Test Of Cosmological Models With Variable G

Ekim Taylan HANIMELI

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Luleå University of Technology
Department of Computer Science, Electrical and Space Engineering
Declaration of Authorship

I, Ekim Taylan Hanimeli, declare that this thesis titled, “Test Of Cosmological Models With Variable G” and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.

- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.

- Where I have consulted the published work of others, this is always clearly attributed.

- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.

- I have acknowledged all main sources of help.

- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed:

Date:
“How should we like it were stars to burn
With a passion for us we could not return?
If equal affection cannot be,
Let the more loving one be me.”

W.H. Auden
This report investigates cosmological models with variable $G$. After a brief review of the standard cosmology, the cosmological constant problem and the cosmological probes used in this work, we begin by deriving the cosmological equations starting from a Newtonian force equation with variable $G$. Then we consider the modification of Einstein’s field equations and demonstrate the consequences of having variable $G$ and show that density dilution becomes unphysical without additional modification. We then develop a way to treat this problem and search for a correspondence between the Einsteinian and Newtonian approaches.

In order to check the validity of the models we use type Ia supernova and baryon acoustic oscillation data. We construct Newtonian models by proposing one exponentially decreasing and one power series $G$ functions. We also consider possible evolution of supernova luminosity as a function of redshift and as a function of $G$. We use a Python implementation of SEAL Minuit to constrain the parameters by minimizing the residuals between the observations and the prediction.

As a result of the computations, we compare the $\chi^2$ values of the models. We find that the best power series models perform better, while exponential models with supernova luminosity evolution perform about as well compared to the standard model.
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1 Introduction

1.1 Standard cosmology and cosmological constant problem

The standard cosmology is derived from Einstein’s field equations:

\[ G^{\mu\nu} = 8\pi GT^{\mu\nu} - \Lambda g^{\mu\nu} \]  

in units where \( c = 1 \). \( G^{\mu\nu} \) in these equations is called Einstein tensor, which represents how the geometry of space behave, while right side represents the energy.

For cosmological purposes we use, \( T^{\mu\nu} = (\rho + p)U^\mu U^\nu + pg^{\mu\nu} \), the stress-energy tensor for perfect fluid with \( \rho \) and \( p \) are energy density and pressure, \( U^\mu = (1, 0, 0, 0) \), and \( g^{\mu\nu} \) is the Friedmann-Lemaître-Robertson-Walker metric given by:

\[ ds^2 = -dt^2 + a(t)^2 \left[ \frac{dr^2}{1 - \kappa r^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] \]  

where \( a(t) = R(t)/R_0 \) is the cosmological scale factor, or, in other words, the radius of the universe measured in units where current radius is one (\( R_0 = 1 \)); \( \kappa \) is the curvature parameter.

Straightforward calculation shows that, with this metric and perfect fluid energy tensor the field equations lead to Friedmann-Lemaître equations. (Carroll, 2004b) The temporal component directly (\( \mu = 0, \nu = 0 \)) gives:

\[ H^2 \equiv \left( \frac{\dot{a}}{a} \right)^2 = \frac{8\pi G}{3} \rho - \frac{\kappa}{a^2} + \frac{\Lambda}{3} \]  

the spatial components (\( \mu = i, \nu = i, i = 1, 2, 3 \)) lead to a single equation due to symmetry. After using the above equation to replace the first derivatives, the spatial components give:

\[ \ddot{a} \left( \frac{\dot{a}}{a} \right) = -\frac{4\pi G}{3} (\rho + 3p) + \frac{\Lambda}{3} \]  

Equations 3 and 4 are the central equations of cosmology that define the expansion of the universe.

The letter \( \Lambda \) in the above equations represent the cosmological constant. It was first invented by Einstein to have his field equations accommodate static solutions for the universe, before of the expansion of the universe was discovered. (Weinberg, 1989)

After Hubble’s discovery, the cosmological constant fell out of favour, until the discovery of the accelerated expansion of the universe by two teams independently in 1998. (Riess et al., 1998) (Perlmutter et al., 1999) The reason can be seen easily by writing the Friedmann equation in another form:

\[ \frac{H^2}{H_0^2} = \Omega_{\text{matter}}a^{-3} + \Omega_{\text{radiation}}a^{-4} + \Omega_{\text{curvature}}a^{-2} + \Omega_\Lambda \]
with, \( H = \dot{a}/a \), \( a \) is the scale factor and, \( \Omega = \rho/\rho_{\text{critical}} = \rho \frac{3H_0}{8\pi G} \) represent the current energy contribution of different components. Since all the components in this expression dilute with a rate at least as fast as \( a^{-2} \) except \( \Omega_\Lambda \), as the size of the universe increases, acceleration can only start with non-zero \( \Omega_\Lambda \). In the standard cosmology, measuring the acceleration directly measures this energy, dubbed as dark energy.

Since its density remains constant as the space expands, dark energy behaves exactly like a vacuum energy would behave, which alleviates some of the mystery attached to it. (Carroll, 2001) Then, what is the problem? This: If this energy is indeed the vacuum energy, then we should be able to calculate the same value using quantum field theory, as well as cosmology. However, the cosmological measurements and theoretical considerations predict energy densities that differ by 120 orders of magnitude! (Weinberg, 1989)(Carroll, 2004a)

From a cosmological standpoint, there are two main approaches to this problem: One approach consists of formulating the dark energy in terms of various dynamical scalar fields that act within general relativity, instead of cosmological constant. These include quintessence models (Martin, 2008)(Tsujikawa, 2013); chameleon models (Waterhouse, 2006); and k-essence models. (Scherrer, 2004)

The second approach tries to explain the recent accelerated expansion period by modifying the theory of gravity itself. These include scalar-tensor theories such as Brans and Dicke, 1961 theory; \( f(R) \) gravity models (Buchdahl, 1970)(Sotiriou and Faraoni, 2010); Gauss-Bonnet gravity (Nojiri, Odintsov, and Sasaki, 2005); and higher dimensional models such as braneworld cosmologies (Langlois, 2002)(Maartens and Koyama, 2010). These start with actions other than Einstein-Hilbert\(^1\) and derive their models by minimizing these actions. In addition to these, there is also the more phenomenological approach of modifying gravity by taking \( G \) as a function of time, directly in general relativity.(Lau, 1985)(Sultana, 2015)(Wright and Li, 2018)

In this work we focus on cosmologies derived from Newtonian force equation with a dynamical gravitational constant. When a modification in the gravitational constant, also known as the Newton’s constant, is to be considered, it is reasonable to start from a Newtonian framework. Good will towards Newton (it is his constant, after all) aside, the Friedmann equations can be actually derived with a purely Newtonian argument, except the the exact form of some terms. Since it is easy to imagine the gravitational force law working in the same way with \( G \) having a different value, even without any underlying theory of how or why it changes, we can consider deriving our cosmological equations starting from a force equation in the Newtonian frame.

