Modeling Optical Parametric Generation in Inhomogeneous Media

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Dedicated to my family
This thesis uses numerical methods to explain experimental results from nonlinear optics. A provided sample of lithium niobate engineered to enhance optical parametric generation (OPG). The second order susceptibility, $\chi^{(2)}$, had a periodic structure inside which the OPG process was intended to be efficient [3]. This periodic structure was along the beam propagation direction, and had a finite transverse width. In experiments previously performed by the group with a low intensity ($\lesssim 0.7 \text{mW/}\mu\text{m}^2$) pulsed beam incident on the edge of the periodic structure, the output showed results which resembled a local variation of the refractive index [21]. Moreover, increasing incident pulse intensities at this edge (to the order of 1W/\mu m^2), the efficiency of the nonlinear process appeared to increase. In this work, these phenomena were explained by a change in the refractive index, and a model based on Maxwell’s equations was solved with the split step method. The numerical results provided good agreement with the experimental for refractive index change values on the order of $5 \cdot 10^{-5}$, and managed to reproduce the results found in both the low- and high-intensity regimes.
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The work presented in this thesis has been performed with the Nonlinear and Quantum Photonics group at the KTH Royal Institute of Technology, under the supervision of Professor Katia Gallo. This thesis marks the conclusion of my Master of Science studies in Engineering Physics and Electrical Engineering at Luleå University of Technology (LTU). The past five years have been highly formative, containing plenty of both good times and bad times, and I would like to thank my friends and classmates at LTU for sharing these years with me - I consider myself lucky to have enjoyed the company of such passionate people. In particular, I extend my gratitude towards Erik Andrén and Felix Trulsson - my roommates during our time working on our respective thesis projects - for turning every single dinner conversation this spring into a discussion on signal processing. It’s been a blast, and without you I would not have felt at home in Stockholm!

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## Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\chi^{(1)}$</td>
<td>First order susceptibility, $n^2 - 1$</td>
</tr>
<tr>
<td>$\chi^{(2)}$</td>
<td>Second order susceptibility</td>
</tr>
<tr>
<td>$\chi_{eff}^{(2)}$</td>
<td>Effective $\chi^{(2)}$ due to periodic poling</td>
</tr>
<tr>
<td>$\Delta \tilde{n}_q$</td>
<td>Renormalized refractive index change, see equation 2.20</td>
</tr>
<tr>
<td>$\Delta k$</td>
<td>Phase mismatch</td>
</tr>
<tr>
<td>$\Delta n(x)$</td>
<td>Refractive index change</td>
</tr>
<tr>
<td>$\mathcal{F}{\cdot}$</td>
<td>Fourier transform, $x$-plane</td>
</tr>
<tr>
<td>$\mathcal{E}$</td>
<td>Electric field amplitude</td>
</tr>
<tr>
<td>$\varepsilon_0$</td>
<td>Vacuum permittivity</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Nonlinear coupling constant</td>
</tr>
<tr>
<td>$\Gamma_q$</td>
<td>Nonlinear coupling constant, see equation 2.22</td>
</tr>
<tr>
<td>$h$</td>
<td>Planck’s reduced constant.</td>
</tr>
<tr>
<td>$\kappa_q$</td>
<td>Envelope renormalization constant, see equation 2.18</td>
</tr>
<tr>
<td>$D$</td>
<td>Electric displacement field, see equation 2.3</td>
</tr>
<tr>
<td>$P$</td>
<td>Polarization density</td>
</tr>
<tr>
<td>$\sigma_q$</td>
<td>Diffraction parameter, see equation 2.17</td>
</tr>
<tr>
<td>$A_s, A_i$</td>
<td>Amplitudes for signal and idler envelopes, see equation 2.36</td>
</tr>
<tr>
<td>$c$</td>
<td>Speed of light in vacuum</td>
</tr>
<tr>
<td>$E_q$</td>
<td>Envelope for the wave $q$, see equation 2.8</td>
</tr>
<tr>
<td>$f$</td>
<td>Lens focal length</td>
</tr>
<tr>
<td>$I_0$</td>
<td>Input pump intensity</td>
</tr>
<tr>
<td>$k_q$</td>
<td>Wavenumber for frequency $\omega_q$, i.e. $\omega_q n(\omega_q)/c$</td>
</tr>
<tr>
<td>$k_x$</td>
<td>Spatial frequency, $x$-direction</td>
</tr>
<tr>
<td>$n_q$</td>
<td>Refractive index at frequency $\omega_q$, i.e. $n(\omega_q)$</td>
</tr>
<tr>
<td>$P_0$</td>
<td>Input pump power</td>
</tr>
<tr>
<td>$P_q(z)$</td>
<td>Optical power of field $q$</td>
</tr>
<tr>
<td>$q$</td>
<td>Arbitrary subscript</td>
</tr>
<tr>
<td>$u_q$</td>
<td>Renormalized envelope, see equation 2.18</td>
</tr>
<tr>
<td>$W$</td>
<td>Pump beam input $1/e^2$ radius</td>
</tr>
<tr>
<td>$W_0$</td>
<td>Gaussian beam waist</td>
</tr>
<tr>
<td>OPG</td>
<td>Optical Parametric generation, see figure 1.3</td>
</tr>
<tr>
<td>PPLN</td>
<td>Periodically Poled Lithium Niobate</td>
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CHAPTER 1

Thesis Introduction

"Waves from moving sources: Adagio. Andante.
Allegro moderato."
- Oliver Heaviside

This report presents the work performed by Daniel Qvarngård at the Nonlinear Quantum Photonics group (NQP), Kungliga Tekniska Högskolan (KTH), Stockholm, during the months of February to May 2019. This chapter introduces the central concepts, as well as presents the aim of the thesis, and concludes with a brief outline.

1.1 Second Order Nonlinear Optics

One of the biggest theoretical developments of science in the past centuries, if not the basic foundation of all of modern physics, was the formulation of Maxwell's equations. Using the now well-known set of equations, relationships between electromagnetic quantities could be studied and predicted - and perhaps the most important prediction of all was that light behaves as an electromagnetic wave of constant vacuum phase velocity. [20] This wavelike nature of light holds the answer to questions like "why is the sky blue?" and "why do rainbows occur?" [21]

It was long believed that a sufficiently good approximation describing the interaction of the electromagnetic waves with matter was a linear one; e.g. if a green beam of light hits a material, then green light will be transmitted. And, if a wave propagates through a slab of matter, the interaction with the material causes the phase velocity to be reduced by a certain factor compared to the vacuum velocity, see figure 1.1. This factor is called the refractive index, and the effect is caused by the interaction of the electric field of the light with the constituents of the material it propagates through.

Ordinary matter consists of charged particles, electrons and atomic nuclei - with an interatomic distance of the order of 1Å (= 10^{-10}m) [12]. As visible light has a wavelength on the order of hundreds of nanometers, the atoms' point of view is that the electric field is approximately spatially uniform. Consider figure 1.2: if the charged constituents of the material are normally arranged as depicted in the top illustration, then the electric field of an incident wave of light propagating through the material displaces different charges in different directions, as depicted in the bottom illustration. The acceleration of the electric charges then induces a back reaction on the electric field - thus affecting the propagation of light. [20]
In a macroscopic picture, the relative charge displacement induced by the electric field shown in figure 1.2 can be characterized by a polarization density. The polarization density can be Taylor expanded in terms of the incident electric field components. However, to avoid unnecessary complication, the vectorial nature of the quantities involved is not important for the current discussion. For details, reader is referred to [5]. "Symbolic" Taylor expansion of the polarization density, $\mathcal{P}$, in terms of the incident electric field amplitude, $\mathcal{E}$, is given as:

$$\mathcal{P}(\mathcal{E}) = \varepsilon_0 \chi^{(1)} \mathcal{E} + \varepsilon_0 \chi^{(2)} \mathcal{E}^2 + \ldots$$

In equation 1.1, the parameters $\chi^{(1)}$ and $\chi^{(2)}$ are called the first and the second order susceptibilities, respectively, and vary depending on which material is under consideration (note: the susceptibilities are in general tensorial quantities). The first order susceptibility being related to the refractive index of the material, $n$, as $n^2 = 1 + \chi^{(1)}$. If the underlying structure of the material under consideration is noncentrosymmetric, i.e. it reacts differently to different signs of $\mathcal{E}$, then $\chi^{(2)}$ can be nonzero [5]. As $\mathcal{E}^2$ reaches zero faster for small electric field amplitudes than $\mathcal{E}$, it is clear from equation 1.1 that the linear theory only involving $\chi^{(1)}$ is a good approximation at low optical intensities. Furthermore, experimental results typically provide $\chi^{(2)}$ on the order of $10^{-12} m/V$, whereas $\chi^{(1)}$ is of the order of unity. However, since the invention of the laser, and pulsed lasers in particular, intensities of light can be achieved experimentally where the higher order
terms start to matter.

These nonlinear interactions of light and matter give rise to many interesting and outright wonderful phenomena, with applications such as:

- All-optical signal processing and telecommunication [5],
- Quantum communication [17],
- Low light imaging [18].

