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Stokes' hypothesis is invalid: A flow-dependent criterion for neglecting bulk viscosity☆

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Dedicated to the memory of Kumbakonam R. Rajagopal

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ABSTRACT

In modeling flows of compressible fluids, it is common practice to set the bulk viscosity equal to zero. This assumption, known as Stokes' hypothesis, has been the subject of debate since its introduction by Stokes in 1845. In this work, we present continuum-mechanical, experimental, and atomistic arguments showing that Stokes' hypothesis is not valid for any real fluid. Nevertheless, it is well known that modeling approaches based on this assumption often yield results of acceptable accuracy in practical applications. We show that this apparent success is not primarily due to the bulk viscosity being zero or negligibly small. Rather, whether bulk-viscous effects may be neglected depends on a careful analysis of the flow under consideration. We present a novel kinematic criterion that replaces Stokes' constitutive hypothesis and determines in which flows bulk-viscous effects may be neglected.

1. Introduction

In the modeling of flow, it is crucial to accurately describe the fluid's response to external stimuli. This is usually achieved through a constitutive relation linking the Cauchy stress tensor $\mathbf{T}$ to the symmetric part of the velocity gradient

$$\mathbf{D} = \frac{1}{2}\left(\nabla \mathbf{v} + (\nabla \mathbf{v})^T\right).$$

The cornerstone of such constitutive relations is the Navier–Stokes fluid model, introduced by George Gabriel Stokes in 1845 (Stokes, 1845). In modern notation it reads

$$\mathbf{T} = -p_{th}\mathbf{I} + \lambda\,(\operatorname{tr}\mathbf{D})\,\mathbf{I} + 2\mu\mathbf{D}, \tag{1}$$

where p_{th} denotes the thermodynamic pressure, λ the second viscosity, and μ the shear viscosity. Note that $\operatorname{tr}\mathbf{D} = \operatorname{div}\mathbf{v}$. In general, the material parameters λ and μ may depend on the fluid density ρ and the temperature θ, while the thermodynamic pressure p_{th} is determined by an equation of state, e.g., the ideal gas law.

It is common to rewrite the constitutive relation (1) by decomposing $\mathbf{D}$ into its isotropic part $\mathbf{D}^i$ and its deviatoric (traceless) part $\mathbf{D}^d$, defined by

$$\mathbf{D}^i = \frac{1}{3}(\operatorname{tr}\mathbf{D})\,\mathbf{I} \quad \text{and} \quad \mathbf{D}^d = \mathbf{D} - \frac{1}{3}(\operatorname{tr}\mathbf{D})\,\mathbf{I}.$$

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Here, $\mathbf{D}^i$ describes changes in volume (expansion or compression), while $\mathbf{D}^d$ describes shear. Substituting $\mathbf{D} = \mathbf{D}^i + \mathbf{D}^d$ into (1) yields the equivalent form

$$\mathbf{T} = \left(-p_{th} + \zeta \operatorname{tr} \mathbf{D}\right) \mathbf{I} + 2\mu \mathbf{D}^d, \tag{2}$$

where $\zeta = \lambda + 2\mu/3$ is the bulk (or volumetric) viscosity. The requirement that the rate of entropy production is non-negative implies that $\mu \geq 0$ and $\zeta \geq 0$.

Taking the trace of the left- and right-hand sides of the constitutive relation (2) yields

$$\operatorname{tr} \mathbf{T} = -3p_{th} + 3\zeta \operatorname{tr} \mathbf{D}. \tag{3}$$

Hence, the bulk viscosity ζ quantifies the viscous contribution to the normal (isotropic) stresses arising from changes in volume, as measured by $\operatorname{tr} \mathbf{D}$. The mechanical pressure p_m is defined as minus the mean value of the normal stresses, i.e., $p_m = -\operatorname{tr} \mathbf{T}/3$, and therefore relation (3) may be written as

$$-p_m = -p_{th} + \zeta \operatorname{tr} \mathbf{D}. \tag{4}$$

In Stokes (1845, page 294), Stokes not only derived a constitutive relation involving two material parameters, but also argued that the coefficients λ and μ in (1) are not independent. More precisely, he proposed that

$$\zeta = \lambda + \frac{2}{3}\mu = 0, \tag{5}$$

a condition that is now known as *Stokes' hypothesis* (Stokes' relation). Owing to (4), Stokes' hypothesis (5) implies equality of the thermodynamic and mechanical pressures, i.e. $p_{th} = p_m$. It should be noted that Stokes' hypothesis does not say anything about the flow. It is purely a constitutive statement about the fluid. Moreover, Stokes hypothesis is only relevant for compressible fluids, since an incompressible fluid can undergo only isochoric motions, i.e. $\operatorname{div} \mathbf{v} = 0$ ($\operatorname{tr} \mathbf{D} = 0$), which implies that the bulk viscosity term vanishes identically.

If Stokes' hypothesis were true, the Navier–Stokes constitutive relation (1) would reduce to

$$\mathbf{T} = -p_{th}\mathbf{I} - \frac{2\mu}{3}(\operatorname{tr} \mathbf{D})\,\mathbf{I} + 2\mu \mathbf{D},$$

that is, to a constitutive relation involving only a single material parameter, the shear viscosity μ. This simplification is practically attractive, since reliable data for the shear viscosity μ are widely available, whereas data for the bulk viscosity ζ are scarce and often uncertain, owing to the difficulty of its experimental determination (Cramer, 2012; Dukhin & Goetz, 2009; Graves & Argrow, 1999; Jaeger et al., 2018).

Using continuum-mechanical arguments Rajagopal has proved that the Stokes' hypothesis is not true for any real fluid, not even for monoatomic gases as often claimed (Rajagopal, 2013). Existing experimental results also contradict Stokes' hypothesis. Moreover, modern atomistic approaches provide no support for its validity. Notably, Stokes himself questioned what is now called Stokes' hypothesis in the 1901 reprint of his 1851 paper motion of pendulums (Stokes, 1901, p. 84).

Despite these facts, Stokes' hypothesis continues to be used or implicitly invoked in much of the compressible flow literature and in computational fluid dynamics software. This practice has several understandable origins. Historically, the reduced form of the Navier–Stokes equations became standard early on and propagated into textbooks and teaching materials. Moreover, reliable data for the bulk viscosity are scarce and difficult to obtain, whereas the shear viscosity is comparatively easy to measure. For many low-Mach-number, shear-driven, or weakly compressible flows, the influence of bulk viscosity is small in the sense that neglecting it leads to errors that are often acceptable, further reinforcing the perception that Stokes' hypothesis is valid. Yet another reason is that the term *pressure* is used in many contexts without specifying its precise meaning, and since Stokes' hypothesis removes the distinction between thermodynamic and mechanical pressure, this ambiguity contributes to the confusion. An interesting discussion concerning the notion of "pressure" can be found in Rajagopal (2015).

Altogether, there appear to be many ambiguities in the literature concerning Stokes' hypothesis (Buresti, 2015; Gad-el-Hak, 1995; Rajagopal, 2013). The purpose of this note is to provide new insights into this issue. In particular, we emphasize that Stokes' constitutive assumption that the bulk viscosity is identically zero, $\zeta = 0$, does not hold for any real fluid. However, we will show that in many flow regimes the effect of the bulk viscosity is negligible. The reason is clearly not that $\zeta = 0$, and in general not even that $\zeta/\mu \ll 1$. Instead, a careful analysis of the flow at hand must be undertaken. This distinction is often missing or blurred in the existing literature. The goal of this work is to present the results clearly and convincingly, and thereby promote a more accurate treatment of bulk-viscosity effects in fluid flows.

In addressing this issue, we note a tension between the scholarly norm of supporting claims with explicit references and the equally important norm of avoiding the unnecessary singling out of specific authors or texts as being "incorrect". Since the aim of the present work is clarification rather than critique, we refrain from cataloguing individual misstatements and instead focus on presenting the correct framework for the treatment of bulk viscosity and on explaining why Stokes' hypothesis, viewed as a constitutive relation, is not valid for any real fluid.

