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Perpetual propagation of singularities for evolutionary Hamilton–Jacobi equations

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Abstract
This paper studies viscosity solutions $S(t, x)$ of the Hamilton–Jacobi equation

$$\partial S/\partial t + H(x, \nabla S) = 0, \qquad (t, x) \in (0, \infty) \times \mathbb{R}^n.$$

The singular set $\Sigma(S)$ for a given solution S consists of all points where S is not differentiable. While it is well known that singularities propagate locally along certain curves, perpetual propagation has been established only in a few special cases. By means of intrinsic characteristics, we establish perpetual propagation of singularities in three different settings. More precisely, from every initial point $(t_0, x_0) \in (0, \infty) \times \mathbb{R}^n$ we construct a curve $\boldsymbol{x}(t)$, defined for $t \in [t_0, \infty)$ and with $\boldsymbol{x}(t_0) = x_0$, such that whenever $t^\bullet \geq t_0$ and $(t^\bullet, \boldsymbol{x}(t^\bullet)) \in \Sigma(S)$ one has $(t, \boldsymbol{x}(t)) \in \Sigma(S)$ for every $t \in [t^\bullet, \infty)$.

Keywords Hamilton–Jacobi equation · Propagation of singularities · Intrinsic characteristic · Tonelli Lagrangian

Mathematics Subject Classification 35F21 · 35D40 · 49L25

1 Introduction

We consider the Hamilton–Jacobi equation in its evolutionary form

$$\partial S/\partial t + H(x, \nabla S) = 0 \quad \text{in } Q = (0, \infty) \times \mathbb{R}^n, \tag{1}$$

$$\lim_{t \downarrow 0} S(t, x) = S_0(x) \quad \text{in } \mathbb{R}^n. \tag{2}$$

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This Cauchy problem is a classical example of one that, in general, does not admit a classical solution. In order to recall the variational feature of (1)–(2), we consider the minimal action

$$A_t(y,x) = \inf_{\xi} \int_0^t L(\xi(s), \dot{\xi}(s))\, ds, \tag{3}$$

where the infimum is taken over all arcs joining the points y and x in time $t > 0$, i.e., $\xi(0) = y$ and $\xi(t) = x$. The Lagrangian $L(x, v)$ and the Hamiltonian $H(x, p)$ are dual to each other with respect to Legendre–Fenchel transformation. The value function associated with (1)–(2) can be represented as

$$S(t,x) = \inf_{y\in\mathbb{R}^n} (S_0(y) + A_t(y,x)) \quad \text{for all} \quad (t,x) \in Q. \tag{4}$$

Under suitable conditions on L and S_0, (4) gives the unique viscosity solution of (1)–(2) [9, 10, 25, 27, 28, 39]. Besides local Lipschitz continuity away from $t = 0$, the principal regularity property of S is local semiconcavity (see Proposition 12 and Example 2 in the appendix).

Definition 1 The *singular set* of S consists of those points in Q where S fails to be differentiable. It is denoted by $\Sigma(S)$ or Σ.

We contribute to the existing literature on (1)–(2) by addressing the following central question concerning the perpetual propagation of singularities for C^2 Tonelli Hamiltonians and Lagrangians [11].

(P) *Given* $(t_0, x_0) \in \Sigma$, *does there exist a curve* $\xi : [t_0, \infty) \to \mathbb{R}^n$ *such that* $\xi(t_0) = x_0$ *and* $(t, \xi(t)) \in \Sigma$ *for all* $t \in [t_0, \infty)$*?*

To the best of the author's knowledge, no counterexample to (P) is known when H and L are smooth. The study of the propagation of singularities along appropriate curves has undergone significant progress during the last decades; consult, e.g., the survey article [12] and the many references therein, the central article [14] as well as recent papers, e.g., [4, 15, 16]. In this branch of research there have emerged two main concepts of characteristics along with some refinements.

1.1 Generalized and intrinsic characteristics

A *generalized characteristic* in an interval $I \subseteq [0, \infty)$ is a locally Lipschitz continuous curve $X : I \to \mathbb{R}^n$ such that

$$\dot{X}(t) \in \operatorname{co} \nabla_p H(X(t), \partial_x^+ S(t, X(t))) \tag{5}$$

holds for almost all $t \in I$. This notion was first defined and studied by Albano and Cannarsa in [3]. The definition of the superdifferential $\partial_x^+ S$ is recalled in (9). In general, the forward generalized characteristic emanating from a given point $(t_0, x_0) \in Q$ is known to be nonunique when $n \geq 2$ (see Example 3.5 in [21] and the paper [45]). Some stronger notions of characteristics appear in the literature [15, 21–23, 32, 46, 49],

known as strict, strong or broken characteristics, replacing the differential inclusion (5) with a discontinuous differential equation for the particular selection of the multivalued mapping of (5) given by

$$v^\sharp(t,x) = \nabla_p H(x, p^\sharp(t,x)),$$

where $p^\sharp(t,x)$ is defined in (7).