In many of these applications, there is a need to generate narrow-band, long-wavelength and intense sources of light with a high degree of coherence. This can typically be achieved by lasers; however, the wavelengths of laser sources are closely tied to the laser gain medium. A nonlinear process which extends the utility of laser sources to specific long wavelength applications is Optical Parametric Generation (OPG).[22] An illustration is given in figure 1.3: a laser beam, shown in green (henceforth referred to as the pump with subscript \( p \)), is shone into a nonlinear device where the dominating terms in equation 1.1 are of first and second order in \( E \). The nonlinear coupling allows for the pump beam to interact with the quantum vacuum fluctuations of two fields oscillating at different frequencies and amplify them (shown in black and yellow in the figure). The fields amplified from their vacuum energy will for historical reasons be named \( \text{signal} \) and \( \text{idler} \), ordered in decreasing frequency, and quantities involving them will be denoted by the subscripts \( s \), \( i \), respectively, e.g., \( \omega_s > \omega_i \) for their frequencies. On the photon level, a way to picture the interaction is given in figure 1.4: reading the diagram from left to right the interaction takes the form of one pump photon being converted into a signal photon and an idler photon. The condition in equation 1.2 reflects conservation of energy in the interaction, where \( h \) is the reduced Planck’s constant,

\[
\hbar \omega_p = \hbar \omega_s + \hbar \omega_i. \tag{1.2}
\]

Conservation of momentum must also be satisfied:

\[
\hbar k_p = \hbar k_s + \hbar k_i. \tag{1.3}
\]

This is also known as the phase-matching condition. However, from the relation \( k = n \omega/c \), where \( n \) is the refractive index at the frequency under study, and \( c \) is the vacuum speed of light, one can see that it is difficult to satisfy both equation 1.2 and 1.3 in dispersive media.[3] In the next subsection, a way to circumvent the issues associated with simultaneously conserving momentum and energy is presented.
Figure 1.4: Three wave mixing. The diagram can be read from any direction; the process from left to right reads as one pump photon is, through the nonlinear medium, converted into a signal and an idler photon.

1.2 Periodic Poling

As nonlinear optical processes require both conservation of energy and conservation of momentum to be efficient, they are in general hard to realize. That is, when the refractive index of the medium is wavelength dependent, it is rare to simultaneously satisfy the two equations below when \( n_p \neq n_s \neq n_i \):

\[
\omega_p = \omega_s + \omega_i \tag{1.4}
\]

\[
k_p = k_s + k_i \iff \frac{n_p \omega_p}{c} = \frac{n_s \omega_s}{c} + \frac{n_i \omega_i}{c}. \tag{1.5}
\]

However, if there is a periodic variation of the nonlinear coupling constant (more information on that is given in section 2.1), the equation for conservation of momentum is modified. Assuming that the largest Fourier component of the nonlinear coupling constant has a spatial period of \( \Lambda \), the conservation of momentum is reformulated as:

\[
k_p = k_s + k_i + \frac{2\pi}{\Lambda}. \tag{1.6}
\]

This is known as the quasi phase-matching condition [3]. Thus, periodically engineering the nonlinear coupling constant can ensure efficient conversion. An example of a material which can be engineered in such a way is Lithium Niobate (LiNbO\(_3\)). LiNbO\(_3\) is a ferroelectric material, which means that its ground state has a finite electric polarization. However, the direction of the static field is not uniquely defined: by applying patterned and strong electric fields (on the order of \( kV/mm \)) to a sample of LiNbO\(_3\), the direction of the electric polarization can be locally altered. This has the effect of changing the sign of the second order susceptibility \( \chi^{(2)} \). By periodically depositing electrodes onto the LiNbO\(_3\)-sample, ferroelectric domains of alternating polarity can be engineered [16]. Thus, \( \chi^{(2)} \) can be made to vary like a square wave, see figures 1.5 and 1.6. The end result is what is known as a Periodically Poled Lithium Niobate (PPLN).

In figure 1.7, the geometry of the problem studied in this thesis is shown. Dark areas symbolize parts of the crystal which have been exposed to strong electric fields, i.e. the darker areas represent the ferroelectric domains discussed earlier. If the poling period, \( \Lambda \), is correctly chosen to satisfy equation 1.6, then the effective \( \chi^{(2)} \) can be considered small outside the poling region. Note that the dark blue areas are designed to only invert the sign of \( \chi^{(2)} \), without affecting the first order susceptibility \( \chi^{(1)} \) (i.e. the refractive index).
Figure 1.5: Illustration of the process of periodic poling. Electrodes are deposited onto a sample of MgO-doped LiNbO$_3$, with a periodicity of $\Lambda$. Applying an electric field of amplitude $\gtrsim 6$kV/mm generates an inversion of the spontaneous polarization direction \cite{16}.

Figure 1.6: The second order susceptibility as a function of $z$.

1.3 Project Goals

Optical parametric generation experiments were in 2019 conducted by Nicklas Bjärnhall Prytz on a PPLN device doped with 5 mol% MgO, with a poling period of 7.5$\mu$m \cite{4}. The pump laser had extraordinary polarization (polarized along the $y$-axis, see figure 1.7), and wavelength $\lambda_p = 532$nm. The device was held in an open-ended oven maintaining a temperature of 75$^\circ$C. Under these conditions and according to the dispersion relationships of this material \cite{10}, equation 1.6 is satisfied for extraordinary signal and idler waves of wavelengths $\lambda_s \approx 810$nm and $\lambda_i \approx 1560$nm, respectively. An unexpected interference pattern at the pump output emerged when the incident pump laser hit the edge of the periodic poling region, c.f. figures 1.8 (a) and (b). The alternating sign of
the second order susceptibility shouldn’t have affected the pump beam appreciably, yet the diffraction pattern persisted also for low pump intensities. Moreover, at higher pump intensities, when the OPG yielded measurable signal and idler output powers, the process was found to be more efficient as compared to the case where the pump laser was incident on the center of the poling region. This despite of the fact that having the pump beam incident on the edge of the poling region, a good portion of the incident power was located outside the nonlinearly active region.

This thesis investigates the hypothesis that there might be an overall refractive index change in the poling region, which is small enough not be noticeable, but strong enough to redirect a substantial portion of the incident pump power inside the PPLN (nonlinearly active) region. The refractive index change is not assumed to be only located at the poled domains, but rather in the entire area in-between the two parallel red lines.

**Aim:**

In order to theoretically study what happens inside the device, nonlinear partial differential equations accounting for diffraction and nonlinear coupling of the waves in the presence of an inhomogeneity in the refractive index have to be solved. As the equations become impractically difficult to find exact solutions for, my work was centered around implementing numerical solutions. The main goals of the analysis were:

(i) to find a value for the change in the refractive index which theoretically replicates the experimental data found in the low input pump intensity limit,

(ii) using a numerical model, provide evidence that a refractive index change can be the cause for the OPG enhancement at the PPLN edge,

(iii) enable quantitative studies and support the experimental studies in describing the role of different parameters on the efficiency of the OPG process, e.g. the role of
the input pump position with respect to the PPLN edge.

To reach these goals, two different methods were used: one based on Fourier optics which was used to estimate the refractive index change in the low input intensity regime, and one known as the split-step method - which is commonly used in modeling nonlinear optics [2]. The split step method used the refractive index change estimated from the linear regime calculations to investigate whether the efficiency of the OPG process would be affected by an inhomogeneous refractive index.

(a) Having the pump incident in the center of the PPLN. (b) Pump output resulting from (a).

(c) Having the pump incident on the edge of the PPLN. (d) Pump output resulting from (c).

**Figure 1.8:** A low intensity pump beam incident on the center of the periodic poling region (bounded by the two red lines) seen in (a) yielded an elliptic output pattern (b). When the pump was incident on the edge of the region (c), the result was found to be modified (d). Note, the mode of incidence of the pump laser seen in (b) gives a more efficient nonlinear process - having higher signal and idler power outputs compared to the case in (a) for similar pump powers. Experiments performed by Nicklas Bjärnhall Prytz.
1.4 Thesis Outline

The main goal of this thesis was to explain the experimental results discussed in the previous section. In the next chapter, 2, a model based on the Maxwell’s equation is presented in subsection 2.1. The model was meant to simulate OPG in a medium with both a transversely varying $\chi^{(2)}$ and an inhomogeneous refractive index. I derived the equations governing the dynamics as a generalization to results given in [21]. In sections 2.2 and 2.4, analytic properties of the model are presented which give tools to better understand and verify the accuracy of the numerical methods provided in chapter 3. In chapter 4, results obtained with the numerical methods are compared with experimental data in order to verify whether the hypothesis that the PPLN has a refractive index change associated with the periodic poling region is a reasonable one. Finally, chapter 5 presents the conclusions drawn from this work and provides an outlook into future developments.
CHAPTER 2

Theory

In this section, the theoretical models are presented. In section 2.1, the coupled wave equations in an inhomogeneous medium are derived, starting from Maxwell’s equations. These equations are then renormalized to dimensionless quantities to facilitate analysis and to give a clearer intuitive picture of the role of the input parameters. In section 2.2, proofs are given for properties of the coupled wave equations which were an integral part of the numerical analysis.