2. On Rajagopal's argument against Stokes' hypothesis

This section is devoted to Rajagopal's continuum-mechanical argument demonstrating that Stokes' hypothesis is not valid for any fluid, including monatomic gases (Rajagopal, 2013). We first recall basic facts concerning implicit algebraic constitutive relations introduced by Rajagopal (2003, 2006, 2023). Subsequently, by analyzing a subclass of such implicit relations, we present a non-standard derivation of the Navier–Stokes fluid model (1), which reveals that imposing Stokes' hypothesis leads to physically inadmissible consequences.

2.1. Implicit algebraic constitutive relations

There are many applications in which a fluid cannot be adequately modeled as a Navier–Stokes fluid, necessitating the use or development of alternative constitutive models. This is often achieved by employing models belonging to the class of Stokesian fluids introduced by Truesdell (1950), for which

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \qquad \mathbf{S} = \mathcal{F}(\rho, \theta, \mathbf{D}), \tag{6}$$

where $\mathbf{S}$ is the viscous stress tensor and $\mathcal{F}$ is an isotropic tensor-valued function. There are, however, flows for which the fluid's response to external stimuli cannot be modeled within the Stokesian framework, and more general constitutive relations must be considered. Rajagopal has shown that such relations can often be justified by viewing them as subclasses of implicit algebraic constitutive relations, where the term algebraic emphasizes that no higher-order spatial or temporal derivatives appear, in contrast to rate-type or differential-type models (Rajagopal, 2003, 2006, 2023).

We therefore consider implicit algebraic constitutive relations of the form

$$\mathbf{F}(\mathbf{T}, \mathbf{D}) = \mathbf{0},$$

where $\mathbf{F}$ is a tensor-valued function. For isotropic fluids, $\mathbf{F}$ must be an isotropic tensor function, and standard representation results (see, e.g., Spencer, 1971) imply that

$$\mathbf{F}(\mathbf{T}, \mathbf{D}) = \alpha_0\mathbf{I} + \alpha_1\mathbf{T} + \alpha_2\mathbf{D} + \alpha_3\mathbf{T}^2 + \alpha_4\mathbf{D}^2 + \alpha_5(\mathbf{T}\mathbf{D} + \mathbf{D}\mathbf{T}) + \alpha_6\left(\mathbf{T}^2\mathbf{D} + \mathbf{D}\mathbf{T}^2\right) + \alpha_7\left(\mathbf{T}\mathbf{D}^2 + \mathbf{D}^2\mathbf{T}\right) + \alpha_8\left(\mathbf{T}^2\mathbf{D}^2 + \mathbf{D}^2\mathbf{T}^2\right) = \mathbf{0}, \tag{7}$$

where the material moduli α_i, $i = 0, \dots, 8$, depend on ρ, θ and the invariants

$$\operatorname{tr}\mathbf{T}, \operatorname{tr}\mathbf{D}, \operatorname{tr}\mathbf{T}^2, \operatorname{tr}\mathbf{D}^2, \operatorname{tr}\mathbf{T}^3, \operatorname{tr}\mathbf{D}^3, \operatorname{tr}(\mathbf{T}\mathbf{D}), \operatorname{tr}\left(\mathbf{T}^2\mathbf{D}\right), \operatorname{tr}\left(\mathbf{T}\mathbf{D}^2\right), \operatorname{tr}\left(\mathbf{T}^2\mathbf{D}^2\right).$$

This class of implicit constitutive relations is significantly richer than the class of incompressible Navier–Stokes fluids or even the class of Stokesian fluids (6), and includes many well-known models discussed by Rajagopal (2003, 2006, 2023).

To assess the validity of Stokes' hypothesis, we restrict attention to the subclass of implicit algebraic constitutive relations of the form

$$\alpha_0\mathbf{I} + \alpha_1\mathbf{T} + \alpha_2\mathbf{D} = \mathbf{0}. \tag{8}$$

In the next subsection, we will use (8) as a starting point to derive the Navier–Stokes fluid model and to show why Stokes' hypothesis is not valid as a constitutive assumption.

2.2. A non-standard derivation of Navier–Stokes fluid model

In classical fluid dynamics it is customary to express the stress tensor $\mathbf{T}$ in terms of the symmetric part of the velocity gradient $\mathbf{D}$. However, as argued by Rajagopal, it is also natural to formulate constitutive relations in the reverse manner, that is, to express $\mathbf{D}$ in terms of $\mathbf{T}$. The latter is more consistent with causality, since stresses are the cause, while deformation is the effect (Rajagopal, 2023). Motivated by this viewpoint, we choose $\alpha_2 = -1$ in (8) and set $\alpha_0 = \alpha(\rho, \theta) + \beta(\rho, \theta)\operatorname{tr}\mathbf{T}$ and $\alpha_1 = \gamma(\rho, \theta)$, which leads to constitutive relations of the form

$$\mathbf{D} = \alpha\mathbf{I} + \beta(\operatorname{tr}\mathbf{T})\mathbf{I} + \gamma\mathbf{T}, \tag{9}$$

where, for notational simplicity, the dependence of the material moduli on ρ and θ has been suppressed. The material moduli in (9) cannot be chosen arbitrarily. They must satisfy certain constraints to ensure realistic and physically meaningful behavior of the fluid. In the following, we show, by contradiction, that physical admissibility of a basic static equilibrium state requires $3\beta + \gamma \neq 0$.

Consider a material point at which the fluid has density ρ and temperature θ. A physically admissible constitutive relation must admit a state of rest at such a point, that is, a state for which $\mathbf{D} = \mathbf{0}$. Consequently, there exists a tensor $\mathbf{T}_0$ such that

$$\mathbf{0} = \alpha\mathbf{I} + \beta\left(\operatorname{tr}\mathbf{T}_0\right)\mathbf{I} + \gamma\mathbf{T}_0. \tag{10}$$

From (10) it follows that $\mathbf{T}_0$ must be of the form $\mathbf{T}_0 = t_0\mathbf{I}$. Substituting this into (10) yields the following relation between the material parameters α, β and γ:

$$0 = \alpha + (3\beta + \gamma)\, t_0, \tag{11}$$

for any fixed thermodynamic state (ρ, θ). Either $3\beta + \gamma = 0$ or $3\beta + \gamma \neq 0$. Assume that $3\beta + \gamma = 0$. Then, according to (11), $\alpha = 0$, which implies that the class (9) of constitutive relations is reduced to relations of the form

$$\mathbf{D} = \beta(\operatorname{tr}\mathbf{T})\mathbf{I} + \gamma\mathbf{T}. \tag{12}$$

Taking the trace of both sides, and using the assumption $3\beta + \gamma = 0$, we obtain

$$\operatorname{tr}\mathbf{D} = (3\beta + \gamma)\operatorname{tr}\mathbf{T} = \mathbf{0},$$

for any $\mathbf{T}$. Hence, a constitutive relation of the type (12) restricts admissible motions to isochoric ones and thus implicitly defines an incompressible material. Thus, the assumption $3\beta + \gamma = 0$ leads to an inadmissible constitutive relation for compressible fluids, and we conclude that $3\beta + \gamma \neq 0$.

The Navier–Stokes constitutive relation can now be derived by solving (9) for $\mathbf{T}$. From (9) we obtain

$$\mathbf{T} = -\frac{\alpha}{\gamma}\mathbf{I} - \frac{\beta}{\gamma}(\mathrm{tr}\,\mathbf{T})\,\mathbf{I} + \frac{1}{\gamma}\mathbf{D}. \tag{13}$$

Here we have used that $\gamma \neq 0$, otherwise the constitutive relation (9) would only allow uniform dilatational motions, which is not acceptable for any realistic model. Taking the trace on both sides in (9) and using that $3\beta + \gamma \neq 0$ yields

$$\mathrm{tr}\,\mathbf{T} = \frac{1}{3\beta+\gamma}\,\mathrm{tr}\,\mathbf{D} - \frac{3\alpha}{3\beta+\gamma}. \tag{14}$$

Substituting this expression into (13) gives

$$\mathbf{T} = -\frac{\alpha}{(3\beta+\gamma)}\mathbf{I} - \frac{\beta}{\gamma\,(3\beta+\gamma)}(\mathrm{tr}\,\mathbf{D})\,\mathbf{I} + \frac{1}{\gamma}\mathbf{D}. \tag{15}$$

With

$$\frac{\alpha}{(3\beta+\gamma)} = p_{th}, \quad -\frac{\beta}{\gamma\,(3\beta+\gamma)} = \lambda, \quad \frac{1}{\gamma} = 2\mu, \tag{16}$$

we recover the classical Navier–Stokes constitutive relation (1).