However, in the present paper we utilize intrinsic characteristics. This class of curves was introduced by Cannarsa and Cheng in [11] for stationary Hamilton–Jacobi equations and subsequently applied and explored, also for evolutionary equations, in [13–15, 47, 48]. The intrinsic approach is further developed in this paper for time-dependent viscosity solutions S of (1). The *intrinsic characteristic* $\boldsymbol{x}\colon [t_0,t_1] \to \mathbb{R}^n$ emanating from a point x_0 is defined by setting $\boldsymbol{x}(t_0) = x_0$ and

$$\{\boldsymbol{x}(t)\} = \arg\max_{x\in\mathbb{R}^n}(S(t,x) - A_{t-t_0}(x_0,x)) \quad \text{when} \quad t \in (t_0,t_1]. \tag{6}$$

The definition is conditional on the existence and uniqueness of a solution of the extremal value problem (6) for each $t \in (t_0,t_1]$. In order to be a candidate for a solution of (P), $\boldsymbol{x}(t)$ needs to be prolonged to $[t_0,\infty)$. This delicate extension problem is addressed and solved in this paper. The evolutionary setting presents additional difficulties, because the extension cannot be obtained directly and requires careful arguments. The detailed proofs are executed in Section 6 after several steps of preparations carried out in Sections 3–5.

1.2 Overview of local and global propagation

There exist a variety of results about singular dynamics in terms of one concept of characteristics or another [12, 14–16]. The basic *local* propagation of singularities along generalized characteristics is captured by the following theorem.

Theorem 1 (Cannarsa–Yu [21]). *Suppose that $H \in C^1$ and that, for each $x \in \mathbb{R}^n$, $p \mapsto H(x,p)$ is convex and such that none of its level sets contains a line segment. Let S be a locally semiconcave viscosity solution of the Hamilton–Jacobi equation (1) in a convex open set $\Omega \subseteq Q$. Then for any $(t_0,x_0) \in \Sigma \cap \Omega$ there exist $t_1 > t_0$ and a generalized characteristic $\boldsymbol{X}$ in the time interval $[t_0,t_1)$ such that $\boldsymbol{X}(t_0) = x_0$, $(t,\boldsymbol{X}(t)) \in \Sigma \cap \Omega$ for all $t \in [t_0,t_1)$,*

$$\dot{\boldsymbol{X}}_+(t_0) = \nabla_p H(x_0, p^\sharp(t_0,x_0)) \quad \textit{and} \quad \lim_{\delta\downarrow 0} \operatorname*{ess\,sup}_{t\in[t_0,t_0+\delta)} \left|\dot{\boldsymbol{X}}_+(t) - \dot{\boldsymbol{X}}_+(t_0)\right| = 0.$$

Here, $p^\sharp(t,x)$ stands for the element of $\partial_x^+ S(t,x)$ that is defined by

$$\{(\omega^\sharp(t,x), p^\sharp(t,x))\} = \arg\min\{\omega + H(x,p)\colon (\omega,p) \in \partial^+ S(t,x)\}. \tag{7}$$

Proof See Theorem 3.2 in [21]. Other similar results include Theorem 8 in [3], Theorem 5.6.6 in [18] and [50]. □

If (t_0, x_0) is not a critical point in the sense that

$$0 \notin \nabla_p H(x_0, \partial_x^+ S(t_0, x_0)), \tag{8}$$

then $\dot{\boldsymbol{X}}_+(t_0) \neq 0$ preventing $\boldsymbol{X}(t)$ from reducing to a single point. Condition (8) expresses that no element of $\partial_x^+ S(t_0, x_0)$ minimizes $p \mapsto H(x_0, p)$ globally.

The extent to which propagation persists indefinitely along generalized characteristics is still a vital research topic for Hamilton–Jacobi equations [4, 12, 15]. We briefly summarize the current state of the general theory of propagation of singularities for (1)–(2). Assuming $L \in C^2$ and $H \in C^2$, the following assertions are known to be true for a generalized characteristic whose initial point is (t_0, x_0).