One advantage of assuming a Newtonian cosmology over general relativity is that, unlike general relativity, the space is strictly Euclidean in Newtonian physics. This is convenient, since the cosmological measurements already indicate a flat space. (Planck Collaboration et al., 2016) The disadvantage, of course, is that it is well known that general relativity works marvelously compared to Newtonian in small scales. So, we have to convince ourselves that the universe behaves Newtonian in cosmological scales, general relativistic in smaller scales, and Newtonian again in yet smaller scales. In this work we simply assume it is the case, but it is not impossible to imagine a mechanism that makes physics better approach Newtonian again with low density or large scale.

\(^1\)Einstein-Hilbert action, \( S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} (R - 2\Lambda) \), is minimized to derive the Einstein’s field equations from a Lagrangian. (Carroll, 2004b) In general, starting from an action principle is the standard way to derive any new physical model.
In addition to Newtonian, we also consider modifying Einstein’s field equations. We demonstrate the effect of variable gravitational constant on the fluid equation, and show a way to overcome this change. We then search for a possible correspondence between the Newtonian and Einsteinian approaches.

There is one glaring problem with changing gravity in the universe: everything else in the universe is also affected by this change. This is a point of concern, ever since change of gravity was first proposed by Dirac, 1937. There are many constraints on gravity from small scale and solar system considerations such as planetary radii, and the stellar evolution. (Canuto, 1981) In this project we largely ignore these concerns by assuming a mechanism that screens the large scale change of gravitational potential from the small scales. These kind of mechanisms exist for a variety of gravitational theories. (Joyce et al., 2015) In this work we assume without proving that such a screening mechanism works in the cases we consider. We relax this assumption briefly when we discuss the possible effects of varying $G$ on supernovae, but otherwise we adopt this assumption very strongly, even allowing $G$ to go negative.

Once the cosmological equations are obtained, we need to compare them against the observation to determine the parameters. The primary sources that are used to obtain information on the expansion of the universe are type Ia supernovae observations, baryon acoustic oscillations (BAO), cosmic microwave background anisotropies, and gravitational lensing. (Amendola and Tsujikawa, 2010) In this work we focus on the first two, namely; Type Ia supernovae and BAO.

1.2 Determination of the cosmological parameters using type Ia Supernovae

The usefulness of Type Ia supernovae as distance indicators have been predicted early on after their discovery even without an understanding of their underlying physics, owing to their similar light curves. (Baade, 1938)(Kowal, 1968) This class of supernova later understood to have a common ignition mechanism that explain the similarity as well as absorption properties of their luminosity curves. (Hoyle and Fowler, 1960) Simply put, this mechanism involves a white dwarf in a binary system that accrete enough matter to exceed the Chandrasekhar limit. The star then discharges the excess gravitational energy in the form of a thermonuclear explosion; the type Ia supernovae. (Woosley and Weaver, 1986)(Langer et al., 2000)

Despite this neat theoretical explanation, more detailed studies showed that the luminosity curves are not uniform but various characteristics of the curves, such as the decline rate and absolute magnitude, are correlated. (Phillips, 1993) So, even though the supernovae are not standard candles per se, these correlations make them standardizable. This allows the use of supernovae in cosmological research, and this lead to the discovery of the accelerated expansion by Riess et al. (1998) and Perlmutter et al. (1999) independently. These observations were consistent with low density, lambda dominated universe. Their results were later extended to include more supernovae and with higher redshift.

These later work include recalibration and compilation of the existing datasets to reduce systematic uncertainties and obtain a common result. This kind of work was undertaken by Betoule et al. (2014) using observations obtained from SDSS-II and SNLS collaborations. This work includes 740 spectroscopically confirmed type Ia supernovae ranging from very low ($z < 0.1$) redshifts to $z \sim 1$. In this project, we
use the calibrated dataset and covariance matrix published by them, and we follow their method of curve standardization which is discussed later in this chapter.

The standardizability of the type Ia supernova data is widely accepted as the standard view. On the other hand, Linden, Virey, and Tilquin (2009) claim that high-redshift supernovae appear brighter than standard cosmology would expect, and this difference is better explained by a redshift dependence to the supernova luminosity. Based on these, Tutusaus et al. (2017) and Tutusaus, Lamine, and Blanchard (2018) show that, if redshift dependence of supernovae is considered, non-accelerated models fit the cosmological data as well as the standard cosmological model with accelerated late time expansion.

Another thing to consider on type Ia supernovae is the possible dependence of the intrinsic luminosity to the variation of gravitational constant. A crude estimate of this kind of a relation (as it was used by Garcia-Berro et al. (2006), for example) can be obtained by a simple reasoning: Since the energy for the explosion comes from the gravitational instability created by the mass above Chandrasekhar limit, and it is reasonable to expect a linear correlation between the energy discharge and the size of the explosion, it follows that luminosity should be correlated with Chandrasekhar mass \( L \propto M_{Ch} \). The Chandrasekhar mass straightforwardly depends on \( G \) as \( M_{Ch} \propto G^{-3/2} \). (Chandrasekhar, 1931) Then we obtain the final relation as \( L \propto G^{-3/2} \).

However, this simple approach may not be very accurate, as the complicated nature of type Ia supernova explosion process is still not well-understood. As it was demonstrated by Wright and Li (2018) in great detail, the actual relation between supernova luminosity and gravity can be quite the opposite, approximately behaving like \( L \propto G \). Of course, representing the supernovae physics accurately is outside the scope of this project. Nonetheless, we will test these two cases in a simplistic way in our calculations.

With this in mind, we can start constructing the equations used to test our gravitational models with supernova data. The simple idea is to calculate the luminosity distance of the objects using the cosmological model, then compute the distance modulus, and compare with the measurements. The luminosity distance in flat space is given by any introductory textbook as:

\[
d_L = c(1 + z) \int_0^z \frac{dz'}{H(z')}
\]  

From this we write the distance modulus:

\[
\mu = 5 \log_{10} \left( \frac{d_L}{c/H_0} \right)
\]  

where \( c \) is the speed of light. The reference distance is chosen as the Hubble radius, \( c/H_0 \), instead of usual 10 kpc. This is done for computational ease, since the supernovae data cannot determine the current value of the Hubble parameter it is best to take it out of the equations as a parameter.