This non-standard derivation of the Navier–Stokes fluid model enables us to assess whether Stokes' hypothesis is valid or not. Indeed, from (16) it follows that

$$3\lambda + 2\mu = \frac{1}{3\beta+\gamma} \neq 0,$$

and thus the Stokes' hypothesis is not valid for any fluid.

Remark. In contrast to classical fluid mechanics, classical linear elasticity theory relates the stress and the infinitesimal strain through a constitutive law that may be written either as stress in terms of strain or, equivalently, strain in terms of stress. As shown above, within the implicit constitutive framework considered here, an analogous inversion is admissible for compressible Navier–Stokes fluids, since physical admissibility requires that $3\lambda + 2\mu \neq 0$.

Remark. An alternative to experimentally determining the bulk viscosity ζ or the second viscosity λ is to first determine β and γ experimentally and thereafter compute λ and ζ using (16).

3. Experimental results and Stokes' hypothesis

There is an extensive experimental literature concerned with bulk viscosity ζ, much of which highlights the intrinsic difficulty of its reliable determination. Bulk viscosity cannot be measured directly and must be inferred indirectly from acoustic attenuation data, which often leads to scattered and frequency-dependent results. This section briefly reviews key experimental findings and shows that, despite the scarcity of unambiguous continuum values, all available evidence contradicts the universal validity of Stokes' hypothesis.

Stokes himself expressed doubts about the validity of what is now known as Stokes' hypothesis. In the 1901 reprint of his 1851 paper on the effect of internal friction of fluids on the motion of pendulums (Stokes, 1901, p. 84), he added the following remark:

> *"Although I have shown (Vol. I., p. 119) that on the admission of a supposition which Poisson would probably have allowed the two constants in his equations of motion are reduced to one, and the equations take the form (1), and although Maxwell obtained the same equations from his kinetic theory of gases (Philosophical Transactions for 1867, p. 81) I have always felt that the correctness of the value $\mu/3$ for the coefficient of the last term in (1) does not rest on as firm a basis as the correctness of the equations of motion of an incompressible fluid, for which the last term does not come in at all. If the supposition made above be not admitted, we must replace the coefficient $\mu/3$ by a different coefficient, which may be written $\mu/3 + \pi$ and π must be positive, as otherwise the mere alternate expansion and contraction, alike in all directions, of a fluid, instead of demanding the exertion of work upon it, would cause it to give out work".*

This shows that, late in his career, Stokes was not fully convinced about the part of his theory in which the hypothesis was invoked, and that he explicitly contemplated a modification involving an additional positive coefficient associated with volumetric changes. It is particularly noteworthy that Stokes expressed these doubts despite Maxwell's pioneering derivation of the same equations from kinetic theory (Maxwell, 1867, p. 81, Eq. (128)). Maxwell's work will be discussed in more detail in Section 4.

Importantly, reservations concerning the hypothesis were already present in Stokes' original 1845 paper (Stokes, 1845). There he emphasized that for many flows of practical interest it is entirely legitimate to set $\zeta = 0$, and that in such cases the resulting predictions would be identical, or nearly identical, whether or not bulk viscosity is included. As a consequence, he warned that agreement between theory and experiment in these regimes cannot be interpreted as confirmation of the hypothesis. In his own words (Stokes, 1845, page 294), using the notation ($\kappa = 3\lambda + 2\mu$), "*Of course we may at once put $\kappa = 0$ if we assume that in the case of a uniform motion of dilatation the pressure at any instant depends only on the actual density and temperature at that instant, and not on the rate at which the former changes with the time. In most cases to which it would be interesting to apply the theory of the friction of fluids the density of the fluid is either constant, or may without sensible error be regarded as constant, or else changes slowly with the time. In the first two cases the results would be the same, and in the third case nearly the same, whether κ were equal to zero or not. Consequently, if theory and experiment should in such cases agree, the experiments must not be regarded as confirming that part of the theory which relates to supposing κ to be equal to zero*".

These remarks already delineate the logical structure of the experimental problem: only flows involving sufficiently large changes in volume can meaningfully test the hypothesis, while agreement in weakly compressible or slowly varying flows is inconclusive.

Stefan (1866) provided an analysis of how shear viscosity leads to decay of the amplitude of a plane harmonic acoustic wave of the form $\exp(-\alpha x)$, where α is the attenuation coefficient. A more complete continuum description of acoustic damping was subsequently provided by Kirchhoff (1868), who also included dissipation due to temperature fluctuations. In modern notation, Kirchhoff's analysis can be expressed as

$$\alpha_K(\omega) = \frac{\omega^2}{2\rho c^3}\left(\frac{4}{3}\mu + \frac{\kappa}{c_p}(\gamma - 1)\right), \tag{17}$$

where κ is the thermal conductivity, c the sound speed, ω the angular frequency of the sound wave, and c_p the specific heat at constant pressure, and γ the ratio of specific heats, i.e. $\gamma = c_p/c_v$, where c_v the specific heat at constant volume.

In the 1930s, increasingly precise acoustic experiments revealed attenuation levels far exceeding the predictions of the Stefan–Kirchhoff theory, a situation comprehensively documented for both for gases and liquids in the review by Richards (1939), which includes 348 references. In Richards' review, excess acoustic attenuation in gases is explained by relaxation of internal molecular modes, whereas Mandelstam and Leontovich attributed it in liquids to relaxation processes associated with delayed equilibration of the fluid's internal state (Mandelstam & Leontovich, 1937). From a modern perspective, both explanations are phenomenological.

Tisza developed the findings concerning the role of relaxation processes in attenuation further into a macroscopic constitutive interpretation within continuum mechanics. In particular, he derived a damped acoustic wave equation from the linearized balance laws (Tisza, 1942, p. 533, Eq. (7)). His derivation, does not include thermal dissipation, since he focused on relaxation-induced dissipation and neglected thermal effects for simplicity. Thermal effects can, however, easily be included, which leads to the following formula for the attenuation coefficient

$$\alpha(\omega) = \frac{\omega^2}{2\rho c^3}\left(\frac{4}{3}\mu + \zeta + \frac{\kappa}{c_p}(\gamma - 1)\right). \tag{18}$$

Bulk viscosity cannot be measured directly. However, by determining the experimental attenuation coefficient α_{exp} it can be found via (17) and (18). Indeed, assuming that $\alpha_{exp} - \alpha_K$ is equal to the part of the attenuation coefficient associated with the bulk viscosity yields

$$\zeta_{exp}(\omega, \rho, \theta) = \frac{2\rho c^3}{\omega^2}\left(\alpha_{exp}(\omega) - \alpha_K(\omega)\right). \tag{19}$$

The quantity $\zeta_{exp}(\omega, \rho, \theta)$ in (19) obtained by inverting attenuation data should not, in general, be identified with the bulk viscosity $\zeta(\rho, \theta)$ of the classical continuum framework. The continuum theory applies in the hydrodynamic regime of sufficiently low frequencies, characterized by $\omega\tau \ll 1$, where τ denotes a characteristic microscopic relaxation time associated with the equilibration of stress or internal energy following changes in volume. In classical continuum mechanics, transport coefficients are assumed to depend only on the thermodynamic state. Attenuation measurements, however, are often carried out at frequencies not small compared with τ^{-1}, since detectable decay over laboratory length scales typically requires ultrasonic or hypersonic regimes. When this condition is not satisfied, the inverted quantity ζ_{exp} represents an effective, frequency-dependent parameter that absorbs relaxation processes beyond those accounted for in classical continuum mechanics. Consequently, ζ_{exp} may vary with frequency and experimental conditions, and different experiments can yield different values even at the same thermodynamic state. Only when the measured attenuation exhibits the expected low-frequency behavior $\alpha \propto \omega^2$ with a frequency-independent prefactor over a sufficiently broad range can $\zeta_{exp}(\omega, \rho, \theta)$ be interpreted as a reliable approximation of the continuum bulk viscosity $\zeta(\rho, \theta)$.