- **(i)** If $(t_0, \boldsymbol{X}(t_0)) \in \overline{\Sigma}$, then $(t, \boldsymbol{X}(t)) \in \overline{\Sigma}$ at all later times $t \in [t_0, \infty)$ [1].
- **(ii)** If L is independent of x and $L(v) = \frac{1}{2}\langle Mv, v\rangle$ is a positive definite quadratic form, then the generalized characteristic is unique and $(t, \boldsymbol{X}(t)) \in \Sigma$ for all $t \in [t_0, \infty)$ if $(t_0, x_0) \in \Sigma$ [17].
- **(iii)** If S_0 is a concave function, H is independent of x and if there exist constants $0 < \alpha \leq \beta$ such that $\alpha I \leq \nabla^2 H(p) \leq \beta I$ for all $p \in \mathbb{R}^n$, then $(t, \boldsymbol{X}(t)) \in \Sigma$ for all $t \in [t_0, \infty)$ if $(t_0, x_0) \in \Sigma$ [2].

Accordingly, (ii) and (iii) provide solutions of our main problem (P). Furthermore, for stationary equations set on manifolds, [5] furnishes a solution of (P) for distance functions and [4] treats Hamiltonians of the form $H(x, p) = \frac{1}{2}|p|^2 + U(x)$.

In the seminal paper [11] on intrinsic characteristics, a general viscosity solution of $H(x, \nabla S(x)) = 0$ was the object of investigation instead of a time-dependent solution of (1)–(2). For background, we recall that for a given Tonelli Hamiltonian $\mathcal{H}$, there exists a minimal constant $c = c_{\mathcal{H}}$, Mañé's critical value, such that $\mathcal{H}(x, \nabla S(x)) = c$ possesses, in $\mathbb{R}^n$, a globally Lipschitz continuous and globally semiconcave viscosity solution $S(x)$ [29]. Therefore, we arrive at $H(x, \nabla S(x)) = 0$ in the viscosity sense for the Hamiltonian $H = \mathcal{H} - c_{\mathcal{H}}$. For this viscosity solution $S(x)$, given a singular point x_0, the intrinsic approach yields a Lipschitz continuous curve $\boldsymbol{x}(t)$ with $\boldsymbol{x}(0) = x_0$ such that $\boldsymbol{x}(t)$ remains singular for all $t \in [0, \infty)$ [11]. Subsequently, the intrinsic approach has been successfully applied to topological investigations of viscosity solutions of Hamilton–Jacobi equations including the characterization of the homotopy type of the singular set [13, 14, 47, 48]. Notably, the pivotal paper [14] on time-dependent Hamilton–Jacobi equations gives a solution of (P) for initial data S_0 that are Lipschitz in the large (see Remark 1 and Subsection 1.4).

1.3 Summary of the main results of the present paper

Under certain conditions on S_0 and L, it turns out to be possible for a given initial time $t_0 > 0$ to find an unbounded increasing sequence of times $t_0 < t_1 < t_2 < \cdots$ such that an extended intrinsic characteristic $x(t)$ can be constructed recursively from (5) for every initial point (t_0, x_0). Owing to Theorem 2, this curve $\boldsymbol{x}(t)$ solves the perpetual propagation problem (P) and, moreover, it satisfies $\dot{\boldsymbol{x}}_+(t_0) \neq 0$ if (8) holds. The evolutionary setting presents additional difficulties as this recursive construction has to be carried out with care regarding the interplay between L and S_0. Our results

apply under three concrete sets of assumptions: a Lipschitz-in-the-large condition on S_0, a nearly quadratic condition on L, and a globally semiconcave S_0.

The main results are presented in Theorems 3–5 in Section 6. These theorems assert that the intrinsic approach answers question (P) affirmatively under each of the following hypotheses on a Tonelli Lagrangian $L(x, v)$ [11] and a lower semicontinuous initial datum $S_0 : \mathbb{R}^n \to (-\infty, \infty]$ which is somewhere finite and such that $S_0(x) \geq -C(1 + |x|)$.

- **(a)** For every $t > 0$ there exists a constant $\varrho_t < \infty$ such that $|x - y(t, x)| \leq \varrho_t$ whenever $(t, x) \notin \Sigma$, where $y(t, x)$ denotes the minimizer in (4). This hypothesis is met, in particular, if S_0 is Lipschitz in the large. See Theorem 3, Corollary 1 and Proposition 8.
- **(b)** $L(x, v)$ is "nearly quadratic" and *no* additional requirement is imposed on S_0 [Theorem 4].
- **(c)** S_0 is globally semiconcave and $L(x, v)$ satisfies appropriate additional conditions [Theorem 5].