2.1 Derivation of The Coupled Wave Equations

In a nonmagnetic medium, Maxwell’s equations are given by [21]:

\[ \nabla \times \mathcal{E} = -\mu_0 \frac{\partial \mathcal{H}}{\partial t}, \quad \nabla \times \mathcal{H} = \frac{\partial \mathcal{D}}{\partial t} \]
\[ \nabla \cdot \mathcal{D} = 0, \quad \nabla \cdot \mathcal{B} = 0. \quad (2.1) \]

Applying the curl operation on the top left equation of 2.1, and using the top right equation together with the commutativity of curl and temporal derivative, the generalized wave equation is obtained:

\[ \nabla \times \nabla \times \mathcal{E} = -\mu_0 \frac{\partial^2 \mathcal{D}}{\partial t^2}. \quad (2.2) \]

The electric displacement field, \( \mathcal{D} \), is related to the electric field via the constitutive relation:

\[ \mathcal{D} = \varepsilon_0 \mathcal{E} + \mathcal{P}. \quad (2.3) \]

The vector field \( \mathcal{P} \) is the polarization field, which can be Taylor expanded about the electric field components. The reader is referred to [5] for the details. Assuming that the \( \mathcal{E} \)-field is linearly polarized in the \( y \)-direction, a scalar approach is sufficient for the analysis. Expanding \( \mathcal{P}(\mathcal{E}) \) about the electric field strength and neglecting the \( \mathcal{O}(\mathcal{E}^3) \) terms, the polarization density can be written as:

\[ \mathcal{P} = \varepsilon_0 \chi^{(1)} \mathcal{E} + \varepsilon_0 \chi^{(2)} \mathcal{E}^2 := \mathcal{P}_0 + \mathcal{P}_{NL}, \quad (2.4) \]
where $\chi^{(2)}$ is the projection of the second order susceptibility on the $y$-axis. Using the vector identity $\nabla \times \nabla \times \mathbf{V} = \nabla(\nabla \cdot \mathbf{V}) - \nabla^2 \mathbf{V}$ for any arbitrary vector field $\mathbf{V}$ [1], equation 2.2 reduces to:
\begin{equation}
\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\frac{\partial^2}{\partial t^2} \left( \frac{1 + \chi^{(1)}}{c_0^2} \mathbf{E} \right) - \mu_0 \frac{\partial^2 \mathcal{P}_{NL}}{\partial t^2}.
\end{equation}

In equation 2.5, the refractive index can be identified as $n^2 := 1 + \chi^{(1)}$. The first term on the left hand side is approximately zero; as it can be rewritten as follows:
\begin{equation}
\nabla \cdot \mathbf{E} = \varepsilon_0 \nabla(\nabla \cdot \mathbf{D} - \nabla \cdot \mathbf{P}),
\end{equation}
where the first term is zero by equations 2.1; and the second term is the divergence of the electric dipole density, which should - by Gauss’s theorem - be approximately zero when considering large enough length scales compared to the charge separation [1, 20]. Thus, one ultimately arrives at the nonlinear wave equation:
\begin{equation}
\nabla^2 \mathbf{E} - \frac{\partial^2}{\partial t^2} \left( \frac{n^2}{c_0^2} \mathbf{E} \right) = \mu_0 \frac{\partial^2 \mathcal{P}_{NL}}{\partial t^2}.
\end{equation}

The nonlinear wave equation, as cast in equation 2.7, is approximately valid for any order of $\mathbf{E}$ accounted for in $\mathcal{P}_{NL}$, and for dispersive media with an inhomogeneous refractive index. In this thesis, the inhomogeneity is assumed to be of the form $n = n_0 + \Delta n(z)$, with $\Delta n \ll n_0$ such that $n^2 \approx n_0^2 + 2n_0 \Delta n(z)$. The $x$-dependence of the refractive index is taken as being zero outside the poling region and constant inside it. Henceforth, the $z-$coordinate will be taken as longitudinal, and the $x-$ and $y-$coordinates will be taken as transverse.

If the $\mathbf{E}$-field has three colinearly propagating components with nonoverlapping spectra and central frequencies at $\omega_p$, $\omega_s$ and $\omega_i$, respectively, $\mathbf{E}$ can be represented as:
\begin{equation}
\mathbf{E} = \frac{1}{2} [E_p \exp(i \omega_p t - ik_p z) + E_s \exp(i \omega_s t - ik_s z) + E_i \exp(i \omega_i t - ik_i z)] + c.c.,
\end{equation}
where the envelopes, $E_q$, vary with position only, and c.c. denotes the complex conjugate of the previous terms. Further, assume that the frequencies for the fields which will henceforth be referred to as pump ($\omega_p$), signal ($\omega_s$) and idler ($\omega_i$) satisfy the relation:
\begin{equation}
\omega_p = \omega_s + \omega_i.
\end{equation}

Then, calculating the $\mathcal{P}_{NL}$ field yields complex amplitudes for the terms oscillating at the different frequencies which are expressed as:
\begin{equation}
\begin{aligned}
P_{NL,p} &= \chi^{(2)} \varepsilon_0 E_i E_s \exp[-i(k_s + k_i)z] \exp[i \omega_p t], \\
P_{NL,s} &= \chi^{(2)} \varepsilon_0 E_i^* E_p \exp[-i(k_p - k_i)z] \exp[i \omega_s t], \\
P_{NL,i} &= \chi^{(2)} \varepsilon_0 E_i^* E_p \exp[-i(k_p - k_s)z] \exp[i \omega_i t].
\end{aligned}
\end{equation}

The Laplacian in equation 2.7 acting on the electric field given in 2.8 can be simplified by assuming that the envelopes vary slowly along their direction of propagation (the $z-$direction) compared to their respective wavelengths:
\begin{equation}
\left| \frac{\partial^2 E_q}{\partial z^2} \right| \ll 2 \varepsilon_0 \varepsilon_0 \frac{\partial E_q}{\partial z}, \quad q = \{p, s, i\},
\end{equation}
which yields
\[ \nabla^2(E_q \exp[-ik_qz]) \approx \left( -2ik_q \frac{\partial E_q}{\partial z} - k_q^2(\omega)E_q + \nabla^2_\perp E_q \right) \exp[-ik_qz]. \tag{2.12} \]

With \( \nabla^2_\perp \) defined as the Laplacian in the \((x, y)\)-plane. As the temporal exponential functions are linearly independent, the wave equation, eq. 2.7, reduces into a system of three coupled partial differential equations:

\[ -2ik_p \frac{\partial E_p}{\partial z} + \frac{2n(\omega_p)\Delta n}{c_0^2} \omega_p^2 E_p + \nabla^2_\perp E_p + \frac{\chi^{(2)} \omega_p^2}{c_0^2} E_s E_i \exp[-i(k_s + k_i - k_p)z] = 0 \]

\[ -2ik_s \frac{\partial E_s}{\partial z} + \frac{2n(\omega_s)\Delta n}{c_0^2} \omega_s^2 E_s + \nabla^2_\perp E_s + \frac{\chi^{(2)} \omega_s^2}{c_0^2} E_p E_i^* \exp[-i(k_p - k_i - k_s)z] = 0 \tag{2.13} \]

\[ -2ik_i \frac{\partial E_i}{\partial z} + \frac{2n(\omega_i)\Delta n}{c_0^2} \omega_i^2 E_i + \nabla^2_\perp E_i + \frac{\chi^{(2)} \omega_i^2}{c_0^2} E_p E_s^* \exp[-i(k_p - k_i - k_s)z] = 0. \]

Of note here is the emergence of the mismatch in momentum, or phase mismatch in the exponential factor of the nonlinear coupling terms: \( k_p - k_s - k_i \). If there is a spatial modulation of \( \chi^{(2)} \) as in figure 1.6, then it can be expanded into a Fourier series as \(15\):

\[ \chi^{(2)}(z) = \sum_m a_m \exp(-i2\pi mz/\Lambda). \tag{2.14} \]

Let the \( a_{m=1} \)-amplitude be known as \( \chi^{(2)}_{\text{eff}} \). The magnitude of the factors of the Fourier series decrease for larger \( |m| \) in order for it to converge \(19\). Hence, QPM is typically realized to first order, i.e. \( k_p - k_s - k_i \approx 2\pi/\Lambda \) \(14\). Assuming that the momentum mismatch is large for the other terms in the Fourier series so that they do not take part in the interaction, the equivalent form of the nonlinear coupling term can be taken as:

\[ \frac{\chi^{(2)}(z)\omega_p^2}{c_0^2} E_p E_i^* \exp[-i(k_p - k_i - k_s)z] \approx \frac{\chi^{(2)}_{\text{eff}} \omega_p^2}{c_0^2} E_p E_i^* \exp[-i(k_p - k_i - k_s - 2\pi/\Lambda)z]. \tag{2.15} \]