Up to the present day, the dominant experimental approach to determining bulk viscosity remains the study of sound attenuation. A variety of experimental techniques have been developed; however, they all rely on the same basic principle: the attenuation (or equivalently the linewidth) of acoustic waves is measured and interpreted within a continuum acoustic framework. The difference between the methods lies mainly in how the acoustic disturbance is generated and detected. Despite the conceptual simplicity of this idea, the experimental determination of bulk viscosity is challenging. Extracting ζ requires reliable knowledge of other material properties, such as the shear viscosity, thermal conductivity, specific heats, and sound speed, each of which introduces additional uncertainty. When the bulk viscosity is small, its contribution to attenuation may be comparable to experimental error, making reliable identification difficult. Further complications arise from frequency-dependent effects and non-ideal wave fields (e.g. boundary losses and scattering). These difficulties are reflected in the literature by the relative scarcity of tabulated bulk-viscosity data and by the often significant variation among reported values obtained by different experiments, see for example Dukhin and Goetz (2009) and Graves and Argrow (1999).

Taken together, the experimental literature provides a clear and consistent picture that Stokes' hypothesis is not generally valid. We highlight a few representative examples. In the early comprehensive review of acoustic attenuation in gases, together with extensive discussion of liquids, by Richards (1939) a systematic discrepancy between theory based on $\zeta = 0$ and experimental observations was documented. Measurements by Greenspan (1956) showed that, for monatomic gases under ordinary conditions, the measured attenuation is consistent with theoretical predictions neglecting bulk viscosity, indicating that any bulk-viscous effects are negligible in these regimes.showed that for monatomic gases under ordinary conditions the effects of bulk viscosity are often difficult to resolve experimentally. Ultrasonic experiments by Madigosky (1967) later demonstrated that for dense monatomic gases the bulk and shear viscosity can be of the same order (in the regimes covered in his experiments on argon), revealing that Stokes' hypothesis fails even for monatomic gases. Greenspan used a phenomenological relaxation model to show that, for diatomic gases, agreement with experimental attenuation requires the inclusion of relaxation processes (Greenspan, 1956). For polyatomic gases acoustic measurements, including those of Prangsma et al. (1973), have reported bulk viscosities comparable to or larger than the shear viscosity. Notably, for carbon dioxide (CO_2) experiments report ζ/μ of order 10^3, depending on frequency and temperature. For liquids, the bulk viscosity is often larger than the shear viscosity, as shown by Dukhin and Goetz (2009, p. 24519–8). Overall, experimental evidence across monatomic gases, polyatomic gases, and liquids overwhelmingly contradicts the universal validity of Stokes' hypothesis.

4. Atomistic perspectives on Stokes' hypothesis

Continuum mechanics and atomistic approaches, such as kinetic theory and statistical mechanics, constitute distinct theoretical frameworks, each based on its own variables, axioms, and modeling assumptions. Atomistic descriptions of fluids rely on assumptions such as explicit molecular collision or interaction models, the existence of a clear separation between fast molecular processes and much slower macroscopic fluid motion, and the use of equilibrium or near-equilibrium concepts. None of these assumptions are intrinsic to continuum mechanics, where constitutive relations are postulated independently of microscopic detail.

In the continuum framework, the viscosities μ and ζ in the Navier–Stokes constitutive relation are assumed to depend on the local thermodynamic state, typically characterized by the density ρ and temperature θ. These material functions are constrained by thermodynamic restrictions and informed by experimental observations. This approach applies to all Navier–Stokes fluids, including gases and liquids, though it does not explain the physical origin of the dependence on ρ and θ. In contrast, atomistic methods provide a first-principles basis for these dependencies by modeling the microscopic behavior of particles. In regimes where the Boltzmann equation is valid (primarily for dilute gases), the dependence of the constitutive parameters in the Navier–Stokes equations on state variables, such as the density and temperature, can be derived via a Chapman–Enskog expansion, see, for example, Farrell et al. (2026) or Stewart (2004). For dense gases and liquids, Green–Kubo relations are used, for example, by Hansen and McDonald (2013). Together, these atomistic tools provide a microscopic interpretation of the dependence of μ and ζ on the thermodynamic state, although the resulting expressions may involve additional details of the underlying molecular model.

Stokes' hypothesis is a constitutive statement within continuum mechanics and the proof by Rajagopal, presented in Section 2, that it is inapt is based solely on arguments within the continuum framework. However, atomistic methods can illuminate regimes in which the bulk-viscous terms may be neglected and thus provide an adequate approximation. Moreover, atomistic methods can be used to assess constitutive relations in continuum mechanics and to provide numerical values of transport coefficients, such as shear viscosity, bulk viscosity, and thermal conductivity. In this sense, atomistic methods may complement or, in some cases, effectively replace experiments. This is particularly important for determining the bulk viscosity, for which only limited data are available. In the following, we review a selection of representative works based on atomistic methods to illustrate that modern atomistic evidence does not support the general validity of Stokes' hypothesis.

Maxwell played a foundational role in the early development of kinetic theory. In his pioneering work (Maxwell, 1867, p. 81, Eq. (128)) he developed a theory for gases and obtained results that were consistent with Stokes' hypothesis. However, later research shows that Maxwell's model is oversimplified, for example, it lacked any internal molecular modes or relaxation mechanisms. Maxwell believed his kinetic theory described all gases, but later developments showed that its molecular assumptions correspond, at best, to an idealized monatomic gas. Truesdell (1952) pointed out that Maxwell's definition of stress and temperature automatically leads to $\zeta = 0$ and that it therefore, in some sense, is more correct to say that Maxwell's kinetic theory assumes Stokes' hypothesis rather than proving it.

Enskog modified Maxwell's definition of stress by including collisional momentum transfer (Enskog, 1922), see also the standard exposition by Chapman and Cowling (1970, Ch. 15–16). As a result, his theory predicts a nonzero bulk viscosity for real monatomic gases, although this contribution becomes asymptotically small in the dilute regime. Thus, Stokes' hypothesis is satisfied exactly only in the hypothetical dilute-gas limit. Curtiss and Dahler developed a kinetic theory for polyatomic gases (Curtiss, 1956; Curtiss & Dahler, 1963), in which a nonzero bulk viscosity arises as a consequence of the relaxation of internal degrees of freedom.

Violations of Stokes' hypothesis have also been demonstrated for gases within statistical mechanics by Sharma et al. (2023), who carried out molecular-dynamics simulations and employed Green–Kubo relations to obtain precise numerical estimates of the bulk viscosity of argon gas under atmospheric conditions. In particular, their analysis shows that the bulk viscosity of dilute monatomic argon gas is nonzero, although very small. In addition, their work provides a useful overview of the subject. In Cramer (2012), Cramer presents numerical estimates for the bulk viscosity of several ideal gases. All of them show a positive ratio of bulk viscosity to shear viscosity (ζ/μ). For several of the gases, $\zeta/\mu \gg 1$, for example, it is reported that $\zeta/\mu \approx 3800$ for carbon dioxide at $300\,\mathrm{K}$.

There are numerous works devoted to numerical computations of the bulk viscosity of liquids based on modern atomistic methods. These studies consistently contradict Stokes' hypothesis in the liquid regime. For example, high-quality equilibrium molecular-dynamics investigations based on Green–Kubo relations, notably those of Jaeger et al. (2018) and Meier et al. (2005), show a positive bulk viscosity that is often comparable to or larger than the shear viscosity. In particular, Meier et al. explicitly demonstrate that $\zeta \to 0$ in the zero-density limit, while $\zeta > 0$ throughout the liquid regime, corresponding to physically realizable liquid states.