Case (a) covers in particular general bounded initial data S_0 as well as unbounded uniformly continuous S_0. In each of Theorems 3–5, it is actually proved that there exists a continuous generating mapping $\mathcal{X}$, whose domain is

$$\operatorname{dom} \mathcal{X} = \{(t, s, x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n : t \geq s > 0\},$$

such that $\boldsymbol{x}(t) = \mathcal{X}(t, t_0, x_0)$, defined for $t \in [t_0, \infty)$, is an extended intrinsic characteristic satisfying $\boldsymbol{x}(t_0) = x_0$, for any initial point $(t_0, x_0) \in Q$. This curve carries singularities indefinitely in the sense that $(t, \boldsymbol{x}(t)) \in \Sigma(S)$ for all $t \in [t^\bullet, \infty)$ if $t^\bullet \geq t_0$ and $(t^\bullet, \boldsymbol{x}(t^\bullet)) \in \Sigma(S)$. None of our results requires S_0 to be smooth and, consequently, the interior of $\overline{\Sigma}$ may be nonempty [37, 51], as recalled in Remark 6 of the appendix.

Remark 1 As regards originality, an analogous result to (a) for initial data S_0 that are Lipschitz in the large appears somewhat implicitly in Sections 5–6 in the important article [14]. The results (b) and (c) for "nearly quadratic" Lagrangians and globally semiconcave initial data, respectively, are novel as far as the author is aware.

Remark 2 The main results which are stated and proved in Section 6 are quite technical. The reader may find it helpful to first read Section 7 which treats the special case of $L(x, v) \equiv L(v)$ with shorter proofs.

Despite avoiding lengthy technical considerations in the introduction, we present the following illustrative special case (which reappears as Corollary 2 in Section 6). Let $K(v)$ and $U(x)$ be C^2 functions such that, on the one hand, K is superlinear and the Hessian $\nabla^2 K(v) > 0$ for every $v \in \mathbb{R}^n$ and, on the other, U is bounded along with its first- and second-order derivatives. Let $S_0 : \mathbb{R}^n \to (-\infty, \infty]$ be lower semicontinuous, somewhere finite and such that $S_0(x) \geq -C(1 + |x|)$. Theorems 3–5 can, in particular, be applied to the Lagrangian

$$L(x, v) = K(v) - U(x),$$

thereby answering question (P) affirmatively, provided *at least one* of the following additional conditions is in force.

- **(a)** See above in this subsection.
- **(b′)** $bI \leq \nabla^2 K(v) \leq \beta I$ for some constants $0 < b \leq \beta$ and all $v \in \mathbb{R}^n$.
- **(c′)** S_0 is globally semiconcave and there exists a constant $b > 0$ such that $\nabla^2 K(v) \geq bI$ for all $v \in \mathbb{R}^n$.

1.4 Relation to the literature

As mentioned in Subsection 1.2, in the stationary case, a general global propagation result along intrinsic characteristics was obtained by Cannarsa and Cheng in [11]. The present paper generalizes the method of [11] to time-dependent Hamilton–Jacobi equations. This extension is somewhat delicate because it relies on the interplay between the properties of the Lagrangian and those of the initial datum. However, the approach of [11] was for the first time extended to evolutionary equations by Cannarsa, Cheng and Fathi in their articles [13, 14]. The emphasis in these papers lies on the topological properties of the cut locus and the singular set of the solutions and the intrinsic characteristics are used for homotopy constructions. In particular, [14] furnishes results on perpetual propagation of singularities analogous to those of the present paper for Lipschitz-in-the-large initial data (cf. Remark 1 above).

Complementing the central reference [14], the contribution of the present paper lies in the systematic extension of intrinsic characteristics to the evolutionary setting involving the nontrivial construction via recursive continuation. In particular, it is demonstrated that these techniques apply to specific classes of Lagrangians and initial data, for which we refer to the summary in Subsection 1.3. In this regard, Theorems 4–5 constitute new results (cf. Remark 1).

1.5 Outline of the paper

For the convenience of the reader, we have made the presentation self-contained to a reasonable degree. Section 2 sets the stage by formulating conditions (A) and (B) for Tonelli Lagrangians and by recalling properties of the variational problem that this paper centers on. Section 3 motivates, defines and reviews the concept of intrinsic characteristics through excess or surplus action. Section 4 is concerned with general technical statements about how extended intrinsic characteristics can be defined. Section 5 is devoted to certain properties of $S(t, \cdot)$ and $A_t(y, \cdot)$ which are crucial for the constructions. Section 6 contains the main results stated in Theorems 3–5. The special case of $L(x, v) \equiv L(v)$ is the topic of Section 7.