The result of this will constitute the prediction of our model when we construct an \( H(z, p_1, ...) \) with \( p_1, ... \) represent the parameters of our model to be determined. Then, we need to compare this prediction to the distance modulus obtained from the supernova measurements, \( \mu_{obs} \), to constrain our model parameters. However, as explained above, type Ia supernovae are not standard candles and therefore we cannot directly use the observed data to find a definite distance modulus; we need
to standardize our candles. Following Betoule et al. (2014) we write:

$$\mu_{\text{obs}} = m - M + \alpha X - \beta C$$  \hspace{1cm} (1.8)$$

Here, $m$ is the apparent magnitude in the B-band, $X$ is the stretch of the light curve and $C$ is the color correction, that are given for each supernova in the dataset. $\alpha$ and $\beta$ are free nuisance parameters and $M$, the absolute magnitude. It depends on the size of the host galaxy ($M_{\text{stellar}}$) on the supernova, given as:

$$M = \begin{cases} M', & \text{if } M_{\text{stellar}} < 10^{10} M_\odot \\ M' + \Delta M, & \text{otherwise} \end{cases}$$  \hspace{1cm} (1.9)$$

$M'$ and $\Delta M$ are additional nuisance parameters while $M_{\text{stellar}}$ is given for each host galaxy in the dataset.

1.3 Determination of the cosmological parameters using baryon acoustic oscillations

The early universe before recombination epoch ($z > 1000$) was a high energy soup of plasma coupled with high energy radiation. This soup was vibrating under the balance of photon pressure and gravitation. As the background expands, the energy density of the photons decrease faster than the density of the matter. This leads to matter combining into molecules and decoupling from the photons. (Peebles and Yu, 1970) (Bond and Efstathiou, 1984) When this combination (which is called recombination for historical reasons) happens, the sound speed decreases abruptly and the vibrational modes in the plasma freeze in the baryons. (Eisenstein and Hu, 1998)

This leads to characteristic length scales to exist in the distribution of baryonic matter, which appears as the excess of galaxies in certain scales, and they can be used as cosmological distance indicators, or standard rulers. (Eisenstein et al., 2005)

Fit to the measurements can be obtained by writing down a distance scale as:

$$D_V = \left( \frac{d_A^2(z)(1+z)^2}{cz} \right)^{1/3}$$  \hspace{1cm} (1.10)$$

where $d_A = \frac{c}{1+z} \int_0^z dz / H(z)$ is the angular diameter distance. This is an isotropic scale. In surveys we can also measure anisotropic scales which are given by two coordinates:

$$\theta = \frac{r_s(z_{\text{drag}})}{(1+z)d_A}$$  \hspace{1cm} (1.11)$$

$$\delta z_s = \frac{r_s(z_{\text{drag}})H(z)}{c}$$  \hspace{1cm} (1.12)$$

where $r_s(z_{\text{drag}}) = \int_{z_{\text{drag}}}^{\infty} \frac{c(z)dz}{H(z)}$, $r_s(z_{\text{drag}})$ is the sound horizon at the drag epoch, and $c_s$ is the sound speed. The drag epoch corresponds to a time when the baryons were released from the Compton drag of the photons. The value for $z_{\text{drag}}$ is given by Eisenstein and Hu (1998) as:

$$z_{\text{drag}} = \frac{1291 \omega_m^{0.251}}{1 + 0.659 \omega_m^{0.828}} (1 + b_1 \omega_m^{b_2})$$  \hspace{1cm} (1.13)$$
where \( b_1 = 0.313\omega_m^{-0.419}(1 + 0.607\omega_m^{0.674}) \), and \( b_2 = 0.238\omega_m^{0.223} \) with \( \omega_m = \Omega_{\text{matter}}(\frac{H_0}{100})^2 \) and \( \omega_b = \Omega_{\text{baryon}}(\frac{H_0}{100})^2 \).

Combining equations for \( \theta \) and \( \delta z_s \) we can obtain \( D_V \) as:

\[
D_V = \frac{r_s(z_{\text{drag}})}{(\theta^2 \delta z_s)^{1/3}}
\] (1.14)

Following Tutusaus, Lamine, and Blanchard (2018), in this project we have used data from Beutler et al. (2011), Kazin et al. (2014), Alam et al. (2017), effective redshifts 0.106, 0.15, 0.44, 0.6, 0.73, 0.38, 0.51, 0.61, 1.19, 1.50, 1.83, as well as the Ly-\( \alpha \) autocorrelation function from Bautista et al. (2017) and Ly-\( \alpha \)-quasar cross correlation from du Mas des Bourboux et al. (2017) at \( z = 2.4 \). Since we did not consider the early universe in this work, we have combined \( H_0 \) and \( r_s(z_{\text{drag}}) \) in the above equations to get one single parameter.
2 Theoretical Considerations

2.1 Newtonian cosmology with variable G

2.1.1 Modified Friedmann equation in the Newtonian frame

We start by a simple force equation using Newton’s second law:

\[- \frac{G(R)M}{R^2} = \ddot{R} \] (2.1)

The over-dots, as usual, denote the time derivative.

In writing down this equation we imagine a thin shell (of mass \( m \), for reasons that will be clear later) at the edge of the universe and how its motion is affected by the gravitational effect of the total mass.

A point worth noting is the \( R \) dependency of \( G \), with \( R \) being the radius of the universe. Of course, a dependency of time would be mathematically identical since we take \( R \) to depend only on time coordinate. But, it is simpler to calculate the following integrals by taking a \( G(R) \) rather than a \( G(t) \).

Another thing to note is that, we are free to measure this radius in any unit we want. A convenient unit of choice is the current value of the radius \( R_0 \), such that \( R_0 = 1 \). Then, when we define a dimensionless scale factor \( a = \frac{R}{R_0} \) which becomes equal to \( R \) in these units. This relation will be used throughout this report with \( a \) and \( R \) being used interchangeably.

Then, after multiplying both sides by \( \dot{R} \) and re-arranging:

\[ M \int Gd(R^{-1}) = \int \left( \frac{1}{2} \dot{R}^2 \right) \] (2.2)

Using integration by parts:

\[ M \left[ \frac{G}{R} - \int \frac{dG}{R} \right] = \frac{1}{2} \dot{R}^2 - C \] (2.3)

With \( M = \frac{4\pi}{3} R^3 \rho \) being the mass, the equation becomes,

\[ H^2 \equiv \frac{\dot{R}^2}{R^2} = \frac{8\pi}{3} \rho \left[ G(R) - R \int \frac{dG}{R} \right] + \frac{2C}{R^2} \] (2.4)

where, primes indicate derivative with respect to \( R \).

This equation is easy to recognize as the Friedmann equation in the Newtonian frame, except the second term in the brackets.

From the famous relation in special relativity, \( E = mc^2 \), we know that mass and energy density are the same thing with \( c^2 \) as the conversion factor. Using units where \( c = 1 \), we can recognize \( \rho \) in the above equation as denoting the energy density.