Summary. Modern atomistic approaches do not provide support for the validity of Stokes' hypothesis as a general constitutive relation, although they can help explain why bulk-viscous effects may be negligible in certain regimes.

5. Criteria for neglecting bulk-viscous effects

We have demonstrated that Stokes' hypothesis is invalid as a constitutive assumption. For many fluids, the bulk viscosity ζ is comparable to or larger than the shear viscosity μ. While the shear viscosity is readily accessible through standard experimental techniques, reliable data for the bulk viscosity are often unavailable. This raises the question: if Stokes' hypothesis $\zeta = 0$ does not hold and the bulk viscosity is unknown, how should one proceed to solve the field equations governing the flow? In this section, we argue that, for certain classes of flows, the influence of bulk viscosity may be neglected on grounds other than Stokes' hypothesis, while in other flow regimes it must be explicitly taken into account.

Compressible flow of a Navier–Stokes fluid with Fourier heat conduction is governed by the compressible Navier–Stokes equations, also referred to as the Navier–Stokes–Fourier system,

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \mathbf{v} = 0, \tag{20}$$

$$\rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{g} - \nabla \left(p_{th} - \zeta \operatorname{div} \mathbf{v}\right) + \operatorname{div}\left(2\mu \mathbf{D}^d(\mathbf{v})\right), \tag{21}$$

$$\rho \frac{De}{Dt} = -p_{th} \operatorname{div} \mathbf{v} + \underbrace{2\mu \mathbf{D}^d(\mathbf{v}) : \mathbf{D}^d(\mathbf{v}) + \zeta \left(\operatorname{div} \mathbf{v}\right)^2}_{=\Phi \text{ Viscous dissipation}} + \operatorname{div}\left(\kappa \nabla \theta\right), \tag{22}$$

where $\mathbf{v} = (u, v, w)$ and $\mathbf{g}$ denotes the body force per unit mass, e the internal energy per unit mass, κ the constant heat conductivity and

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

The system is closed by specifying an equation of state $p_{th} = f(\rho, \theta)$, for example the ideal gas law $p_{th} = R\rho\theta$, together with a constitutive relation for the internal energy $e = f(\rho, \theta)$, such as $e = c_v\theta$ for ideal gases or, to a good approximation, $e \approx c_p\theta$ for liquids.

As discussed, available data for the bulk viscosity ζ are extremely limited and often restricted to specific regimes. It is therefore tempting to invoke Stokes' hypothesis, which reduces the system (20)–(22) to

$$\frac{D\rho}{Dt} + \rho \operatorname{div} \mathbf{v} = 0, \tag{23}$$

$$\rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{g} - \nabla p_{th} + \operatorname{div}\left(2\mu \mathbf{D}^d(\mathbf{v})\right), \tag{24}$$

$$\rho \frac{De}{Dt} = -p_{th} \operatorname{div} \mathbf{v} + 2\mu \mathbf{D}^d(\mathbf{v}) : \mathbf{D}^d(\mathbf{v}) + \kappa \nabla^2 \theta. \tag{25}$$

However, this approach cannot be justified on constitutive grounds, since Stokes' hypothesis is not valid for any fluid and the magnitude of ζ is often comparable to, or even larger than, that of the shear viscosity μ. Nevertheless, there exist many flows for which the reduced system (23)–(25), which neglects bulk-viscous effects, provides a good approximation to the full system (20)–(22). The correct justification for this, however, is not that it follows from Stokes' hypothesis $\zeta = 0$.

Before deriving a general criterion for when the effects of bulk viscosity may be neglected, we note that, although a compressible fluid has a positive bulk viscosity, the bulk-viscous terms vanish in isochoric flows, i.e., when $\operatorname{div} \mathbf{v} = 0$ (for example, in simple unidirectional shear flows).

The mechanical pressure p_m is by definition minus the mean value of the normal stresses, i.e., $p_m = -\operatorname{tr} \mathbf{T}/3$. Thus the Cauchy stress can be decomposed into $\mathbf{T} = \mathbf{T}^d + p_m\mathbf{I}$, where $\mathbf{T}^d$ is the deviatoric part of $\mathbf{T}$ and $p_m\mathbf{I}$ is the isotropic part. An equivalent formulation of the constitutive relation (2) is

$$p_m = -p_{th} + \zeta \operatorname{div} \mathbf{v}, \qquad \mathbf{T}^d = 2\mu \mathbf{D}^d.$$

This shows that, in a Navier–Stokes fluid, the deviatoric part of $\mathbf{D}$ generates the deviatoric (shear) stress through the shear viscosity μ, while the isotropic part $(\operatorname{div} \mathbf{v})\mathbf{I}$ generates an isotropic viscous contribution to the Cauchy stress $\mathbf{T}$ through the bulk viscosity ζ, in addition to the thermodynamic pressure p_{th}. Both the thermodynamic pressure p_{th} and the bulk-viscous term $\zeta \operatorname{div} \mathbf{v}$ contribute to the isotropic part of the Cauchy stress $\mathbf{T}$. However, the bulk-viscous contribution is kinematic in nature and vanishes when $\operatorname{div} \mathbf{v} = 0$. Consequently, the term $\nabla(\zeta \operatorname{div} \mathbf{v})$ in the equation for balance of linear momentum may be neglected whenever

$$|\nabla (\zeta \operatorname{div} \mathbf{v})| \ll \left|\nabla p_{th}\right|. \tag{26}$$

The bulk viscosity ζ contributes to the internal-energy balance through the bulk-viscous dissipation term $\zeta (\operatorname{div} \mathbf{v})^2$, which represents irreversible heating associated with changes in volume. The bulk-viscous dissipation is negligible if

$$\zeta (\operatorname{div} \mathbf{v})^2 \ll \left|\rho \frac{De}{Dt}\right|, \ \left|p_{th} \operatorname{div} \mathbf{v}\right|, \ 2\mu \mathbf{D}^d : \mathbf{D}^d \text{ or } |\operatorname{div}(\kappa \nabla \theta)|, \tag{27}$$

depending on whether the evolution of internal energy is governed primarily by storage, pressure–volume work, shear dissipation, or thermal diffusion in the flow regime of interest. In the momentum balance, the bulk-viscous term and the pressure gradient are of the same tensorial character, since both arise from isotropic stresses, and comparing them is therefore natural and direct. In the energy balance, by contrast, the bulk-viscous contribution appears as a scalar dissipation term and must be compared with the other scalar terms appearing in that equation.

The conclusion is that, for certain classes of flows, it is justified to neglect the bulk-viscous terms in the governing field equations. However, this simplification cannot be justified by appealing to Stokes' hypothesis, which is not valid for any fluid, even though such an appeal is nevertheless frequently made in the literature. As shown above, a careful analysis of the flow under consideration is required. More precisely, one must compare the magnitude of the bulk-viscous terms with that of the other relevant terms in the field equations, that is, checking that the criteria (26) and (27) are fulfilled.

We emphasize that the conditions (26) and (27) are *kinematic criteria*, while Stokes' hypothesis is a *constitutive* relation. The flow-dependent criteria (26) and (27) are conceptually related to the notion of activated fluids, in which the constitutive response depends on whether a prescribed activation criterion is satisfied. In the present case, however, it is not the constitutive relation that changes (ζ is never zero), but rather the relevance of the bulk-viscous terms in the governing equations. These terms may be neglected when these criteria are satisfied, even though the bulk viscosity itself is not zero.

An interesting question is whether the assumptions underlying the Chapman–Enskog expansion in the small-Knudsen-number, near-equilibrium regime imply restrictions on the class of admissible flows under which the criteria (26) and (27) would be automatically satisfied. A detailed investigation of this connection, however, lies beyond the scope of the present work.

For many classes of flows, the effects of bulk viscosity can be neglected. This is the case, for example, in many low-Mach-number flows, as discussed by Papalexandris (2020), in shear-driven flows, as described by Szeri (2010, p. 74), or when the bulk viscosity itself is sufficiently small. It is worth recalling that Stokes himself emphasized that when the density of a fluid is constant, when it may be regarded as constant without sensible error, or when it varies only slowly in time, the resulting motion is nearly the same whether the bulk viscosity is equal to zero or not, as explained in Section 3.