In equation 2.15, a modified version of the wavevector mismatch has emerged, so that a convenient quantity to define is \( \Delta k := k_p - k_i - k_s - 2\pi/\Lambda \). In order to facilitate interpretation of the terms in eqs. 2.13 and the analysis of the solutions to the equations, dimensionless quantities are introduced. Assuming that the input profile is approximately a Gaussian beam of \(1/e \) radius \( W_0 \), the transverse coordinates are normalized according to \((x, y) = (W_0 x, W_0 y)\). With \( L \) denoting the crystal (or interaction) length, the longitudinal coordinate is normalized to \( z = LZ \). Moreover, the dependence on the \( y \)-coordinate is neglected for the sake of simplicity and without loss of generality, as the \( \Delta n \) and the \( \chi^{(2)}_{\text{eff}} \) are taken as strictly \( x \)-dependent. Then the equation for e.g. the pump envelope is rewritten as:

\[ -i \frac{\partial E_p}{\partial z} + \frac{L}{2k_p} \frac{2n(\omega_p)\Delta n}{c_0^2} \omega_p^2 E_p + \]

\[ + \frac{L}{4(k_p W_0^2/2)} \frac{\partial^2 E_p}{\partial \tilde{x}^2} + \frac{L}{2k_p} \chi^{(2)}_{\text{eff}} \omega_p^2 E_s E_i \exp[-i\Delta k z] = 0. \tag{2.16} \]
Identifying $k_pW_0^2/2$ as the Rayleigh range of the pump beam, i.e. the distance away from the waist at which the width of a Gaussian beam has increased by a factor of $\sqrt{2}$ [21], the diffraction parameter can be introduced as:

$$\sigma_q = \frac{L}{4(k_qW_0^2/2)}. \quad (2.17)$$

Thus, the transverse Laplacians in eqs. 2.13 can be interpreted as diffraction terms. The envelopes are normalized as:

$$E_q = \kappa_qu_q, \quad \kappa_q = \sqrt{\frac{2\eta_0}{n_q}I_0}, \quad (2.18)$$

where $A$ is the relevant input area, $P_0$ the input power and $\eta_0$ is the vacuum impedance. With this normalization, $|u_p|^2 + |u_s|^2 + |u_i|^2 = P/P_0$. [21]

The nonlinear coupling parameter is defined as:

$$\gamma = \chi_{eff}^{(2)} \sqrt{\frac{\eta_0I_0}{2\eta_pn_sn_i}}. \quad (2.19)$$

Renormalizing the refractive index change as:

$$\Delta\tilde{n}_q = Lk_q\Delta n/n_p, \quad (2.20)$$

the coupled wave equations, eqs. 2.13, can be rewritten in the following form:

$$i\frac{\partial u_p}{\partial z} = \Delta\tilde{n}_pu_p + \sigma_p\frac{\partial^2 u_p}{\partial z^2} + \frac{Lk_p}{n_p} \gamma u_su_i \exp[i\Delta kz],$$

$$i\frac{\partial u_s}{\partial z} = \Delta\tilde{n}_su_s + \sigma_s\frac{\partial^2 u_s}{\partial z^2} + \frac{Lk_s}{n_s} \gamma u_i^*u_p[-i\Delta kz], \quad (2.21)$$

$$i\frac{\partial u_i}{\partial z} = \Delta\tilde{n}_iu_i + \sigma_i\frac{\partial^2 u_i}{\partial z^2} + \frac{Lk_i}{n_i} \gamma u_p^*u_p[-i\Delta kz].$$

The equations can be further simplified by redefining $u_p \to u_p \exp[i\Delta kz]$. Furthermore, abbreviated coupling constants can be introduced:

$$\Gamma_q = \chi_{eff}^{(2)} \frac{Lk_q}{n_q} \sqrt{\frac{\eta_0I_0}{2n_pn_sn_i}}, \quad q \in \{p, s, i\}. \quad (2.22)$$

The end result of this abbreviation is given by:

$$i\frac{\partial u_p}{\partial z} = (\Delta\tilde{n}_p + \Delta k)u_p + \sigma_p\frac{\partial^2 u_p}{\partial z^2} + \Gamma_p u_su_i,$$

$$i\frac{\partial u_s}{\partial z} = \Delta\tilde{n}_su_s + \sigma_s\frac{\partial^2 u_s}{\partial z^2} + \Gamma_s u_i^*u_p,$$

$$i\frac{\partial u_i}{\partial z} = \Delta\tilde{n}_iu_i + \sigma_i\frac{\partial^2 u_i}{\partial z^2} + \Gamma_i u_p^*u_p. \quad (2.23)$$
If the wavevector mismatch \( \Delta k \) is larger than \( \pi/L \), then the nonlinear coupling terms in equation 2.21 will over the course of the propagation of the waves from 0 to \( L \) pick up a phase larger than \( \pi \), and start to destructively interfere. Thus, the bandwidth of the process, also known as the \textit{QPM-condition}, is given by [3, 14, 21]:

\[
|k_p - k_s - k_i - 2\pi/L| \leq \frac{2\pi}{L}.
\]

(2.24)

The relationship in equation 2.24 will be used extensively in section 2.3.

2.2 Conservation of Total Power

\textbf{Lemma.} The total optical power is conserved for solutions to the coupled wave equations, eqs. 2.21, which vanish sufficiently quickly at large \( x \) and \( y \).

\textbf{Proof.} The normalization given in equation 2.18 means that the total electric field intensity, \( I \), is proportional to \(|u_p|^2 + |u_s|^2 + |u_i|^2\). Differentiating with respect to \( z \) gives:

\[
\frac{\partial I}{\partial z} \propto u_p \frac{\partial u_p}{\partial z} + u_s \frac{\partial u_s}{\partial z} + u_i \frac{\partial u_i}{\partial z} + \text{c.c.}
\]

If the fields \( u_p \) and \( u_s \) and \( u_i \) are solutions to eqs. 2.21, the derivatives w.r.t. \( z \) can be expressed in terms of the RHS of 2.21. Integrating over \( x \) and \( y \), the evolution of the total optical power is found to be:

\[
\frac{\partial P}{\partial z} \propto \iint_{\mathbb{R}^2} dx dy \left[ \left( -i\sigma_p u_p^* \nabla_\perp^2 u_p + i\sigma_p u_p \nabla_\perp^2 u_p^* \right) + \right.
\]

\[
\left. + \left( -i\sigma_s u_s^* \nabla_\perp^2 u_s + i\sigma_s u_s \nabla_\perp^2 u_s^* \right) + \right.
\]

\[
\left. + \left( -i\sigma_i u_i^* \nabla_\perp^2 u_i + i\sigma_i u_i \nabla_\perp^2 u_i^* \right) + iL\gamma \left( \frac{k_p}{l} - \frac{k_s}{n_s} - \frac{k_i}{n_i} \right) u_i u_s^* u_p + iL\gamma \left( \frac{k_s}{n_s} + \frac{k_i}{n_i} - \frac{k_p}{l} \right) u_s u_i u_p^* \right],
\]

where the terms proportional to \( \Delta n \) and \( \Delta k \) cancel themselves due to the complex conjugation. If the fields \( u_p, u_s, \) and \( u_i \) have the property that their magnitude, and the magnitude of their spatial derivatives, go sufficiently fast to zero for large \( (x, y) \) such that the above integral converges, then the first three terms in parentheses exactly cancel themselves by the self-adjoint property of the Laplace operator - analogous to integrating e.g. \(-i\sigma_p u_p^* \nabla_\perp^2 u_p\) by parts twice. The terms proportional to \( \gamma \) are also zero, which can be seen from:

\[
c \cdot \left( \frac{k_s}{n_s} + \frac{k_i}{n_i} - \frac{k_p}{l} \right) = \omega_s + \omega_i - \omega_p = 0.
\]

The above expression is zero due to the assumption made in eq. 2.9. Thus \( P \) is independent of \( z \). \hfill \Box
2.3 Input Conditions of Optical Parametric Noise

In order to find the input conditions for the signal and idler fields, Kleinman [13] showed that a correct input power of the fields is found when adding the energy of half a photon in the signal and idler modes into a timescale corresponding to the inverse of the spectral bandwidth:

$$P_s(0) \approx \frac{1}{2} h \nu_s \delta \nu,$$  \hspace{1cm} (2.25)

where $h$ is Planck’s constant, $\nu_s$ is the signal frequency, and $\delta \nu$ is the frequency bandwidth. A completely analogous expression is found for the idler input power $P_i(0)$. The condition can be motivated by the fact that $h \nu/2$ is the zero point energy of the photon field oscillating at $\nu$, which can be thought of as a quantum harmonic oscillator [9]. $\delta \nu$ can be found from the quasi phase-matching condition; assuming that $\delta \nu \ll \nu_s$, and keeping the pump wavevector fixed along the $z$-direction, the signal and idler wavevectors parallel to that of the pump, the QPM-condition in equation 2.24 can be written as [8]:

$$\left| k_p - k_s(\nu_s + \delta \nu) - k_i(\nu_p - \nu_s - \delta \nu) - \frac{2\pi}{\Lambda} \right| \approx$$

$$\approx \left| k_p - k_s(\nu_s) - \frac{\delta \nu}{\nu_s} \frac{\partial k_s}{\partial \nu} - k_i(\nu_i) + \frac{\delta \nu}{\nu_s} \frac{\partial k_i}{\partial \nu} - \frac{2\pi}{\Lambda} \right| \leq \frac{2\pi}{L},$$  \hspace{1cm} (2.26)