Another relativistic correction in the above equation would be to recognition of \( C \) as representing curvature. However, we are considering a cosmology fully Newtonian, therefore the space is assumed to be Euclidean, or geometrically flat. So, here \( C \) represents the boundary condition for the energy of the universe.
2.1.2 Pressure, and the acceleration equation

Re-arranging the force equation we can obtain the usual force equation except the pressure term coming from radiation.

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3} \rho$$  \hspace{1cm} (2.5)

The Friedmann equation represents an energy relation, as it is especially evident by the equation obtained after the integration by parts. The first term on the left hand side is clearly the gravitational potential energy per mass, and the first term on the right is the kinetic energy per mass. Then, the second term on the left should also represent some sort of energy per unit mass \( m \). More specifically, if we rewrite the term under the integral sign such that:

$$Mm \frac{dG}{R} = \frac{4\pi}{3} R^3 \rho m G' \frac{dR}{R} = \frac{1}{3} \rho m G' dV = dE$$  \hspace{1cm} (2.6)

Where we used the first law of thermodynamics, \( dE = pdV \), in the absence of heat transfer out of the universe. This tells us that \( \frac{1}{3} \rho m G' \) is the pressure that corresponds to the energy, given by the integral in the Friedman equation.

To make more sense out of this, realize that we could have started with an energy equation directly and obtained the force equation from that.

$$\frac{1}{2} R^2 - \frac{MG}{R} + W = Constant$$ \hspace{1cm} (2.7)

Here, we anticipated an extra term in the energy equation with \( W \). After taking a time derivative, we have:

$$\dot{R} \ddot{R} - \frac{MG}{R} + \frac{MGR}{R^2} + \dot{W} = 0$$  \hspace{1cm} (2.8)

This becomes the same as our initial force equation if we take:

$$\dot{W} = -\frac{MG}{R}$$ \hspace{1cm} (2.9)

which leads to:

$$W = M \int \frac{dG}{R} + Constant$$  \hspace{1cm} (2.10)

However, we could have, instead, closed the system by choosing \( W = 0 \). In that case we would need a force equation such that,

$$\ddot{R} = \frac{M \dot{G}}{RR} - \frac{MG}{R^2}$$  \hspace{1cm} (2.11)

The first term on the right-hand side can again be seen as a force per mass which leads to a pressure:

$$\frac{4\pi R^2 p}{m} = \frac{4\pi R^3 \rho G'}{3}$$  \hspace{1cm} (2.12)

which gives,

$$p = \frac{1}{3} \rho m G'$$  \hspace{1cm} (2.13)
This shows integrating the gravitational force with variable $G$ is analogous to having an extra pressure force doing work on the system. We can write this system of equations as Friedmann equations as well. Let us be more general and keep the $W$ in our equations. Expanding the mass $M$, the $\dot{R}$ equation gives:

$$H^2 = \frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3} \left( \rho - \frac{3}{4\pi G} \frac{W}{R^2} \right) + \frac{2C}{R^2}$$

(2.14)

and the derivative of it gives:

$$\frac{\ddot{R}}{R} = \frac{4\pi G}{3} \left( \rho - \rho R \frac{G'}{G} + \frac{3}{4\pi G} \frac{W'}{R} \right)$$

(2.15)

This also exposes what the earlier $m$ should be. Since the relativistic cosmological equation including pressure is $\ddot{R} \frac{R}{G} = -\frac{4\pi G}{3} (\rho + 3p)$ it follows that $m = R/G$ in the earlier equations.

### 2.1.3 Critical density and the dimensionless Friedmann equation

We can rewrite the Friedmann equation in a different form by defining a critical density. In standard cosmology, this value is $\rho_{cr} = \frac{3H_0^2}{8\pi G}$. It is evident from the $H^2$ equation derived above, in the constant $G$ case this definition of critical density makes $C = 0$ at $\rho = \rho_{cr}$.

Following this idea we define a critical density for the variable $G$ case, such that $C = 0$ at $\rho = \rho_{cr}$. First, we define $\bar{G} = G(R) - R \int \frac{dG}{R}$ for simplicity. Then, we can write the Friedmann equation in a compact form.

$$H^2 = \frac{\dot{R}^2}{R^2} = \frac{8\pi}{3} \rho \bar{G} + \frac{2C}{R^2}$$

(2.16)

We define $\rho_{cr} = \frac{3H_0^2}{8\pi G_0}$. One curiosity about this definition is that, not only its value will be entirely different from the standard value, therefore leading to odd values for $\Omega$, it can even be negative when $G$ is positive leading to negative $\Omega$ values. In general we will not force $\bar{G}$ to be positive either with the assumption that small scale gravitational constant could be different from that of cosmological scales, but we make matter energy density to be strictly positive.

We progress by noting that, as the universe expands the matter density will decrease as $\rho \propto R^{-3}$ so we can write $\rho = \rho_0 a^{-3}$ with $\rho_0$ being the current value of the matter density and $a = R$ being the dimensionless scale factor, numerically equal to $R$. (Ryden, 2003) Also, defining $\Omega = \rho_0 / \rho_{cr}$ we write the above expression in a different form.

$$\frac{H^2}{H_0^2} = \frac{\dot{\bar{G}}}{C_0 \Omega a^{-3}} + \frac{2C}{a^2 H_0^2}$$

(2.17)

Evaluating this expression at $a = 1$ we get $\frac{2C}{H_0^2} = 1 - \Omega$. Then our final expression becomes:

$$\frac{H^2}{H_0^2} = \frac{\dot{\bar{G}}}{C_0} \Omega a^{-3} + (1 - \Omega) a^{-2}$$

(2.18)

This expression is useful since it measures the rate of expansion of the universe in terms of dimensionless quantities. This can also be written in terms of redshift,
using the fact that 

\[ a = \frac{1}{1+z} \]

\[ \frac{H^2}{H_0^2} = \frac{\bar{G}(z)}{\bar{G}_0} \Omega(1+z)^3 + (1-\Omega)(1+z)^2 \]  

(2.19)

The differences between this and the standard expression worth noting. First, the \( \bar{G}(z) / \bar{G}_0 \) term directly appears as a result of the variation of \( G \) and the choice of the critical density. Second, this expression has neither radiation nor dark energy density parameters. The exclusion of latter is obvious, this is what is being tested. On the other hand radiation was omitted because, since its energy dilution rate is very high (\( \rho_{\text{rad}} \propto R^{-4} \)), it is only relevant for early universe behaviour, and therefore it was not necessary for the late universe probes considered in this project.

### 2.2 Modifying gravitational constant in Einstein’s field equations

In this section we will generalize the Newtonian approach to a general relativistic context by modifying the gravitational constant directly in the Einstein’s field equations. We will search for the condition to obtain the cosmological equations we derived in the previous section.