However, bulk viscosity may play a significant role in low-Mach-number flows when strong changes in volume are induced by thermal or compositional effects. For example, neglecting the effect of bulk viscosity in buoyancy- and mixing-driven flows requires assumptions beyond low Mach number alone. When the bulk-to-shear viscosity ratio ζ/μ is large, the bulk-viscous effects are non-negligible in turbulent flows, as shown by Pan and Johnsen (2017), and in boundary-layer analyses, as demonstrated by Cramer and Bahmani (2014) and Zheng et al. (2024). Moreover, in flows involving shock waves, strong compression produces very large values of $|\operatorname{div}\mathbf{v}|$ and its gradients, so bulk-viscous effects in both the momentum and energy balances may become significant, as shown by Chikitkin et al. (2015). As discussed in Section 3, bulk viscosity must also be accounted for in acoustics.

6. Isothermal gas lubrication

To assess whether the effect of bulk viscosity can be neglected in a given flow, a scaling and dimensional analysis is required. As an illustrative and practically relevant example, we consider a typical gas-lubrication flow.

6.1. The fluid domain

We consider a gas-lubricated bearing in which the fluid domain is

$$\Omega = \left\{(x,y,z) : 0 < x < L_x,\ 0 < y < L_y,\ 0 < z < h(x,y)\right\},$$

that is, a domain between two surfaces. The lower surface, $z = 0$, moves with the velocity $\mathbf{v}_l = (u_l, v_l, 0)$, while the upper surface, $z = h(x,y)$, is stationary, so that the local gap between the surfaces is given by $h(x,y)$. In typical lubrication flows, the gap between the surfaces is very small compared to the characteristic size of the surfaces, that is, $h(x,y) \ll L_x$ and $h(x,y) \ll L_y$. We denote the maximal distance between the surfaces by h_{max}.

6.2. The governing equations for isothermal gas lubrication

We model the gas as an ideal gas, i.e., the equation of state is $p_{th} = R\rho\theta$, where R is the specific gas constant. Under normal operating conditions, it is justified to assume that the flow is isothermal (Szeri, 2010, p. 451), that is, the temperature is constant. Consequently, variations in density depend only on the pressure. Moreover, under normal operating conditions the gas is in the dilute regime, for which kinetic theory predicts that the shear viscosity μ is independent of the density and therefore constant in isothermal flows (Bird et al., 2002, p. 27). No specific assumption is required regarding the dependence of the bulk viscosity ζ on density or pressure, since the subsequent scaling analysis shows that bulk-viscous effects are asymptotically negligible in thin-film gas-lubrication flows.

The equations for balance of mass and linear momentum are

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \operatorname{div}(\rho\mathbf{v}) = 0 & \text{(Balance of mass)} \\ \rho\dfrac{D\mathbf{v}}{Dt} = \rho\mathbf{g} - \nabla p_{th} + \operatorname{div}\mathbf{S} & \text{(Balance of linear momentum)} \end{cases} \tag{28}$$

where $\mathbf{S}$ is the viscous stress tensor given by

$$\mathbf{S} = (\zeta \operatorname{div} \mathbf{v}) \mathbf{I} + 2\mu \mathbf{D}^d(\mathbf{v}).$$

The Knudsen number, defined as $\mathrm{Kn} = \lambda/h^*$, where λ is the molecular mean free path and h^* the characteristic film thickness, characterizes the degree of rarefaction of the gas. In the continuum regime ($\mathrm{Kn} \lesssim 10^{-3}$), the Navier–Stokes equations with no-slip boundary conditions are appropriate. In the slip-flow regime ($10^{-3} \lesssim \mathrm{Kn} \lesssim 10^{-1}$), continuum equations remain valid in the bulk, but slip boundary conditions must be imposed at solid surfaces. For larger Knudsen numbers ($\mathrm{Kn} \gtrsim 10^{-1}$), non-equilibrium effects become significant and kinetic descriptions based on the Boltzmann equation are generally required. In the present work, we consider the continuum regime, and therefore the fluid satisfies no-slip and impenetrability conditions at the surfaces, i.e., $\mathbf{v}(x, y, 0) = \mathbf{v}_l$ and $\mathbf{v}(x, y, h(x, y)) = \mathbf{0}$.

6.3. Scaling and dimensional analysis

The purpose of the following analysis is to assess whether the effect of bulk viscosity can be neglected in gas-lubrication flows, based on scaling and dimensional considerations rather than on an a priori assumption.

Define the following dimensionless variables:

$$\bar{x} = x/x^*, \quad \bar{y} = y/y^*, \quad \bar{z} = z/z^*, \quad \bar{t} = t/t^*.$$

$$\bar{S}_{ij} = S_{ij}/S^*_{ij}, \quad \bar{p} = p_{th}/p^*, \quad \bar{u} = u/u^*, \quad \bar{v} = v/v^*, \quad \bar{w} = w/w^*,$$

Here, the bar denotes that it is a dimensionless variable, while the superscript $*$ indicates that the scalar represents a characteristic scale. Appropriate characteristic scales must now be determined to reflect the physical aspects of the flow at hand. In particular, the thin nature of the domain requires that the characteristic length scale across the fluid film be much smaller than the characteristic length scale associated with the dimension of the surfaces.

- Since the fluid domain is thin, there are naturally two different length scales involved: one related to the size of the surfaces and one related to the gap between them. If we consider the case when the side lengths of the surfaces, L_x and L_y, are of comparable magnitude, we can choose a common characteristic length scale L for the x- and y-directions. More precisely, we set $x^* = y^* = L$, where $L = \min\{L_x, L_y\}$. Moreover, we chose $z^* = h_{max}$. Since $h_{max} \ll 1$, we write h_{max} must be much smaller than L. This can be formulated as $h_{max} = \varepsilon L$, with $\varepsilon \ll 1$, that is, $z^* = \varepsilon L$. Hence, ε is a small parameter related to the thickness of the fluid domain.
- From the problem formulation it follows that the characteristic speed in the x- and y-directions can be chosen as $V = \max\{u_l, v_l\}$, Thus, we set $u^* = v^* = V$.
- The characteristic shear stresses S^*_{13} and S^*_{23} follow from the problem formulation. Using $S_{13} = \mu \partial u/\partial z$ together with $u^* = V$ and $z^* = \varepsilon L$, we obtain

  $$S^*_{13} = \frac{\mu V}{\varepsilon L} = S\varepsilon^{-1},$$

  where $S = \mu V/L$. Similarly, $S^*_{23} = S\varepsilon^{-1}$. For the remaining components of the viscous stress tensor, we introduce the characteristic stresses

  $$S^*_{12} = S\varepsilon^{a}, \quad S^*_{11} = S\varepsilon^{d_1}, \quad S^*_{22} = S\varepsilon^{d_2}, \quad S^*_{33} = S\varepsilon^{d_3},$$

  where a and d_i, $i = 1, 2, 3$, are to be determined. Owing to symmetry, $S^*_{ij} = S^*_{ji}$.

It remains to determine w^*, p^*, a, d_1, d_2 and d_3.