Assuming perfect phase matching at the central frequencies, the inequality reduces to:

$$\left| \delta \nu \left( \frac{\partial k_i}{\partial \nu} - \frac{\partial k_s}{\partial \nu} \right) \right| \leq \frac{2\pi}{L},$$  \hspace{1cm} (2.27)

which can be evaluated in terms of the group velocities of the medium at the signal and idler fields, respectively denoted as $v_{g,s}$ and $v_{g,i}$. From [21], equation 2.27 is rewritten as:

$$\left| 2\pi \delta \nu \left( \frac{1}{v_{g,s}} - \frac{1}{v_{g,i}} \right) \right| \leq \frac{2\pi}{L} \Rightarrow |\delta \nu| \leq \frac{1}{ \frac{L}{v_{g,i}} - \frac{L}{v_{g,s}}}. \hspace{1cm} (2.28)$$

Considering both the negative and positive range for $\delta \nu$, it is found to be twice that of the far right hand side of equation 2.28:

$$\delta \nu = \frac{2}{ \frac{L}{v_{g,i}} - \frac{L}{v_{g,s}}}. \hspace{1cm} (2.29)$$

With the input power of the signal and idler fields as given by equation 2.25, there’s still the matter of the angular distribution of the input fields. If the pump field is given by a Gaussian beam of waist $W_0$, then the $1/e^2$ radius of its distribution in reciprocal space is given by $2/W_0$:

$$\mathcal{F}\{E_p\}(k_x) = \int_{-\infty}^{\infty} dx E_p \exp(i k_x x) \propto \exp(-k_x^2/(2/W_0)^2). \hspace{1cm} (2.30)$$

Thus, the distribution of $k_x$ elements of the wavevectors in the pump field has a width $\delta k_x := 2/W_0$. Consistent with the assumptions made in deriving equations 2.21, $\delta k_x$ is
assumed to be much smaller than the magnitude of the pump wavevector $|k_p| = 2\pi/\lambda_p$. Therefore, the change in wavevector can be taken as strictly in the $x$-direction so that the largest angle wavevector of the pump field is approximately $|k_p| \hat{z} + \delta k_x$.

With this distribution of $k_x$-elements in the wavevectors of the input pump field, the corresponding width of the signal and idler fields can be found from the quasi phase-matching condition:

$$
|k_p + \delta k_x - k_s - \delta k_s - k_i - \frac{2\pi}{\Lambda}| \leq \frac{2\pi}{L}.
$$

(2.31)

Once more assuming perfect phase-matching, or $\Delta k = 0$, in the center of the distributions, the condition for the width in the signal (and idler) distribution is found to be:

$$
|\delta k_x - \delta k_s| \leq \frac{2\pi}{L} \Rightarrow \delta k_s = \frac{2}{W_0} + \frac{2\pi}{L}.
$$

(2.32)

Thus, the square absolute value of the angular distributions can be assumed to be of the form:

$$
|\mathcal{F}\{E_s\}(k_x)|^2 = |A_s|^2 \exp\left(-\frac{2k_x}{W_0 + \frac{2\pi}{L}}\right).
$$

(2.33)

With the distribution in equation 2.33, the magnitude of the constant $A_s$ can be found from the condition 2.25. From [21], the power in an electric field $E(x, y)$ propagating through a medium of refractive index $n$ is given by:

$$
P = \frac{n}{2\eta_0} \int_{-\infty}^{\infty} dx dy |E(x, y)|^2 = \frac{n}{2\eta_0} \sqrt{\frac{\pi W^2}{2}} \int_{-\infty}^{\infty} dx |E(x)|^2,
$$

(2.34)

where $\eta_0$ is the vacuum impedance, $\approx 377$ Ohms [20], and $W$ is the radius of the beam which is included to account for the integration over $y$. According to Parseval’s identity [15], the above can be reformulated as:

$$
P = \frac{n}{2\eta_0} \sqrt{\frac{\pi W^2}{2}} \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} |\mathcal{F}\{E\}(k_x)|^2.
$$

(2.35)

Entering equation 2.33 into equation 2.35, and requiring that the power satisfies the condition in 2.25, the magnitude of the constant $A_s$ is found:

$$
|A_s|^2 = \frac{2\eta_0}{\pi n_s} \frac{(\frac{1}{2}h\nu_s)\delta \nu}{W \delta k_s}.
$$

(2.36)

As for the input phase of $\mathcal{F}\{E_s\}(k_x)$, the method of estimation is given in section 3.1.

### 2.4 The Non-depleted Pump Approximation

An approximate solution to the coupled wave equations, eqs. 2.21, in the case of $\Delta n = 0$ can be found by assuming that the power of the pump remains constant throughout the propagation.[14] That is, it’s not appreciably affected by the nonlinear coupling to the other fields. Moreover, it is assumed that each wave can be described by a plane wave so that the distribution in the transverse plane is constant. In that case, the equation for the pump reduces to:

$$
\frac{\partial u_p}{\partial z} = 0.
$$

(2.37)
This is known as the non-depleted pump approximation. Given equation 2.37, the equations for the signal and idler fields decouple (via differentiating w.r.t. \( z \)) into:

\[
\frac{d^2 u_s}{dz^2} = \Gamma_s \Gamma_i u_s(z) |u_p|^2 - i\Delta k \frac{du_s}{dz},
\]

\[
\frac{d^2 u_i}{dz^2} = \Gamma_s \Gamma_i u_i(z) |u_p|^2 - i\Delta k \frac{du_i}{dz},
\]

(2.38)

where, as previously discussed, \( |u_p|^2 \) is a constant (unity in this normalization). The solutions of these decoupled ordinary differential equations are of a hyperbolic nature, with a prefactor (gain) multiplying \( z \) in the exponent given by:

\[
b = \sqrt{\Gamma_s \Gamma_i |u_p|^2 - (\Delta k/2)^2}.
\]

(2.39)

The solutions to equation 2.38 can therefore be written as:

\[
u_s(z) = \left[ u_s(0) \cosh(bz) + \frac{1}{b} \left( \frac{i\Delta k}{2} u_s(0) - i\Gamma_s u_s^*(0) u_p \right) \sinh(bz) \right] \exp(-i\Delta k z/2),
\]

\[
u_i(z) = \left[ u_i(0) \cosh(bz) + \frac{1}{b} \left( \frac{i\Delta k}{2} u_i(0) - i\Gamma_i u_i^*(0) u_p \right) \sinh(bz) \right] \exp(-i\Delta k z/2).
\]

(2.40)

The intensity as a function of the propagation distance for the signal and idler fields can be found by taking the absolute values squared of the solutions in 2.40. Assuming \( \Delta k = 0 \):

\[
|u_s(z)|^2 = |u_s(0)|^2 \cosh^2(bz) + \frac{\Gamma_s^2}{b^2} |u_i(0)|^2 |u_p|^2 \sinh^2(bz) +

+ \frac{\Gamma_i^2}{b^2} |u_p||u_s(0)||u_i(0)| \sinh^2(bz) \sin(\Delta \varphi),
\]

\[
|u_i(z)|^2 = |u_i(0)|^2 \cosh^2(bz) + \frac{\Gamma_i^2}{b^2} |u_s(0)|^2 |u_p|^2 \sinh^2(bz) +

+ \frac{\Gamma_s^2}{b^2} |u_p||u_s(0)||u_i(0)| \sinh^2(bz) \sin(\Delta \varphi),
\]

(2.41)

where \( \Delta \varphi = \angle u_p - \angle u_s(0) - \angle u_i(0) \) is the input phase difference. The form of the intensity growth in equation 2.41 shows that the signal and idler fields scales versus the pump at large, positive \( z \) as:

\[
|u|^2 \sim \exp(2bz) = \exp(2\sqrt{\Gamma_s \Gamma_i |u_p|^2 - (\Delta k/2)^2} z).
\]

(2.42)

The above relation shows that in the absence of a wavevector mismatch \( \Delta k \), plotting the logarithm of the intensity at a fixed \( z \) versus the square root of the input pump intensity yields a linear graph. Assuming that the pump power can be regarded as uniformly distributed over an area of \( \pi W_p^2/2 \), a graph of \( \log[P_s(L)/P_s(0)] \) versus the square root of the input pump power, \( \sqrt{P_0} \), should have a slope of approximately:

\[
\frac{L_{\chi_{eff}}^{(2)}}{n_s n_i} \sqrt{\frac{k_s k_i}{\eta_0 n_p}} \sqrt{\frac{\pi W_p^2}{2}}.
\]

(2.43)
Further information is given in section 4.2 - where the above result was used to validate the behavior of the numerical solution. What is also clear is that the presence of a phase mismatch reduces the efficiency of the nonlinear process in this approximation, highlighting the importance of maintaining a small $\Delta k$. 
CHAPTER 3

Method

In this chapter, the focus is on the numerical implementations that have been used to solve the equations modeling the wave dynamics. In section 3.1, the input conditions used for the problem are presented. The pump beam input condition was given from the experimental conditions, and the signal and idler input amplitudes were estimated as described below. The methods used to propagate the input amplitudes are presented in section 3.2. Moreover, a method used to validate the results is presented in section 3.2.3. It is based on the conservation of total optical power as shown in section 2.2.