#### 2.2.1 Energy conservation in variable \( G \)

In Einstein’s field equations \( G^{\mu\nu} = 8\pi G T^{\mu\nu} - \Lambda g^{\mu\nu} \) there is a geometrical identity, named Bianchi identity, that implies the conservation of the stress-energy tensor. (Carroll, 2004b) Bianchi identity states that the covariant derivative (shown by \( D_\mu \)) of the Einstein tensor \( G^{\mu\nu} \) is zero.

\[ D_\mu G^{\mu\nu} = 0 \Rightarrow D_\mu T^{\mu\nu} = 0 \]  

(2.20)

Since Bianchi identity is independent of physics, it should hold whatever we do to the left hand side of the field equations. Therefore, in order to set \( G \) as a dynamical variable directly in the field equations, we need to add another dynamical variable to satisfy the Bianchi identity. If we also want to conserve the energy tensor, the remaining options are variable \( c \), variable \( \Lambda \), or both, as shown by Sultana (2015). He shows that if the speed of light is taken as constant, Bianchi identity leads to:

\[ \dot{\Lambda} = -8\pi \bar{G} \rho \]  

(2.21)

Another option to satisfy the Bianchi identity is changing the energy conservation so that \( D_\mu T^{\mu\nu} \neq 0 \). This, then, is the only way to proceed to have a model with constant \( c \) and no cosmological constant. We consider a such model with \( G \equiv G(t) \):

\[ G^{\mu\nu} = 8\pi G(t) T^{\mu\nu} \]  

(2.22)

By Bianchi identity, this leads to:

\[ D_\mu (G(t) T^{\mu\nu}) = T^{\mu\nu} \partial_\mu G(t) + G(t) D_\mu T^{\mu\nu} = 0 \]  

(2.23)

Obviously, this breaks the energy conservation since

\[ D_\mu T^{\mu\nu} = -\frac{T^{\mu\nu} \partial_\mu G(t)}{G(t)} \neq 0 \]  

(2.24)
More specifically, if we take the \( \nu = 0 \) equation, we get:

\[
\rho \dot{G} + G \left( \dot{\rho} + 3 \frac{\dot{a}}{a} (\rho + p) \right) = 0 \tag{2.25}
\]

The part within the parentheses is the usual fluid equation and the \( \rho \dot{G} \) term comes from varying \( G \). Equation 6 can be solved directly to obtain:

\[
\rho \propto G^{-1} a^{-3(1+w)} \tag{2.26}
\]

this leads to a matter dilution \( \rho \propto G^{-1} a^{-3} \) instead of the usual \( \rho \propto a^{-3} \). Having the density for usual matter coupled to \( G \) does not make physical sense since it implies the matter is created or destroyed as \( G \) changes its value. What we need is then the addition of new dynamical parameters to the energy tensor to counteract the extra term that comes from varying \( G \). Physically this means defining a new dark fluid with dynamical energy density.

### 2.2.2 Addition of the new fluid

We define a new stress energy tensor with the addition of new parameters to replace \( T^{\mu \nu} \) in the field equations.

Let \( \tilde{T}^{\mu \nu} \) be a tensor of the form:

\[
\tilde{T}^{\mu \nu} = \begin{pmatrix}
\rho + L & g^{\mu \nu} (p + M) \\
g^{\mu \nu} (p + M) & g^{\mu \nu} (p + M)
\end{pmatrix} \tag{2.27}
\]

with \( L \) and \( M \) are two unknown functions. We will now try to find a relation between them, which will serve as the equation of state of the added fluid.

Taking this as our new stress energy tensor on the right side of the first equation we have the new Einstein equations:

\[
G^{\mu \nu} = 8 \pi G(t) \tilde{T}^{\mu \nu} \tag{2.28}
\]

With the Bianchi identity this gives:

\[
\tilde{T}^{\mu \nu} \partial_{\mu} G + GD_{\mu} T^{\mu \nu} = 0 \tag{2.29}
\]

The \( \nu = 0 \) equation gives:

\[
G \left[ \dot{\rho} + \dot{L} + 3 \frac{\dot{a}}{a} (\rho + L) + 3 \frac{\dot{a}}{a} (p + M) \right] + (\rho + L) \dot{G} = 0 \tag{2.30}
\]

When we impose \( D_{\mu} T^{\mu \nu} = 0 \) condition, this equation becomes:

\[
\dot{L} + (\rho + L) \frac{\dot{G}}{G} + 3 \frac{\dot{a}}{a} (L + M) = 0 \tag{2.31}
\]

This equation can be solved for either \( L \) or \( M \) in terms of the other variables. For \( M \) we directly get:

\[
M = \frac{a}{3a} L - \left( \frac{\dot{G}a}{3Ga} + 1 \right) L - \frac{\dot{G}a}{3Ga} \rho \tag{2.32}
\]
Or, it can be solved for $L$ by rewriting it as a first order ordinary differential equation and defining a new function $Q(t)$ such that:

\[
L + \left( \frac{\dot{G}}{G} + 3 \frac{\dot{a}}{a} \right) L + \left( \frac{\dot{G}}{G} \rho + 3 \frac{\dot{a}}{a} M \right) = 0
\]
(2.33)

\[
\frac{\dot{Q}}{Q} = \frac{\dot{G}}{G} + 3 \frac{\dot{a}}{a}
\]
(2.34)

\[
\frac{d}{dt}(LQ) = -Q \left( \frac{\dot{G}}{G} \rho + 3 \frac{\dot{a}}{a} M \right)
\]
(2.35)

Which leads to,

\[
L = -\frac{1}{Ga^3} \int dt \left( \rho a^3 \dot{G} + MG \frac{da^3}{dt} \right)
\]
(2.36)

The above relations could be interpreted as a dynamical dark energy equation of state with $L$ as energy and $M$ as pressure. It is easy to see in the case with $L$ and $G$ constant, this relation becomes the same as the usual dark energy equation-of-state. The dependency of this EOS on Hubble parameter, matter density, and $G$ function suggests the added energy would also depend on these parameters, even without any other preconception about the nature of this energy. This dependence of the proposed fluid energy to the density of matter is also suggested by Solà (2011) from quantum field theoretical considerations.

In this section, we have shown that a variable $G$ model should also include an additional dark fluid for consistency. What we have done with the addition of new tensor is, in spirit, similar to what is done in Brans-Dicke models as they also require definition of new tensors in order to preserve the conservation of usual matter. However, our definition is not equivalent to that of Brans-Dicke. (Weinberg, 1972)

2.2.3 Modified Friedmann equations and the Newtonian correspondence

Together with FLRW metric, as discussed above $G^{\mu\nu} = 8\pi G(t)\tilde{T}^{\mu\nu}$ leads straightforwardly to modified versions of Friedmann equations.

\[
H^2 = \frac{\dot{R}^2}{R^2} = \frac{8\pi G}{3} (\rho + L) - \frac{\kappa}{a^2}
\]
(2.37)

\[
\frac{\dot{a}}{a} = -\frac{4\pi G}{3} (\rho + 3p + 3M)
\]
(2.38)

where $G$ is dynamical and $L$ and $M$ are related as discussed above.