6.4. Dimensionless form of the equation for the balance of mass

We determine the characteristic velocity w^* by considering the scaled form of the balance-of-mass equation,

$$\frac{L}{t^* V}\frac{\partial \bar{p}}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}}(\bar{p}\bar{u}) + \frac{\partial}{\partial \bar{y}}(\bar{p}\bar{v}) + \frac{w^*}{\varepsilon V}\frac{\partial}{\partial \bar{z}}(\bar{p}\bar{w}) = 0.$$

This choice reflects the standard lubrication scaling, in which the transverse velocity adjusts to ensure mass conservation in a thin domain. To avoid unrealistic motions or unnecessary restrictions, the characteristic velocity w^* is chosen so that all terms are of the same magnitude, uniformly in the small parameter ε, that is, regardless of how thin the fluid domain is. A detailed discussion of this choice can be found in Almqvist et al. (2019) and Dowson (1962). Choosing $w^* = \varepsilon V$ and $t^* = L/V$, which corresponds to the required for a typical fluid particle to travel from the inlet to the outlet, satisfies this criterion and leads to the following dimensionless balance-of-mass:

$$\frac{\partial \bar{p}}{\partial \bar{t}} + \frac{\partial}{\partial \bar{x}}(\bar{p}\bar{u}) + \frac{\partial}{\partial \bar{y}}(\bar{p}\bar{v}) + \frac{\partial}{\partial \bar{z}}(\bar{p}\bar{w}) = 0. \tag{29}$$

6.5. Dimensionless form of the equation for balance of linear momentum

In component form the scaled form of the equation for balance of linear momentum in (28) reads

$$\begin{aligned}
\frac{p^*V^2}{R\theta L}\bar{p}\frac{D\bar{u}}{D\bar{t}} &= \frac{p^*g_x}{R\theta}\bar{p} - \frac{p^*}{L}\frac{\partial \bar{p}}{\partial \bar{x}} + \frac{S}{L}\left(\varepsilon^{d_1}\frac{\partial \bar{S}_{11}}{\partial \bar{x}} + \varepsilon^{a}\frac{\partial \bar{S}_{12}}{\partial \bar{y}} + \frac{1}{\varepsilon^2}\frac{\partial \bar{S}_{13}}{\partial \bar{z}}\right)\\
\frac{p^*V^2}{R\theta L}\bar{p}\frac{D\bar{v}}{D\bar{t}} &= \frac{p^*g_y}{R\theta}\bar{p} - \frac{p^*}{L}\frac{\partial \bar{p}}{\partial \bar{y}} + \frac{S}{L}\left(\varepsilon^{a}\frac{\partial \bar{S}_{21}}{\partial \bar{x}} + \varepsilon^{d_2}\frac{\partial \bar{S}_{22}}{\partial \bar{y}} + \frac{1}{\varepsilon^2}\frac{\partial \bar{S}_{23}}{\partial \bar{z}}\right)\\
\frac{p^*V^2\varepsilon}{R\theta L}\bar{p}\frac{D\bar{w}}{D\bar{t}} &= \frac{p^*g_z}{R\theta}\bar{p} - \frac{p^*}{\varepsilon L}\frac{\partial \bar{p}}{\partial \bar{z}} + \frac{S}{L}\left(\frac{1}{\varepsilon}\frac{\partial \bar{S}_{31}}{\partial \bar{x}} + \frac{1}{\varepsilon}\frac{\partial \bar{S}_{32}}{\partial \bar{y}} + \varepsilon^{d_3-1}\frac{\partial \bar{S}_{33}}{\partial \bar{z}}\right).
\end{aligned}$$

Any realistic simplified lubrication model must include the terms which include the pressure $\bar{p}$ and the shear stress $\bar{S}_{13}$, independently of the value of ε. This implies that they must be in balance, i.e., p^* must be of order ε^{-2}. We choose

$$p^* = \frac{S}{\varepsilon^2} = \frac{\mu V}{L\varepsilon^2}. \tag{30}$$

With these characteristic scales for the velocity and the pressure we have that

$$\nabla(\zeta\,\mathrm{div}\,\mathbf{v}) = \frac{\zeta^* V}{\varepsilon L^2}\begin{pmatrix} \varepsilon\frac{\partial}{\partial \bar{x}}\left(\bar{\zeta}\left[\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{w}}{\partial \bar{z}}\right]\right)\\ \varepsilon\frac{\partial}{\partial \bar{y}}\left(\bar{\zeta}\left[\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{w}}{\partial \bar{z}}\right]\right)\\ \frac{\partial}{\partial \bar{z}}\left(\bar{\zeta}\left[\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{w}}{\partial \bar{z}}\right]\right)\end{pmatrix}, \qquad \nabla p_{th} = \frac{\mu V}{L^2\varepsilon^3}\begin{pmatrix}\varepsilon\frac{\partial \bar{p}}{\partial \bar{x}}\\ \varepsilon\frac{\partial \bar{p}}{\partial \bar{y}}\\ \frac{\partial \bar{p}}{\partial \bar{z}}\end{pmatrix},$$

which yields

$$\frac{|\nabla(\zeta\,\mathrm{div}\,\mathbf{v})|}{|\nabla p_{th}|} = O\left(\frac{\zeta^*\varepsilon^2}{\mu}\right). \tag{31}$$

Thus, the contribution of the bulk viscosity term to the momentum balance is much smaller than that of the pressure gradient for $\varepsilon \ll 1$. From (31) we deduce that the condition (26) for neglecting the effect of the bulk viscosity ζ in the flow at hand is satisfied as $\varepsilon \ll 1$. This was the main aim, but let us complete the analysis.

To determine a we consider the part S_{12} of the extra stress in the constitutive relation:

$$S_{12} = S\varepsilon^a\bar{S}_{12} = \frac{\mu V}{2L}\left(\frac{\partial \bar{u}}{\partial \bar{y}} + \frac{\partial \bar{v}}{\partial \bar{x}}\right).$$

The left- and right hand side must be of the same order of ε, which implies that $a = 0$.

It remains to determine d_1, d_2 and d_3. To do this we study the diagonal elements in the extra stress in scaled version of the constitutive relation. By taking into account that the effects of the bulk viscosity may be neglected we have that

$$\begin{aligned}
S\varepsilon^{d_1}\bar{S}_{11} &\approx \frac{2\mu V}{L}\left[-\frac{1}{3}\left(\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{w}}{\partial \bar{z}}\right) + \frac{\partial \bar{u}}{\partial \bar{x}}\right]\\
S\varepsilon^{d_2}\bar{S}_{22} &\approx \frac{2\mu V}{L}\left[-\frac{1}{3}\left(\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{w}}{\partial \bar{z}}\right) + \frac{\partial \bar{v}}{\partial \bar{y}}\right]\\
S\varepsilon^{d_3}\bar{S}_{33} &\approx \frac{2\mu V}{L}\left[-\frac{1}{3}\left(\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial \bar{w}}{\partial \bar{z}}\right) + \frac{\partial \bar{w}}{\partial \bar{z}}\right].
\end{aligned}$$

From this we deduce that $d_1 = d_2 = d_3 = 0$.

We have shown that in component form the scaled form of the equation for balance of linear momentum in (28) is:

$$\frac{V^2}{R\theta}\bar{p}\frac{D\bar{u}}{D\bar{t}} = \frac{Lg_x}{R\theta}\bar{p} - \frac{\partial \bar{p}}{\partial \bar{x}} + \left(\varepsilon^2\frac{\partial \bar{S}_{11}}{\partial \bar{x}} + \varepsilon^2\frac{\partial \bar{S}_{12}}{\partial \bar{y}} + \frac{\partial \bar{S}_{13}}{\partial \bar{z}}\right) \tag{32}$$

$$\frac{V^2}{R\theta}\bar{p}\frac{D\bar{v}}{D\bar{t}} = \frac{Lg_y}{R\theta}\bar{p} - \frac{\partial \bar{p}}{\partial \bar{y}} + \left(\varepsilon^2\frac{\partial \bar{S}_{21}}{\partial \bar{x}} + \varepsilon^2\frac{\partial \bar{S}_{22}}{\partial \bar{y}} + \frac{\partial \bar{S}_{23}}{\partial \bar{z}}\right) \tag{33}$$

$$\frac{V^2\varepsilon^2}{R\theta}\bar{p}\frac{D\bar{w}}{D\bar{t}} = \frac{L\varepsilon g_z}{R\theta}\bar{p} - \frac{\partial \bar{p}}{\partial \bar{z}} + \left(\varepsilon^2\frac{\partial \bar{S}_{31}}{\partial \bar{x}} + \varepsilon^2\frac{\partial \bar{S}_{32}}{\partial \bar{y}} + \varepsilon^2\frac{\partial \bar{S}_{33}}{\partial \bar{z}}\right). \tag{34}$$