The appendix recalls and discusses some properties of the value function and the associated minimizing arcs under weaker hypotheses than (A) and (B).

2 Conditions and prerequisites

The Hamiltonian H appearing in (1) is the Legendre–Fenchel transform of the Lagrangian L, i.e.,

$$H(x,p) = \sup_{v\in\mathbb{R}^n} (\langle p, v\rangle - L(x,v)) \quad \text{for all} \quad (x,p) \in \mathbb{R}^n \times \mathbb{R}^n.$$

2.1 Conditions

The following conditions link (1)–(2) to a well-behaved Bolza problem in the calculus of variations.

(A) The Lagrangian L belongs to $C^2(\mathbb{R}^n \times \mathbb{R}^n)$ and satisfies

$$\ell_0(v) \le L(x,v) \le \ell_1(v)$$

for all $(x,v) \in \mathbb{R}^n \times \mathbb{R}^n$ where $\ell_k(v)/|v| \to \infty$ as $|v| \to \infty$ for $k = 0, 1$.

(B) There exist a nonincreasing function $g\colon [0,\infty) \to (0,\infty)$ and a nondecreasing function $h\colon [0,\infty) \to (0,\infty)$ such that

$$\nabla_v^2 L(x,v) \ge g(|v|)I \quad \text{and} \quad |D^\kappa L(x,v)| \le h(|v|)$$

for all $(x,v) \in \mathbb{R}^n \times \mathbb{R}^n$ and every multi-index $|\kappa| = 1, 2$.

(C) The initial datum $S_0\colon \mathbb{R}^n \to (-\infty,\infty]$ is lower semicontinuous, somewhere finite and such that $S_0(x) \ge -C(1+|x|)$ for some constant $C \ge 0$.

Conditions (A), (B) and (C) are vital for the extremal problems (3), (4) and (6) and the subsequent global-in-time construction of intrinsic characteristics via recursive continuation. Section 5 demonstrates relevant semiconcavity and uniform convexity properties of S and A_t, respectively, which are used in this construction. Conditions (A) and (B) are adopted from [11].

Definition 2 L and H are referred to as a *Tonelli Lagrangian* and a *Tonelli Hamiltonian*, respectively, if (A) and (B) above are fulfilled.

Unless otherwise stated, (A), (B) and (C) are in force in the sequel.

2.2 Sub- and superdifferentials and viscosity solutions

For an open subset Ω of $\mathbb{R}^N$ and a function $f\colon \Omega \to \mathbb{R}$, the *superdifferential* of f is the multivalued mapping $\partial^+ f\colon \Omega \rightrightarrows \mathbb{R}^N$ defined by

$$\partial^+ f(x) = \left\{ p \in \mathbb{R}^N \colon \limsup_{|h|\to 0} \frac{f(x+h) - f(x) - \langle p, h\rangle}{|h|} \le 0 \right\}. \tag{9}$$

The definition of the *subdifferential* $\partial^- f$ is obtained by changing "lim sup" and "≤ 0" in (9) to "lim inf" and "≥ 0", respectively. Let f be locally semiconcave, i.e., assume that for every compact convex subset $\mathcal{K} \subset \Omega$ there exists a constant $c \geq 0$ such that $f - \frac{1}{2}c|\cdot|^2$ is concave in $\mathcal{K}$. Then, for any $x \in \Omega$, $\partial^+ f(x)$ is a compact convex nonempty subset of $\mathbb{R}^N$ which is a singleton if and only if f is differentiable at x. We recall a serviceable characterization of global semiconcavity.

Lemma 1 *The following conditions are equivalent for a given continuous function $f : \mathbb{R}^N \to \mathbb{R}$ and a constant c.*

(i) $f - \frac{1}{2}c|\cdot|^2$ *is globally concave.*

(ii) f *satisfies*

$$f(x+h) + f(x-h) - 2f(x) \leq c|h|^2$$

for all $x \in \mathbb{R}^N$ and all $h \in \mathbb{R}^N$.

(iii) *There exists $\varrho > 0$ such that*

$$f(x+h) + f(x-h) - 2f(x) \leq c|h|^2$$

for all $x \in \mathbb{R}^N$ and all $h \in B(0, \varrho)$.