3.1 Modeling the Input Conditions

To mimic the experimental pump laser conditions, the input condition for the pump amplitude was set as in figure 3.1: as a Gaussian beam, with its waist at $z = \frac{L}{2}$ in the absence of any nonlinear coupling or refractive index change.

![Image](image-url)

**Figure 3.1:** In the absence of any refractive index change or nonlinear coupling to the other fields, the pump beam should behave as a Gaussian beam with its waist in the center of the OPG device.

As the input phase distributions of the signal and idler fields in the reciprocal space were unknown, a method of estimation was developed. The idea was to distribute their input power within the input pump beam $1/e^2$ radius, $W$. From equation 2.33, the square absolute value of the distribution was assumed to be given by:
\begin{equation}
|\mathcal{F}\{E_s\}(k_x)|^2 \propto \exp\left(-\frac{2k_x^2}{(2\pi/L + 2/W_0)^2}\right).
\end{equation}

The phase was estimated so that the input power distribution of the signal and idler fields was located within the width of the pump. In order to be able to analytically solve the integrals required, the input condition for the signal envelope was assumed to be of the form:

\begin{equation}
\mathcal{F}\{E_s\}(k_x) = A_s \exp(-i\alpha_0 - ia_1 k_x - ia_2 k_x^2) \exp\left(-\frac{k_x^2}{(2\pi/L + 2/W_0)^2}\right).
\end{equation}

Performing an inverse Fourier transformation and solving the resulting Gaussian integral, the result is given by:

\begin{equation}
E_s = \frac{A_s}{2\pi} \exp(-i\alpha_0) \sqrt{\frac{\pi}{ia_2 + 1/\Delta k_s^2}} \exp\left(-\frac{(x + a_1)^2}{4(ia_2 + 1/\Delta k_s^2)}\right).
\end{equation}

Thus, the parameter $a_0$ sets an overall phase, $a_1$ is a measure of translation in real space, and $a_2$ modifies the width of the input beam. From equation 2.41, the output power depends approximately sinusoidally on the parameter $a_0$ in the nondepleted pump approximation. Therefore, $a_0$ was instead chosen to a zero input phase difference. This would then, by equation 2.41, yield the mean value and since the experimental result is the average over a large number of optical cycles these values should then be comparable. The other parameters were set randomly with a uniform distribution so that the full $1/e^2$ width of the signal (and similarly for the idler) input intensity occurred within the $1/e^2$ width of the pump intensity. See equations 3.4 and 3.5 for the bounds of the distributions, and figure 3.2 for an example of the resulting inputs.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure3.2.png}
\caption{Example distribution of the input signal and idler intensities. Note that most of the intensity of the input conditions is distributed inside the $1/e^2$-width of the pump, shown by the red dashed lines.}
\end{figure}
\[-\sqrt{\frac{1}{2\delta k_s^2} \left( \frac{W^2}{2} - \frac{2}{\delta k_s^2} \right)} \leq a_2 \leq \sqrt{\frac{1}{2\delta k_s^2} \left( \frac{W^2}{2} - \frac{2}{\delta k_s^2} \right)} \]  

(3.4)

\[W - \sqrt{4 \left( \frac{1}{\delta k_s^2} + \delta k_s^2 a_2^2 \right)} \leq a_1 \leq \sqrt{4 \left( \frac{1}{\delta k_s^2} + \delta k_s^2 a_2^2 \right)} - W\]  

(3.5)

### 3.2 Simulation of the Beam Propagation

As the differential equations presented in section 2.1 were difficult to solve analytically, two numerical approaches were implemented in order to investigate the behavior of the model and whether the hypothesis of a refractive index change inside the poling region was a reasonable one. Common to both approaches was a discretization of the solution domain, as shown in figure 3.3. \(\Delta z\) denotes the step size used in \(z\), and \(\Delta x\) analogously for \(z\). Moreover, \(\chi_{\text{eff}}^{(2)}\) was just as \(\Delta n\) modeled to be zero outside the poling region, as the wavevector mismatch is so large in this region that the nonlinear coupling will be very inefficient; practically nonexistent [3].

![Figure 3.3: Illustration of the numerical grid, and the imaging lens](image)

When the fields propagated numerically as per the methods given below, the action of the lens used to image the OPG output at the end facet of the PPLN device was also simulated in order to compare with the patterns recorded experimentally. As the imaging lens focal plane was placed at the output facet (see figure 3.3), the imaging plane was simply related to the output plane \(z = L\) by a Fourier transformation, albeit with a re-scaling in the \(z\)–coordinate. If the discretization in the \(z\)–plane had a spacing \(\Delta x\) and a number of points \(N_x\), the corresponding coordinates in the imaging plane was chosen to be sampled by spatial increments of [21]:

\[\Delta \xi = \frac{\lambda f}{N_x \Delta x},\]  

(3.6)
where $\lambda$ is taken as the wavelength inside the material, and $f$ is the focal length of the lens.

### 3.2.1 Fourier Optics Approach

The simulation is in this instance based on linear propagation in the medium using a scalar wave model and Fourier optics. This method does not incorporate the nonlinear coupling terms in the coupled wave equations, 2.21, but the utility of using this method was two-fold. Firstly, the nonlinearities were time-consuming, and part of the problem only concerned investigating whether a small refractive index change could reproduce the outputs found in the experiment for low pump powers, which according to the model derived in section 2.1 reduce to a linear problem. Secondly, the method was later used to bug-test the program which did incorporate the nonlinearities in the limit of low powers, as the different approaches should behave similarly in that limit.

The treatment of the Fourier optics approach to the problem given here is based on chapter four of [21]. Let $\mathcal{F}$ denote Fourier transformation in the plane perpendicular to the propagation axis. Then, the spatial frequencies of a complex amplitude $u(x, z)$ in some plane, say $z = 0$, can be found from $\hat{u}(k_z) = \mathcal{F}\{u(x, 0)\}(k_z)$. In other words, the Fourier transform decomposes any complex amplitude into its plane wave components. Defining the kernel for translation as:

$$K(k_z; z) = \exp\{-i \sqrt{k^2 - k_z^2} z\}, \quad (3.7)$$

the complex amplitude in any plane along the propagation direction can be found from that at $z = 0$ via:

$$u(x, z) = \mathcal{F}^{-1}\{K(k_z; z)\mathcal{F}\{u(x, 0)\}\}. \quad (3.8)$$

The interpretation of equation 3.8 is that the complex amplitude in the initial plane is decomposed into its plane wave components, each of which is then advanced in phase by an amount corresponding to the equivalent $k$-vector element in the $z$-direction times the propagation distance before returning to real space coordinates with the inverse Fourier transform.

In order to model the refractive index difference in the two domains, the complex amplitude at the input plane was advanced in short steps $\Delta z$, as in figure 3.3, and the resulting amplitude was corrected in phase by multiplying with a factor of $\exp\{-2\pi i \Delta z \Delta n(x)/\lambda\}$. Note: for short steps $\Delta z$, $k_z \Delta z \approx k \Delta z$.

### 3.2.2 Split-Step Method

The split-step method was implemented in order to account for the nonlinear coupling between the pump, signal and the idler. The treatment of the method given here is based on [2] and [7].

The equations in 2.21 can be written in the form:

$$\frac{\partial u}{\partial z} = (\hat{D} u + \hat{N}(u)), \quad (3.9)$$

where $\hat{D}$ is a linear differential operator (the diffraction terms), and $\hat{N}$ is an operator containing the inhomogeneity terms and the nonlinear mixing terms. Both operators can
be cast as matrix operators, the details can be found in [7]. If the operators in the above expression - equation 3.9 - were numbers instead, it would imply the following solution to the first-order differential equation in $z$:

$$u(z) = \exp(\hat{D}z + \hat{N}z)u(0). \quad (3.10)$$

Indeed, this can be seen to be the solution to eq. 3.9, with the operator $\hat{D} + \hat{N}$ acting as a generator of translation along $z$. If the operators commuted, the analysis would be simplified as one could then exponentiate them separately. However, as $\hat{N}$ depends on $u$, which in turn depends on $x$, it doesn’t commute with $\hat{D}$. However, for small step sizes $\Delta z$, and with the Baker-Hausdorff formula, the problem might be simplified [2]:

$$\exp(\hat{N}\Delta z)\exp(\hat{D}\Delta z) = \exp\left(\hat{D}\Delta z + \hat{N}\Delta z + \frac{\Delta z^2}{2}[\hat{N}, \hat{D}] + \frac{\Delta z^3}{12}[\hat{N}, [\hat{N}, \hat{D}]] + \ldots\right). \quad (3.11)$$

The split step method utilizes this relationship by assuming that for small step sizes the evolution of the pump, signal and idler fields can be treated as purely diffractive ($\exp(\hat{D}\Delta z)$), followed by a purely nonlinear step ($\exp(\hat{N}\Delta z)$). This scheme, then, neglects the commutator between the two operators which is a term proportional to $\Delta z^2$ in the exponent. The second order term can be included by a procedure discussed in [2]: by taking first a diffractive step of half a step length, followed by a series of alternating steps and ending with a linear half-step backwards:

$$u(x, z) \approx \exp(-\hat{D}\Delta z/2) \prod \exp(\hat{D}\Delta z)\exp(\hat{N}\Delta z)\exp(\hat{D}\Delta z/2)u(x, 0). \quad (3.12)$$

The diffractive steps can be evaluated by means of Fast Fourier Transformation. As the Fourier transform of the linear differential operator acting on the fields is just a number, the action of the diffractive operator $D_q = -i \sigma_q \partial^2 / \partial x^2$ can be estimated via:

$$\exp(\hat{D}_q\Delta z)u_q(x, z) = \mathcal{F}^{-1}\{\exp(i\sigma_q k_q^2 \Delta z)\mathcal{F}\{u_q(x, z)\}\}, \quad (3.13)$$

where the "Fourier transform" of the diffraction operator has been written explicitly.