We can relate these equations with the ones derived with a Newtonian approach to try finding a correspondence between the two. Direct comparison of the above equations with the Newtonian counterparts lead to:

\[
L = -\frac{3}{4\pi G} \frac{W}{R^2}
\]
(2.39)

\[
3M = -\rho R \frac{G'}{G} + \frac{3}{4\pi G} \frac{W'}{R}
\]
(2.40)
Remembering that \( a = R \) in two notations and replacing these in \( M = \frac{\dot{a}}{3a} L - \left( \frac{G a}{3ca} + 1 \right) L - \frac{\dot{a}}{3a} \rho \) gives:

\[
\frac{1}{3} \frac{\rho R^3}{G} + \frac{1}{4\pi G} \frac{W'}{R} = \frac{R}{4\pi R} \left( \frac{W}{G R^2} - \frac{W G}{G^2 R^2} - 2 \frac{W \dot{R}}{G R^3} \right) + \left( \frac{\dot{G} R}{3 G R} + 1 \right) \left( \frac{3}{4\pi G} \frac{W R}{G R^2} \right) - \frac{\dot{G} R}{3 G R} \rho
\]

with primes denoting the derivative with respect to \( R \). This equation is trivially satisfied if \( W = 0 \). Assuming \( W \neq 0 \) we can simplify this equation.

\[
W' = W' - \frac{W G'}{G} - 2 \frac{W}{R} + \frac{G' W}{G} + 3 \frac{W}{R}
\]

(2.42)

\[
\frac{1}{R} = 0
\]

(2.43)

This shows that, with a finite scale factor, the only correspondence between the two approaches is the case with \( L = 0 \) and \( M = \frac{1}{3} \rho R \frac{\dot{G}}{G} \).

This makes sense, since the Newtonian relations were derived from a gravitational force equation, which does not exist in general relativity in the first place. However, it is interesting to see that it is possible to get a set of equations that are same in both cases under one unique condition.
3 Models, Results and Conclusion

3.1 Models and Methodology

3.1.1 Treatment of supernova evolution

In addition to the discussion in the introduction, we also want to consider the possible evolution of the supernova luminosity with redshift, and under the influence of evolving $G$. For the former, following Tutusaus et al. (2017) and Linden, Virey, and Tilquin (2009) we define $m_{evo}$ such that:

$$ m_{evo} = \begin{cases} \epsilon z^\delta, & \text{for case 1} \\ \epsilon (\ln(1 + z))^\delta, & \text{for case 2} \end{cases} $$  \(3.1\)

$$ \mu_{obs} = m - M + \alpha X - \beta C - m_{evo} $$

In these equations $\delta$ and $\epsilon$ are free parameters that describe the empirical evolution of supernovae with redshift. Since $m_{evo}$ becomes degenerate with $M$ for $\delta = 0$ that parameter was constrained to be in the range $(0.2, 2.5)$ for the computation.

Next, we investigate how to consider the effect of gravity on the intrinsic luminosity of the supernovae in the computation. This contradicts with the assumption that the variation on gravity is only effective in the cosmological scales, but it is an assumption we would be willing to abandon if the outcome of the tests favours so.

From the flux luminosity relation, $F = L/4\pi d_L^2$. Since the flux we receive is constant, we have $L \propto d_L^2$, where $L$ is luminosity and $F$ is flux. This means if $L$ evolves as proportional to a function $f$ such that $L \propto f$ then, $d_L \propto \sqrt{f}$. Then the distance modulus, $\mu = 5\log_{10}(d_L/c/H_0)$, is modified:

$$ \mu = 5\log_{10}(\frac{d_L}{c/H_0}) - \frac{5}{2} \log_{10}(f(z)) $$  \(3.2\)

(i) $L \propto G^{-3/2}$ leads to $f(z) = (G(z)/G_0)^{-3/2}$:

$$ \mu = 5\log_{10}(\frac{d_L}{c/H_0}) + \frac{15}{4} \log_{10}(G(z)/G_0) $$  \(3.3\)

(ii) $L \propto G$ leads to $f(z) = G(z)/G_0$:

$$ \mu = 5\log_{10}(\frac{d_L}{c/H_0}) - \frac{5}{2} \log_{10}(G(z)/G_0) $$  \(3.4\)
3.1.2 Tested models

In this project the above tests are used to constrain the parameters of the previously given Newtonian equation.

\[ H(z) = H_0 \left[ \frac{\tilde{G}(z)}{G_0} \Omega(1 + z)^3 + (1 - \Omega)(1 + z)^2 \right]^{1/2} \]  

(3.5)

where \( \tilde{G}(R) = G(R) - R \int \frac{dG}{R} \) as previously given. We want to test three models:

(i) \( G(R) = \tilde{G}(1 + R/\tilde{R}) \exp(-R/\tilde{R}) \). \( \tilde{G} \) and \( \tilde{R} \) are free parameters. Then \( \tilde{G} \) becomes, after the integration and using \( \tilde{R} = 1/(1 + z) \):

\[ \tilde{G}(z) = \tilde{G}_0 e^{-\frac{1}{1 + z}} \]  

(3.6)

From this, it will be easy to see that \( \tilde{G} \) will be indeterminate as \( \tilde{G}(z) \) becomes indeterminate.

(ii) For any well-behaved function of \( G(R) \) we can write a power series expansion around \( R = 1 \) such that:

\[ G(R) = \sum_{n=0}^{\infty} a_n (1 - R)^n \]  

(3.7)

Here only terms up to third order are developed. The parameters of the fit from this model are \( a_1, a_2, \) and \( a_3 \) which will be added one by one. As before, \( G_0 \) cancels out in the final expression.

(iii) In addition to these, the flat standard model is fitted to the data to have a reference value.

\[ H(z) = H_0 \left[ (1 - \Omega_{mat}) + \Omega_{mat}(1 + z)^3 \right]^{1/2} \]  

(3.8)

where, \( \Omega_{rad} \) is omitted in this case as well, since it is negligible in the tested redshift range.