Proof Let $g(x) = f(x) - \frac{1}{2}c|x|^2$. The equivalence between (i) and (ii) stems from the fact that the continuous function g is concave if and only if it is midpoint concave. Similarly, if (iii) holds, then g is locally concave hence, concave on $\mathbb{R}^N$. □

Definition 3 *(Viscosity solution).* A *viscosity solution* of (1) is a continuous function $S : Q \to \mathbb{R}$ such that

$$\omega + H(x, p) \leq 0 \quad \text{if} \quad (\omega, p) \in \partial^+ S(t, x)$$

and

$$\omega + H(x, p) \geq 0 \quad \text{if} \quad (\omega, p) \in \partial^- S(t, x)$$

for any $(t, x) \in Q$.

The monograph [18] is an excellent source for the theory of semiconcave functions and viscosity solutions of Hamilton–Jacobi equations.

Let $g : \mathbb{R}^N \to (-\infty, \infty]$ be a proper function in the sense that its essential domain $\operatorname{dom} g = \{x \in \mathbb{R}^N : g(x) < \infty\}$ is nonempty. The *Legendre–Fenchel transform* or conjugate function $g^* : \mathbb{R}^N \to (-\infty, \infty]$ of g is the convex and lower semicontinuous function which assigns to each $y \in \mathbb{R}^N$ the value

$$g^*(y) = \sup_{x \in \mathbb{R}^N} (\langle x, y \rangle - g(x)).$$

The biconjugate g^{**} coincides with the largest lower semicontinuous convex minorant of g provided g has an affine minorant. If g itself is convex and lower semicontinuous, then the Fenchel subdifferential ∂g is defined by

$$\partial g(x) = \{y \in \mathbb{R}^N : \forall z \in \mathbb{R}^N \; g(z) \geq g(x) + \langle z - x, y \rangle\} \quad \text{for every } x \in \operatorname{dom} g,$$

while $\partial g(x) = \emptyset$ otherwise. In this case, $g^{**} = g$ and $\partial g^* = (\partial g)^{-1}$.

Remark 3 If L is a Tonelli Lagrangian, then $H \in C^2$ and $\nabla_p^2 H(x, p)$ is positive definite everywhere. By applying the Legendre–Fenchel transform to (A) we get

$$\ell_1^*(p) \le H(x, p) \le \ell_0^*(p)$$

and, once more, we see that

$$\ell_0^{**}(v) \le L(x, v) \le \ell_1^{**}(v).$$

The convex lower semicontinuous envelopes ℓ_k^{**} are still superlinear. Therefore, ℓ_0 and ℓ_1 of condition (A) can and will be assumed to be convex continuous functions. Consequently, by Jensen's inequality for integrals,

$$t\ell_0\left(\frac{x-y}{t}\right) \le A_t(y, x) \le t\ell_1\left(\frac{x-y}{t}\right) \tag{10}$$

for all $t > 0$ and all $x, y \in \mathbb{R}^n$.

2.3 Lagrangian action and the value function

The Lagrangian *action* functional is defined by

$$J(\boldsymbol{\xi}; 0, t) = S_0(\boldsymbol{\xi}(0)) + \int_0^t L(\boldsymbol{\xi}(s), \dot{\boldsymbol{\xi}}(s))\, ds \tag{11}$$

for Lipschitz continuous arcs $\boldsymbol{\xi} \in \mathrm{Lip}([0, t]; \mathbb{R}^n)$. The *value function* is defined by

$$S(t, x) = \inf_{\boldsymbol{\xi} \in \mathbb{A}(t,x)} J(\boldsymbol{\xi}; 0, t), \qquad (t, x) \in Q, \tag{12}$$

where $\mathbb{A}(t, x)$ denotes the set of admissible arcs ending at (t, x), i.e.,

$$\mathbb{A}(t, x) = \{\boldsymbol{\xi} \in \mathrm{Lip}([0, t]; \mathbb{R}^n) : \boldsymbol{\xi}(t) = x\}.$$

The Bolza problem (11)–(12) possesses at least one solution $\boldsymbol{\xi} \in \mathbb{A}(t, x)$ for any $(t, x) \in Q$. The value function is a locally semiconcave viscosity solution of (1) [9, 10, 18–20, 39]. The initial condition (2) holds pointwise. Furthermore, the value function is the unique viscosity solution of (1)–(2) [28].

A point (t, x) is singular exactly if it has more than one minimizing arc [30, 34].