The nonlinear step can be performed by a Runge-Kutta solver. An example of such a solver can be found in [11]; consider a first order differential equation of the form:

$$\frac{dy}{dx} = f(x, y). \quad (3.14)$$

Then, for a small step $\Delta x$, the advancement of $y$ can be approximated by:

$$y(x + h) = y(x) + \Delta x[f(x, y) + f(x + \Delta x, y + \Delta x f(x, y))]/2 + \mathcal{O}(\Delta x^3). \quad (3.15)$$

In order to apply the Runge-Kutta approach to the equations in 2.23 modifications have to be made in order to account for the coupling between the equations. In analogy to equation 3.15, the interaction terms and the refractive index change terms for the pump, signal and idler fields are denoted by $N_p$, $N_s$, and $N_i$, respectively. For example,

$$N_s(u_p, u_s, u_i) := \Delta n_s(x)u_s + \Gamma_s u_i^* u_p. \quad (3.16)$$
Henceforth, \( u_q(x, z) \) is denoted \( u_q \) for brevity. The Runge-Kutta implementation of \( \exp(\hat{N}\Delta z)u \) is given by:

\[
\begin{align*}
  u_p(x, z + \Delta z) & \approx u_p + \frac{\Delta z}{2} [N_p(u_p, u_s, u_i) + \\
  & + N_p(u_p + \Delta z N_p(u_p, u_s, u_i), u_s + \Delta z N_s(u_p, u_s, u_i), u_i + \Delta z N_i(u_p, u_s, u_i))], \\
  u_s(x, z + \Delta z) & \approx u_s + \frac{\Delta z}{2} [N_s(u_p, u_s, u_i) + \\
  & + N_s(u_p + \Delta z N_p(u_p, u_s, u_i), u_s + \Delta z N_s(u_p, u_s, u_i), u_i + \Delta z N_i(u_p, u_s, u_i))], \\
  u_i(x, z + \Delta z) & \approx u_i + \frac{\Delta z}{2} [N_i(u_p, u_s, u_i) + \\
  & + N_i(u_p + \Delta z N_p(u_p, u_s, u_i), u_s + \Delta z N_s(u_p, u_s, u_i), u_i + \Delta z N_i(u_p, u_s, u_i))].
\end{align*}
\]

(3.17)

The specific form of the Runge-Kutta advancement has been chosen so as to have the same leading order error (\( \mathcal{O}(\Delta z^2) \)) as the split-step scheme in 3.12.

### 3.2.3 Rate of Convergence for the Numerical Schemes

In order to quantify the error induced by the finite step lengths, lemma 2.2 was used. As optical power is conserved in the coupled wave equations, eqs. 2.21, a measure of the quality of the numerical solution was defined in terms of the output power difference. Let \( P_0 \) denote the input power, and \( P = \int dx(|u_p(x, L)|^2 + |u_s(x, L)|^2 + |u_i(x, L)|^2) \) denote the output power, then the fractional error in power is defined as:

\[
\varepsilon_P = \frac{|P_0 - P|}{P_0}.
\]

(3.18)

Computing the fractional power error for the Fourier optics approach in the linear regime (without signal and idler fields), and with the split step code running with the parameters presented in table 3.1, the result was the convergence rates seen in figure 3.4.

<table>
<thead>
<tr>
<th>Pump</th>
<th>Signal</th>
<th>Idler</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sigma_p = 0.42 )</td>
<td>( \sigma_s = 0.66 )</td>
<td>( \sigma_i = 1.28 )</td>
</tr>
<tr>
<td>( \Gamma_p = 32.1 )</td>
<td>( \Gamma_s = 21.1 )</td>
<td>( \Gamma_i = 11.0 )</td>
</tr>
<tr>
<td>( \Delta n_p = 11.3 )</td>
<td>( \Delta n_s = 7.45 )</td>
<td>( \Delta n_i = 3.87 )</td>
</tr>
</tbody>
</table>

### 3.3 Estimating the Refractive Index Change

In order to estimate a value for the refractive index change, a collection of experimental results for low input powers were compared with a set of simulated outputs for varying \( \Delta n \) parameters, until a roughly corresponding functional dependence on \( x \) was found between
3.3. Estimating the Refractive Index Change

Figure 3.4: Convergence rate for the numerical schemes. Since the split step had a third order error in the step size, the logarithmically scaled error graph has a slope of three.

the two sets of data. As both numerical methods only solved for the $y = 0$ plane, cross-sections of the transverse $(x, y)$ experimental diffraction patterns were taken, see figure 3.5. Since the actual values for each experimentally recorded pixel depended on the exposure time setting used, the numerical and experimental pump intensities were normalized to their peak values to simplify analysis and to avoid unnecessary complication.

Figure 3.5: Example of one of the recorded diffraction patterns after the imaging system at low input pump intensities. (a) shows an intensity distribution in the transverse plane, with a red line indicating the position of the graph shown in (b).

(a) Diffraction image. Red line indicates cut at $y = 0$.

(b) Intensity distribution.
CHAPTER 4

Results and Discussion

4.1 Estimation of the Refractive Index Change

Investigations of the behavior of the pump beam caused by the inhomogeneous refractive index - neglecting the nonlinear interactions - were performed via both the Fourier optics approach and the split step method, both simulating a Gaussian beam of waist $W_0 = 30\mu m$. In figure 4.2, the top-down view of the intensity distribution is shown. In between the two red lines the refractive index is $4.8 \times 10^{-5}$ higher than outside. A diffraction pattern clearly emerges along the direction of propagation.

Mapping the distribution at the output facet ($z = 2cm$) to the imaging plane resulted in figure 4.3, where the results of simulation (solid line) had been overlaid onto experimental results (dotted line) for ease of comparison. The correspondence between the two sets of points was found to be quite good for $\Delta n = (5 \pm 1) \times 10^{-5}$, especially for the first two peaks from the right. There was slight disagreement further away from the center ($x = 0$) which might be attributed to aberrations in the optical system becoming more prominent away from the optic axis of the imaging system, but the order of magnitude of the refractive index appeared to be reasonable and consistent with data from the literature [6].
**Figure 4.1:** Propagation of the pump beam intensity when incident on the center of the poling region, whose edges are shown as the two red lines. Since only the linear behavior is considered in this instance the color bar has been normalized to the peak intensity at the beam waist ($I_0$). Simulation parameters: waist radius $W_0 = 30\mu m$, vacuum wavelength $\lambda_p = 532nm$, extraordinary refractive index according to [10] for temperature 75°C.

**Figure 4.2:** Pump beam diffraction resulting from a small linear refractive index change and the pump incident right on the edge of the poling region (shown in red, and situated at $x = 250\mu m$). Here the value $\Delta n = 4.8 \times 10^{-5}$. It is noteworthy that the peak intensity (at $z \approx 0.5cm$) is over 1.5 times larger than the input peak intensity ($I_0$). Simulation parameters: waist radius $W_0 = 30\mu m$, vacuum wavelength $\lambda_p = 532nm$, extraordinary refractive index according to [10] for temperature 75°C.
Figure 4.3: Experimental (dotted line) and simulated (solid) diffraction patterns. Simulation parameters: waist radius \( W_0 = 30 \mu m \), vacuum wavelength \( \lambda_p = 532nm \), extraordinary refractive index according to [10] for temperature 75°C.

4.2 OPG Enhancement

Having obtained an estimate of the refractive index change associated with the edge of the periodic poling region from fitting simulations to experimental data, the next step was to investigate what impact this effect would have on the optical parametric generation process and gain insights into the experimentally observed OPG gain enhancement.

In figure 4.4, comparisons are made between having the pump centered on the middle of the periodic poling region and displaced by 230\( \mu m \) towards the edge. Figure 4.4 (a) and (b) show the pump intensity propagation for the two cases, respectively. The most notable difference between (a) and (b) is that the pump laser was found to have a region (between \( z = 0.5 \) and \( z = 1cm \)) where the pump intensity was noticeably higher. As the process is nonlinear, this increase in pump intensity caused by the refractive index change might explain the experimentally observed enhancement of the OPG process. Comparing the two cases to the signal intensity distributions seen in (c) and (d), an apparent increase in the signal intensity was found, similar to the experimental results discussed in section 1.3. Highlighting the output facet (\( z = 2cm \)) signal intensity distributions yielded figure 4.4 (e), where the increase in signal intensity as the pump is grazing the periodic poling region is clear.