3.1.3 Fitting the data

The model parameters are obtained by minimizing the \( \chi^2 \) function constructed using the model predictions as well as the data and the covariance matrices supplied by different studies.

\[ \chi^2 = (r_{\text{observation}} - r_{\text{prediction}})^T C^{-1} (r_{\text{observation}} - r_{\text{prediction}}) \]  

(3.9)
This is minimized by the use of migrad function in iminuit package of Python. This package is a Python implementation of SEAL Minuit\(^1\) which is a minimizer developed by CERN physicist Fred James\(^2\). (James and Roos, 1975)

We assume type Ia supernova data and BAO data are statistically independent. Therefore, we minimize the sum of their \(\chi^2\) values when the two datasets are considered together.

### 3.2 Results and discussion

The previously discussed two G evolution models are tested in combination with various supernova luminosity evolution possibilities. The \(\chi^2\) values from minimization of these are given in Table 1. In this table, exponential denotes the one-parameter \(G \propto (1 + R/\tilde{R})\exp(-R/\tilde{R})\), and PS denotes the power series expansion models as discussed in the previous section.

<table>
<thead>
<tr>
<th>G Model</th>
<th>SN Evolution</th>
<th>(\chi^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0. (\Lambda)CDM</td>
<td>No evolution</td>
<td>698.63</td>
</tr>
<tr>
<td>1. Exponential</td>
<td>No evolution</td>
<td>740.34</td>
</tr>
<tr>
<td>2. Exponential</td>
<td>(e \ast z^\delta)</td>
<td>699.33</td>
</tr>
<tr>
<td>3. Exponential</td>
<td>(e \ast \ln(1 + z)^\delta)</td>
<td>702.87</td>
</tr>
<tr>
<td>4. Exponential</td>
<td>(L \propto G^{-3/2})</td>
<td>752.13</td>
</tr>
<tr>
<td>5. Exponential</td>
<td>(L \propto G)</td>
<td>752.48</td>
</tr>
<tr>
<td>6. PS (1-parameter)</td>
<td>No evolution</td>
<td>701.15</td>
</tr>
<tr>
<td>7. PS (1-parameter)</td>
<td>(e \ast z^\delta)</td>
<td>695.79</td>
</tr>
<tr>
<td>8. PS (1-parameter)</td>
<td>(e \ast \ln(1 + z)^\delta)</td>
<td>695.98</td>
</tr>
<tr>
<td>9. PS (1-parameter)</td>
<td>(L \propto G^{-3/2})</td>
<td>730.87</td>
</tr>
<tr>
<td>10. PS (1-parameter)</td>
<td>(L \propto G)</td>
<td>751.40</td>
</tr>
<tr>
<td>11. PS (2-parameter)</td>
<td>No evolution</td>
<td>696.39</td>
</tr>
<tr>
<td>12. PS (2-parameter)</td>
<td>(e \ast z^\delta)</td>
<td>695.80</td>
</tr>
<tr>
<td>13. PS (2-parameter)</td>
<td>(e \ast \ln(1 + z)^\delta)</td>
<td>696.01</td>
</tr>
<tr>
<td>14. PS (2-parameter)</td>
<td>(L \propto G^{-3/2})</td>
<td>728.42</td>
</tr>
<tr>
<td>15. PS (2-parameter)</td>
<td>(L \propto G)</td>
<td>729.43</td>
</tr>
<tr>
<td>16. PS (3-parameter)</td>
<td>No evolution</td>
<td>696.37</td>
</tr>
<tr>
<td>17. PS (3-parameter)</td>
<td>(e \ast z^\delta)</td>
<td>695.81</td>
</tr>
<tr>
<td>18. PS (3-parameter)</td>
<td>(e \ast \ln(1 + z)^\delta)</td>
<td>695.97</td>
</tr>
<tr>
<td>19. PS (3-parameter)</td>
<td>(L \propto G^{-3/2})</td>
<td>727.15</td>
</tr>
<tr>
<td>20. PS (3-parameter)</td>
<td>(L \propto G)</td>
<td>729.57</td>
</tr>
</tbody>
</table>

**Table 3.1: SN+BAO fit results**

From Table 1 we can compare different G models as well as the effects of different supernova luminosity evolutions within each model. Starting with the latter, we can immediately see that, for any G model, the gravity dependent luminosity models have much larger \(\chi^2\), therefore perform worse, than other alternatives, especially compared to no evolution models that have the same number of parameters. Since it is hard to think the variation of gravity in small scales not affecting the supernova luminosity, this shows that a mechanism for screening the gravitational interaction from small scales is needed to make the studied models work. Redshift dependent

\(^1\)https://iminuit.readthedocs.io/en/latest/

\(^2\)http://seal.web.cern.ch/seal/work-packages/mathlibs/minuit/
luminosity models, on the other hand, have the lowest $\chi^2$. However, they involve two additional parameters, which might make them undesirable due to Occam’s razor.

When the power series model is compared to the exponential, the former clearly gives better results with the same number of parameters. We can confidently say that the polynomial fit is a much better approach with only one caveat: The expansion is centered around $z = 0$. Since the data used in this comparison probe low-redshift ($z < 2.5$) regime this approach works very well, but it might create computational complications with high-redshift probes, such as CMB.

In comparison with $\Lambda$CDM, three of the models give the lower $\chi^2$ with the least possible number of parameters. These are numbered 7, 8 and 11. Since these latter models are almost the same in every aspect we will only consider one of them, $\epsilon \times z^5$, since its $\chi^2$ is marginally lower. The parameter results for these are given in Table 2. We will also consider the best exponential model (2) which still has a very good fit even though somewhat worse than the power series models. Its fit results are shown in Table 3. The errors throughout are presented as one sigma deviation.

For the selected models (2, 7, 11) and for the standard model we also give the supernova residual graphs in Figure 1. These graphs show the difference between the observed and the predicted distance moduli for each supernova, shown as blue points. The systematic error associated with the measurements are shown as red lines. These errors are obtained by taking the square root of the corresponding elements of the provided covariance matrix. These show are used to see how well the individual supernova fit the considered models. From the similarity of these graphs we can conclude that the tested models perform almost the same as the standard model, and the difference is within the measurement error range of the supernovae.