From the Navier–Stokes constitutive relation it follows that

$$\bar{S}_{13} = \left(\frac{\partial \bar{u}}{\partial \bar{z}} + \varepsilon^2\frac{\partial \bar{w}}{\partial \bar{x}}\right), \qquad \bar{S}_{23} = \left(\frac{\partial \bar{v}}{\partial \bar{z}} + \varepsilon^2\frac{\partial \bar{w}}{\partial \bar{y}}\right).$$

Substituting $\bar{S}_{ij}$ into the system (32)–(34) and neglecting the terms including ε or ε^2 lead to:

$$\frac{V^2}{R\theta}\bar{p}\frac{D\bar{u}}{D\bar{t}} = \frac{Lg_x}{R\theta}\bar{p} - \frac{\partial \bar{p}}{\partial \bar{x}} + \frac{\partial^2 \bar{u}}{\partial \bar{z}^2} \tag{35}$$

$$\frac{V^2}{R\theta}\bar{p}\frac{D\bar{v}}{D\bar{t}} = \frac{Lg_y}{R\theta}\bar{p} - \frac{\partial \bar{p}}{\partial \bar{y}} + \frac{\partial^2 \bar{v}}{\partial \bar{z}^2} \tag{36}$$

$$\frac{\partial \bar{p}}{\partial \bar{z}} = 0. \tag{37}$$

In many realistic applications, the non-dimensional parameters satisfy

$$\frac{V^2}{R\theta} \ll 1, \qquad \frac{Lg_x}{R\theta} \ll 1, \qquad \frac{Lg_y}{R\theta} \ll 1.$$

A representative example for air is given by: $V = 10\,\mathrm{m\,s^{-1}}$, $R = 287\,\mathrm{J\,kg^{-1}\,K^{-1}}$, $\theta = 300\,\mathrm{K}$, $g_x = g_y = 0\,\mathrm{m\,s^{-2}}$, $L = 0.01\,\mathrm{m}$, and $h_{max} = 10^{-4}\,\mathrm{m}$. This yields $\varepsilon = h_{max}/L = 10^{-2}$, which justifies neglecting terms of order ε and higher. Moreover, for these values

$$\frac{V^2}{R\theta} = 1.16 \times 10^{-3}, \qquad \frac{Lg_x}{R\theta} = 0, \qquad \frac{Lg_y}{R\theta} = 0.$$

In such regimes, the system (35)–(37) reduces to

$$\frac{\partial^2 \bar{u}}{\partial \bar{z}^2} = \frac{\partial \bar{p}}{\partial \bar{x}}, \qquad \frac{\partial^2 \bar{v}}{\partial \bar{z}^2} = \frac{\partial \bar{p}}{\partial \bar{y}}, \qquad \frac{\partial \bar{p}}{\partial \bar{z}} = 0. \tag{38}$$

From these three equations and Eq. (29) for balance of mass one can derive a Reynolds type equation for the pressure. Indeed, the third equation in (38) implies that $\bar{p}$ does not depend on z. This fact can be used to integrate the first equation in (38) twice with respect to $\bar{z}$ to obtain an explicit expressions for $\bar{u}$ and $\bar{v}$, where the constants are determined with help of the no-slip condition at the surfaces. Inserting the expressions for $\bar{u}$ and $\bar{v}$ into Eq. (29) and integrating across the fluid film leads to the dimensionless compressible Reynolds equation for the pressure $\bar{p}(\bar{x}, \bar{y})$. In dimensional form, the equation reads

$$\frac{\partial}{\partial x}\left(\frac{h^3}{12\mu} p \frac{\partial p}{\partial x} - \frac{hu_l}{2} p\right) + \frac{\partial}{\partial y}\left(\frac{h^3}{12\mu} p \frac{\partial p}{\partial y} - \frac{hv_l}{2} p\right) = 0, \tag{39}$$

where the pressure p approximates the thermodynamic pressure with a relative error of order $O(\varepsilon^2)$, i.e. $\left(p_{th} - p\right)/p_{th} = O(\varepsilon^2)$.

The Reynolds equation (39) for an ideal gas is well-known; see, for example, the classical book (Hamrock et al., 2004). It is commonly derived by initially invoking Stokes' hypothesis, $\zeta = 0$, and subsequently omitting the bulk viscosity term in the balance-of-linear momentum equation. However, as shown in Section 2, the constitutive assumption that $\zeta = 0$ does not hold for any real fluid. Thus, although this approach leads to the correct result, the justification for it should be modified. As demonstrated in this section, the neglect of bulk viscosity must be justified by an analysis of the specific flow under consideration.

Remark. In Almqvist et al. (2019) the flow between two adjacent surface which are in relative motion was studied when the fluid is modeled by a density-pressure relationship where

$$\rho = \rho_a \left(p_{th}/p_a\right)^{1-\alpha/2},$$

where $\alpha \leq 2$. The case $\alpha = 0$ corresponds to that the fluid is considered as an ideal gas. However, the starting point in the analysis was to assume that the classical Stokes' hypothesis holds.

7. Conclusions

Stokes' hypothesis, understood in its original constitutive form $\zeta = \lambda + 2\mu/3 = 0$, is not valid for any real fluid. Using Rajagopal's continuum-mechanical framework based on implicit algebraic constitutive relations, together with a non-standard derivation of the Navier–Stokes constitutive relation, it was shown that physical admissibility requires $\zeta \neq 0$ for compressible fluids. Experimental evidence, primarily from acoustic attenuation, likewise shows that bulk viscosity is generally nonzero and may be comparable to, or substantially larger than, the shear viscosity. Modern atomistic studies are consistent with this picture in that they provide no support for the universal validity of Stokes' hypothesis, not even for monatomic gases.

In numerical simulations of flows, where the fluid is modeled as a compressible Navier–Stokes fluid, it is common to invoke Stokes' hypothesis $\zeta = 0$ as a constitutive assumption, that is, all effects of the bulk viscosity are a priori neglected. We have, however, shown that $\zeta \neq 0$. There are flows in which bulk-viscous effects may be neglected, but not because $\zeta = 0$, nor, in general, because $\zeta/\mu \ll 1$. The correct argument is that, in the flow at hand, the rate of change of volume, as measured by $\operatorname{div} \mathbf{v}$, and its gradients are small compared with the dominant mechanisms in the governing equations. Neglect of bulk viscosity should therefore be justified by explicit, flow-dependent estimates, by verifying that

$$|\nabla (\zeta \operatorname{div} \mathbf{v})| \ll \left|\nabla p_{th}\right|$$

in the linear-momentum balance equation and that $\zeta(\operatorname{div} \mathbf{v})^2$ is negligible compared with the dominant terms in the internal-energy balance equation. The presented isothermal gas-lubrication example illustrates this approach: thin-film scaling yields

$$\frac{|\nabla (\zeta \operatorname{div} \mathbf{v})|}{\left|\nabla p_{th}\right|} = O\left(\frac{\zeta^* \varepsilon^2}{\mu}\right),$$

so bulk-viscous stresses are negligible for $\varepsilon \ll 1$, independently of the precise value of ζ.

The main message is thus twofold: Stokes' hypothesis is invalid as a constitutive relation, yet neglect of bulk viscosity can still be appropriate when justified by the physics and scaling of the flow under consideration. We have demonstrated that any omission of bulk-viscous effects should therefore be justified explicitly by the kinematic (regime-dependent) criteria (26) and (27), and not by invoking Stokes' constitutive hypothesis. Continued reference to Stokes' constitutive hypothesis as a justification for neglecting bulk viscosity obscures this distinction and contributes to a persistent ambiguity in the literature.

CRediT authorship contribution statement

Andreas Almqvist: Writing – review & editing, Writing – original draft, Validation, Methodology, Investigation, Formal analysis, Conceptualization. **Evgeniya Burtseva:** Writing – review & editing, Writing – original draft, Validation, Methodology, Investigation, Formal analysis, Conceptualization. **Peter Wall:** Writing – review & editing, Writing – original draft, Validation, Methodology, Investigation, Formal analysis, Conceptualization.

Declaration of competing interest

The authors hereby declare that there are no competing interests.

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Data availability

No data was used for the research described in the article.

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