If the pump wave is not depleted, then from the scaling law in equation 2.42, a plot of the logarithm of the signal power versus the square root of the pump input power should give a linear segment. Indeed, this behavior was found in figure 4.4 (f), up to the regime indicated by the vertical black line where the non-depleted pump approximation ceased to be valid \((P_0 \gtrsim 3kW)\). Moreover, the numerical method was - in contrast to the approximate analytical solution - able to resolve the impact of the diffraction operators \( \nabla_1^2 \). This ability to resolve the \( x \)-dependence showed that when the pump was incident on the poling edge as in figure 4.4 (b), there was in effect an increase in the gain of the device which can be seen from the slightly different slopes in figure 4.4 (f). Computing the slopes of the linear segments seen for pump powers less than 3\( kW \), it was found that the gain factors for the two cases in 4.4 (a) and (b) were \( 0.6234W^{-0.5} \).
and $0.6379 W^{-0.5}$, respectively. This can be compared with the gain factor found from equation 2.42: $0.6063 W^{-0.5}$, which agrees quite well with the numerical values despite the crudeness of the approximations made in section 2.4. In figure 4.5, experimental data given by Nicklas Bjärrhäll Prytz shows the signal output power versus the peak input pump power. Having the pump centered in the middle of the poling region (circles) and 20 $\mu m$ from the edge of the poling region (squares) gave two different responses. Fittings to first order polynomials were performed (solid lines), so that the slopes could be computed. It was found that the experimental gain factors for the two different cases in figure 4.4 (a) and (b) were, respectively, $0.4103 W^{-0.5}$ and $0.4365 W^{-0.5}$. The fractional change was found to be $\approx 6.38\%$. The disagreement between numerical and simulated gain factors could stem from e.g.:

- uncertainty in the pump waist radius, input power and incidence angle compared to the normal of the input facet,

- the pump approaching the depletion regime, c.f. the range of the abscissa in figure 4.5 to the depletion region indicated by the vertical black line in figure 4.4 (f).

Despite the disagreements, the behavior of the parameters appeared similar in both simulation and experiment. Thus it was concluded that the numerical model could replicate the data.
Figure 4.4: Role of the transverse pump input coordinate. Parameters used: Pump waist and power, respectively: $W_0 = 30 \mu m$, $P_0 = 2700 W$. Wavelengths of pump, signal and idler: 532.0, 808.2, 1557 nm, respectively. Extraordinary refractive index at each wavelength taken according to [10] for a temperature of 75°C. $\Delta n = 4.8 \cdot 10^{-5}$. $\chi_{eff}^{(2)} = 24 \mu V/m$, with a poling period of $\Lambda = 7.5 \mu m$. 

(a) Pump incident in the middle. 

(b) Pump centered at $x = 230 \mu m$. 

(c) Resulting signal from (a). 

(d) Resulting signal from (b). 

(e) Signal intensity distribution, $z=2 cm$. Blue corresponds to (c) and orange to (d).

(f) $P_0$-dependence of signal output power. Pump depletion to the right of the vertical black line.
Figure 4.5: Signal power versus the square root of the peak input pump power for the case of the pump incident on the edge (blue) and in the middle (orange). Pump waist $W_0 \approx 30\mu m$, vacuum wavelength $\lambda_p \approx 532nm$ and a bandpass filter for $\lambda \approx 810 \pm 5nm$ used for measuring the signal power. $P_s(0)$ defined as per equation 2.25.

After verifying that the model was able to capture the essential features of the nonlinear wave interaction in the reference case and at the edge of the PPLN, the next part of the project was to extract quantitative indications of the sample performance.

Experimental data for the signal output power as a function of the pump center coordinate along the transverse ($x$) direction - i.e. from the center of the periodic poling region towards the edge - is shown as dotted lines in figure 4.6. Overlaid is the corresponding simulation. Since the experiment used a pulsed laser, whereas the simulation neglected the time dependence, the numerical result is shown in arbitrary units. However, the similar functional dependencies of the different sets of data was considered a solid piece of evidence that the refractive index change found in section 4.1 was a reasonable explanation for the increase in the OPG efficiency.
Figure 4.6: Experimental (dotted lines) and simulated (solid, purple line) data describing the output signal power versus the x-coordinate of the center of the pump beam. Similar functional dependence was found between the sets of data. PPLN edge at $x = 250 \mu m$. Parameters used: Pump waist and power, respectively: $W_0 = 30 \mu m$, $P_0 = 2700 W$. Wavelengths of pump, signal and idler: 532.0, 808.2, 1557 nm, respectively. Refractive index at each wavelength taken according to [10] for a temperature of 75°C. $\Delta n = 4.8 \cdot 10^{-5}$. $\chi^{(2)}_{eff} = 24 \mu V/m$, with a poling period of $\Lambda = 7.5 \mu m$. 
In this thesis, I have developed a model to explain counterintuitive experimental results obtained from a nonlinear optical device intended for optical parametric generation (OPG). In OPG, nonlinear interactions couple an incident high-intensity pump laser beam to quantum vacuum fluctuations in the electric field at other frequencies (signal and idler) which are then amplified along the propagation direction.

The intention of the fabrication process of the OPG-device was to locally alter the second order susceptibility only (as opposed to altering the first order as well), which should only affect incident light of high intensities. However, it was found that if a low intensity laser grazes the local modifications as in figure 1.8 (b), the end result resembles effects which are linear in nature, and thus the hypothesis proposed was that there is indeed a modification of the first order susceptibility induced by the fabrication process. Thus I developed a model as presented in section 2.1 which incorporated the phenomenon, in the form of a refractive index change. Furthermore, it was found that for certain planes of incidence the OPG process was made more efficient with an increase in the output power of the generated waves. To explain the observed phenomena, the goals I set out to achieve were:

(i) to find a value for the change in the refractive index which theoretically replicates the experimental data found in the low input pump intensity limit,

(ii) using a numerical model, provide evidence that a refractive index change can be the cause for the OPG enhancement at the PPLN edge,

(iii) enable quantitative studies and support the experimental studies in describing the role of different parameters on the efficiency of the OPG process, e.g. the role of the input pump position with respect to the PPLN edge.

Below, each goal is treated in turn:

(i): Numerically simulating the pump wave propagation for a set of different refractive index changes, it was found that a local variation of $4.8 \cdot 10^{-5}$ gave an output similar to that found in experiments, as seen in figure 4.3. There, the experimental (dotted line) and the simulated (solid line) have both been included for ease of comparison. The slight disagreements might be explained by aberrations in the optical system, as well as the fact that the spatial distribution of the pump laser was not exactly Gaussian in the experiments, see for instance figure 1.8 (b). The disagreements, the correlation was nevertheless
considered good enough for the purpose of reliably estimating a small refractive index change; and since the estimate might have been difficult to acquire experimentally, the method was considered successful. The refractive index change found using the numerical approach was used to investigate whether this model would provide an enhancement in the efficiency of the nonlinear process.

(ii): Simulating the nonlinear wave propagation at an input intensity of $\approx 2W/\mu m^2$, and having the input pump profile translated along the input facet relative to the refractive index change showed that, indeed, a more efficient optical parametric generation process took place when the pump was centered 20$\mu m$ within the edge of the periodically poled region. (iii): The results presented in figure 4.4 then led to a specific set of measurements which were then compared to the theoretical model. In figure 4.6, experimental (dotted lines) and simulated (solid, purple line) results for the signal output power versus the translation of the input pump plane compared to the refractive index change boundary. The functional behavior of the output signal power was found to be quite similar in both cases, and it was seen that as the pump was incident on the refractive index change boundary the output signal power increased by almost a factor of two compared to the reference case. Thus, all the original targets of the thesis were successfully accomplished and the work presented here will together with experimental results be part of an upcoming publication [4].

Suggestions for further work are given along two branches: experimental material science, and improving the numerical model.

In terms of further work in material science, the suggestion is to determine the cause for the inhomogeneity as this knowledge might be more useful for studying other devices engineered from PPLN. Due to the doping of the lithium niobate by magnesium oxide, one possible explanation is the accumulation of impurities at the ferroelectric domain walls.

Of a more theoretical nature, the model could further be improved by including the pulse shape of the input laser in order to be able to study e.g. pulse distortions and output pulses and how these phenomena are affected by an inhomogeneous refractive index. Moreover, modeling different pump incidence angles could possibly give a larger increase in the efficiency gain at the edge of compared to the center of the periodic poling region, due to the reflection of the pump beam at the boundary. As it has been presented in this work, the model could in future investigations be used to study the interplay between different input pump profiles and inhomogeneous refractive index domains. The input pump profile may be shaped via diffractive optical components, and the numerical model presented in this thesis might be used to find input conditions which - coupled with an inhomogeneous refractive index - give rise to new spatial phenomena, and e.g. more efficient nonlinear processes.
Bibliography