Lemma 2 *Let $\boldsymbol{\xi}$ be a minimizing arc of (11)–(12) for a given point $(t, x) \in Q$. Then $\boldsymbol{\xi} \in C^2([0, t])$ and $(s, \boldsymbol{\xi}(s)) \notin \Sigma(S)$ for every $s \in (0, t)$. Moreover, $(t, x) \notin \Sigma(S)$ if and only if the minimizer of (12) for (t, x) is unique if and only if $S(t, \cdot)$ is differentiable at x.*

Proof See, e.g., Theorems 6.2.8, 6.4.7 and Corollary 6.4.10 in [18]. □

3 Excess action and intrinsic characteristics

The excess action gauges how far a given curve is from being a minimizer of action. The concept of intrinsic characteristics arises when the excess action functional is minimized.

3.1 Minimization of excess action

Definition 4 The *excess action* for a locally Lipschitz continuous curve $\boldsymbol{\xi}\colon (0,\infty)\to\mathbb{R}^n$ and an initial time $t_0>0$ is defined by

$$E(\boldsymbol{\xi};t_0,t)=S(t_0,\boldsymbol{\xi}(t_0))+\int_{t_0}^{t}L(\boldsymbol{\xi}(s),\dot{\boldsymbol{\xi}}(s))\,ds-S(t,\boldsymbol{\xi}(t)) \tag{13}$$

for all $t\in[t_0,\infty)$.

A curve $\boldsymbol{\xi}$ is an *action minimizer* in the interval $[t_0,t_1]\subset[0,\infty)$ if

$$S(t,\boldsymbol{\xi}(t))=J(\boldsymbol{\xi};t_0,t):=S(t_0,\boldsymbol{\xi}(t_0))+\int_{t_0}^{t}L(\boldsymbol{\xi}(s),\dot{\boldsymbol{\xi}}(s))\,ds$$

for all times $t\in[t_0,t_1]$.

Proposition 1 *Let $\boldsymbol{\xi}\colon(0,\infty)\to\mathbb{R}^n$ be locally Lipschitz continuous. Then the following assertions hold true for any $t_0>0$.*

(i) $E(\boldsymbol{\xi};t_0,t)\geq 0$ *whenever* $t_0\leq t$.
(ii) $E(\boldsymbol{\xi};t_0,t_2)=E(\boldsymbol{\xi};t_0,t_1)+E(\boldsymbol{\xi};t_1,t_2)$ *if* $t_0\leq t_1\leq t_2$.
(iii) *The function* $t\mapsto E(\boldsymbol{\xi};t_0,t)$ *is nondecreasing and locally Lipschitz continuous on* $[t_0,\infty)$.
(iv) *For any* $t_0<t_1$, $E(\boldsymbol{\xi};t_0,t_1)=0$ *if and only if* $\boldsymbol{\xi}$ *is an action minimizer in* $[t_0,t_1]$. *In this case,* $\boldsymbol{\xi}$ *extends in a unique way to an action minimizing arc* $\boldsymbol{x}$ *for* $S(t_1,\boldsymbol{\xi}(t_1))$.

Proof See Proposition 1 in [49]. □

We approach the concept of intrinsic characteristics by investigating the minimization of the excess action functional

$$E(\boldsymbol{\xi};t_0,t_1)=S(t_0,\boldsymbol{\xi}(t_0))+\int_{t_0}^{t_1}L(\boldsymbol{\xi}(s),\dot{\boldsymbol{\xi}}(s))\,ds-S(t_1,\boldsymbol{\xi}(t_1)) \tag{14}$$

over all Lipschitz arcs $\boldsymbol{\xi}$ subject to the condition that $\boldsymbol{\xi}(t_0)=x_0$ for a fixed point (t_0,x_0) and $t_0<t_1$ with $\boldsymbol{\xi}(t_1)$ free. If $\boldsymbol{\xi}(\cdot)$ is a solution of this minimization problem, then $\boldsymbol{\xi}(\cdot)$ solves the Euler–Lagrange equation and satisfies the transversality condition [24]

$$\nabla_v L(\boldsymbol{\xi}(t_1),\dot{\boldsymbol{\xi}}(t_1))\in\partial_x^+ S(t_1,\boldsymbol{\xi}(t_1)). \tag{15}$$

Following Cannarsa and Cheng [11], we are primarily focused on the terminal point $\boldsymbol{\xi}_1(t_1)$ of a minimizing curve $\boldsymbol{\xi}_1(\cdot)$. This extremal value problem reduces to finding the minimizer $x_1 \in \mathbb{R}^n$ of

$$x \mapsto A_{t_1-t_0}(x_0, x) - S(t_1, x). \tag{16}$$

Changing signs in (16), the minimization problem is converted to one of maximization. To recapitulate, the definition of intrinsic characteristics is based on the problem of globally maximizing

$$\mathbb{R}^n \ni x \mapsto S(t, x) - A_{t-t_0}(x_0, x) \tag{17}$$

for times $t > t_0$.

