the fundamental flexural mode

\[ m_e := \begin{pmatrix} 8.93 \times 10^4 \\ 7.344 \times 10^4 \\ 1.115 \times 10^5 \\ 1.931 \times 10^5 \\ 3.789 \times 10^5 \\ 4.108 \times 10^5 \\ 3.405 \times 10^5 \\ 1.773 \times 10^5 \\ 1.274 \times 10^5 \\ 1.71 \times 10^5 \\ 2.09 \times 10^5 \\ 1.454 \times 10^5 \\ 1.55 \times 10^5 \\ 2.476 \times 10^5 \\ 2.157 \times 10^5 \\ 3.086 \times 10^5 \end{pmatrix} \]

the equivalent mass per unit length of the building
\[
\sigma_X := \left( \frac{3I_\nu R_\nu q_p b_{\text{building}} c_f \Phi_1}{m_c} \right) = \begin{pmatrix}
0.026 \\
0.064 \\
0.024 \\
0.021 \\
0.013 \\
0.006 \\
0.015 \\
0.017 \\
0.01 \\
0.023 \\
0.009 \\
0.043 \\
0.05 \\
0.006 \\
0.016 \\
0.034 \\
\end{pmatrix} \quad \text{m/s}^2
\]

the standard deviation of the acceleration

The peak acceleration at the top of the building

\[
X_{\text{max}} := \left( k_p \sigma_X \right) = \begin{pmatrix}
0.093 \\
0.223 \\
0.086 \\
0.073 \\
0.047 \\
0.02 \\
0.053 \\
0.059 \\
0.035 \\
0.082 \\
0.032 \\
0.148 \\
0.172 \\
0.022 \\
0.055 \\
0.116 \\
\end{pmatrix} \quad \text{m/s}^2
\]