<table>
<thead>
<tr>
<th>$a_1$</th>
<th>$a_2$</th>
<th>$\Omega$</th>
<th>$\epsilon$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>11. $-4.144 \pm 0.6029$</td>
<td>3.745 $\pm 1.109$</td>
<td>7.61 $\pm 3.8566$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>7. $-2.295 \pm 0.1817$</td>
<td>-</td>
<td>$-0.4382 \pm 0.1574$</td>
<td>$0.2485 \pm 0.1305$</td>
<td>$0.2014 \pm 1.945$</td>
</tr>
<tr>
<td>8. $-2.277 \pm 0.1651$</td>
<td>-</td>
<td>$-0.4567 \pm 0.1528$</td>
<td>$0.2628 \pm 0.1191$</td>
<td>$0.2001 \pm 2.033$</td>
</tr>
</tbody>
</table>

**Table 3.2:** Fit parameters for best power series models

<table>
<thead>
<tr>
<th>$\dot{R}$</th>
<th>$\Omega$</th>
<th>$\epsilon$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.08786 $\pm 0.07255$</td>
<td>$2.175E - 5 \pm 0.0002473$</td>
<td>$0.3214 \pm 0.07403$</td>
<td>$0.4161 \pm 0.2423$</td>
</tr>
</tbody>
</table>

**Table 3.3:** Fit parameters for the best exponential model

In order to better visualize these results in comparison to the standard model, we produce graphs that illustrate the expansive behaviour of the universe. Figure 2 shows the change radius of the universe as a function of time for each model superposed onto the standard model. This can be readily calculated from $H(z) = \dot{a}/a$ as $t = \int \frac{dz}{H(z)(1+z)}$. This graph is normalized so that $R_{0,\Lambda CDM} = R_{0,\text{model}} = 1$ at $t = H_{0,\Lambda CDM} = H_{0,\text{model}}^{-1}$. Here we can see that the two parameter power series model replicates the late history almost perfectly while also tracing the same path into the immediate future. For the others, we can see the blue and red curves starting close to parallel in the early times, but diverging differently. This divergence indicates a difference in the acceleration, which is reduced in these models by the redshift dependence of the supernova.
Figure 3.1: Supernovae modulus distance errors for $\Lambda$CDM, and models 2, 11, 7 from top to bottom. Blue dots represent each supernova and red lines are supernova measurement errors.
Figure 3 is given to show this difference in acceleration more clearly. Since the derivative of $H(z)/(1+z)$ is proportional to the acceleration, the point where the slope changes sign indicate the regime of accelerated expansion. This graph is again normalized so that $H_0_{\Lambda CDM} = H_0_{\Lambda model}$ and for $\Lambda CDM$, the blue line shows the transition between the matter dominated to $\Lambda$ dominated (accelerated) region. Power series models also show a similar behaviour, albeit with less acceleration for model 7, which is obviously explained by the addition of supernova luminosity evolution. The exponential model, on the other hand, cannot accelerate by design, and the acceleration the supernova data normally imply is fully neutralized by the luminosity evolution of the supernovae. This can also be seen from the difference between the $\epsilon$ and $\delta$ values in Tables 2 and 3.

The $\Omega$ values in these tables beg some explanation: They look very odd, especially for those who are familiar with the standard cosmology. We need to keep in mind that $\Omega$, by definition, is the fraction of the energy density, which is the meaningful quantity, of the critical density. Since our critical density will not have the same value as the standard model, it is not meaningful to compare the $\Omega$ values here to the usual one, or to each other.

Moreover, the parameters that define the $G$ function in these tables do not allow us to calculate the actual value of $G$, as the normalization parameter ($\hat{G}$ for exponential and $G_0$ for power series) cancels out in both cases. This factor can be constrained with additional assumptions, but the more important part, the evolution behaviour of $G$, is fully determined. In Figure 4, this evolution is shown for each model, with red lines showing the best fit function, and cyan lines indicating the error. The error lines are constructed by generating random parameters from Gaussian distribution centered at the best fit values.

One crucial information about the normalization parameter ($\hat{G}$ for $G_0$) is its sign. This can be determined easily from the sign of $\Omega/G_0$. Since $\rho_0 = \Omega_0 H_0^2$ in our definition, and we know that the energy density $\rho_0$ should be strictly positive, the sign
FIGURE 3.3: $\frac{H(z)}{1+z}$ vs $z$ for models 2, 11, 7 from top to bottom. Blue lines are the standard model and the red lines are the considered models.
of the scale factor should be such that it will make $\Omega / G_0$ positive. The exponential function is already non-negative, so $\tilde{G}$ will be positive. However, for the power series models, carrying out the above-mentioned calculation show that $G_0$ should be negative for both models.

With this in mind, we can interpret the graphs in Figure 4. As suspected, the exponential model that approaches zero fits the data almost as well as the standard model. When the term connected to gravity becomes close to zero at low redshifts the integration constant dominates and the expansion function ($H(z)$) starts to behave like a power law. This is in accordance with (Tutusaus et al., 2017).

On the other hand, both of the power series models show better fit to data compared to exponential model, with both having negative late time $G$, which generates accelerated expansion according to $\ddot{R} = -\frac{4\pi G}{3} \rho$. This indicates that, even when the supernova luminosity is allowed to vary with redshift, supernova and BAO data are more consistent with accelerated expansion.

### 3.3 Closing remarks and future work

In this work we started with a Newtonian force equation and derived the modified Friedmann-Lemaître equations. Then we studied various variable $G$ models using those equations and tested their compatibility with the cosmological data, namely type Ia supernova and BAO. For the supernova, we considered, along with unmodified data, the possibility of modification of luminosity with redshift and gravity.

From these considerations we obtained the following conclusions:

- Variable $G$ models can fit the cosmological data as well as, or better than the standard model. However, they involve additional parameters which may decrease their preferability.
- Screening of small scale gravitational interaction from large scale $G$ behaviour is needed in all cases for the results to make sense. Not only because gravity dependent models are disfavoured by the data, but also some models predict negative gravitational constant today.
- In the tested models, late time acceleration is better compatible with the data even when the SN luminosity is allowed to vary with redshift.

In addition, we investigated the possibility of having the same cosmological expressions in Einsteiinan and Newtonian cases. We showed that it is only possible in one specific case, in which a negative pressure term is introduced to the acceleration equation. We also demonstrated that, if $G$ is taken as dynamical in the Einstein equations, the fluid equation should be modified in a way that leads to unphysical matter dilution. We then showed a way to remedy this problem by introducing a new energy tensor and we derived the equation of state relation between the elements of this tensor. This relation essentially acts as an equation of state for a dynamical dark energy.

One immediately obvious way to extend this work would be the addition of other cosmological probes, especially cosmic microwave background. This would both help to constrain the $G$ functions further, and determine additional parameters for cosmology. It would also require investigating the early universe effects of variation in gravity.

Another interesting addition would be to explicitly find a mechanism that would screen the change of gravitational constant in cosmological scales from small scales.
Figure 3.4: G graphs for models 2, 11, 7 from top to bottom. Red line represents the best fit curve and the blue lines represent the random curves for one sigma error.
The existence of this was assumed for most of the project without proof. and even though this would not have changed any of the obtained results, it would definitely increase the sensibleness of the negative gravity conclusion.

Furthermore, a similar work can be carried out starting from Einstein’s field equations, using the results from section three, to test different functions for $G$ and $M$. Then it would also be possible to find a more constrained $G$ by using the small scale considerations as well as the cosmological ones. This would help to better understand the nature of $M$ function, which is essentially a dynamical dark energy.