Definition 5 Assume that, for certain times $0 < t_0 < t_1$, the function (17) has a unique maximizer for each $t \in (t_0, t_1]$. Then, the *intrinsic characteristic* $\boldsymbol{x}\colon [t_0, t_1] \to \mathbb{R}^n$ is defined by setting $\boldsymbol{x}(t_0) = x_0$ and

$$\{\boldsymbol{x}(t)\} = \operatorname*{arg\,max}_{x\in\mathbb{R}^n}(S(t, x) - A_{t-t_0}(x_0, x)) \quad \text{when} \quad t \in (t_0, t_1]. \tag{18}$$

The intrinsic characteristic starting from a nonsingular point coincides with a minimizing arc for S as long as the former remains nonsingular.

Proposition 2 *For given times $0 < t_0 < t_1 < \infty$ and a given point $x_0 \in \mathbb{R}^n$, suppose that*

$$x_1 \in \operatorname*{arg\,max}_{x\in\mathbb{R}^n}(S(t_1, x) - A_{t_1-t_0}(x_0, x)) \tag{19}$$

and that (t_1, x_1) is nonsingular. Let $\boldsymbol{x}_1\colon [0, t_1] \to \mathbb{R}^n$, $\boldsymbol{x}_1(t_1) = x_1$, be the unique minimizing arc for $S(t_1, x_1)$. Then, the intrinsic characteristic $\boldsymbol{x}\colon [t_0, t_1] \to \mathbb{R}^n$ emanating from (t_0, x_0) is well-defined and coincides in $[t_0, t_1]$ with $\boldsymbol{x}_1$, i.e., $\boldsymbol{x}(t) = \boldsymbol{x}_1(t)$ when $t \in [t_0, t_1]$.

Proof Since (t_1, x_1) is a nonsingular point, $\nabla_x S(t_1, x) \in \partial_x^- A_{t_1-t_0}(x_0, x)$ at $x = x_1$. Thus, as $A_{t_1-t_0}(x_0, \cdot)$ is locally semiconcave, also $\nabla_x A_{t_1-t_0}(x_0, x)$ exists at $x = x_1$ and so

$$p_1 := \nabla_x S(t_1, x)|_{x=x_1} = \nabla_x A_{t_1-t_0}(x_0, x)|_{x=x_1}.$$

On the one hand, by Lemma 2 there exists a unique minimizing arc $\boldsymbol{x}_1(s)$, $s \in [0, t_1]$, for $S(t_1, x_1)$. On the other hand, there exists a curve $\boldsymbol{\xi}_1(s)$, satisfying $\boldsymbol{\xi}_1(t_0) = x_0$ and $\boldsymbol{\xi}_1(t_1) = x_1$, that minimizes (14). This curve must also minimize $\int_{t_0}^{t_1} L(\boldsymbol{\xi}(s), \dot{\boldsymbol{\xi}}(s))\, ds$ over all Lipschitz curves with fixed endpoints, $\boldsymbol{\xi}(t_0) = x_0$ and $\boldsymbol{\xi}(t_1) = x_1$. In particular, the curves $\boldsymbol{\xi}_1$ and $\boldsymbol{x}_1$ are extremals satisfying $\boldsymbol{\xi}_1(t_1) = x_1 = \boldsymbol{x}_1(t_1)$. Moreover, $\dot{\boldsymbol{\xi}}_1(t_1) = \nabla_p H(x_1, p_1) = \dot{\boldsymbol{x}}_1(t_1)$. We briefly recall the argument for $\boldsymbol{x}_1$. Differentiation of

$$(0, t_1] \ni t \mapsto S(t, \boldsymbol{x}_1(t)) = S_0(\boldsymbol{x}_1(0)) + \int_0^t L(\boldsymbol{x}_1(s), \dot{\boldsymbol{x}}_1(s))\, ds$$

at $t = t_1$, invoking the Hamilton–Jacobi equation, yields

$$\langle \nabla_x S(t_1, \boldsymbol{x}_1(t_1)), \dot{\boldsymbol{x}}_1(t_1)\rangle = L(\boldsymbol{x}_1(t_1), \dot{\boldsymbol{x}}_1(t_1)) + H(\boldsymbol{x}_1(t_1), \nabla_x S(t_1, \boldsymbol{x}_1(t_1)))$$

from which $\dot{\boldsymbol{x}}_1(t_1) = \nabla_p H(x_1, p_1)$ follows. Similarly, or directly from (15), we confirm that $\dot{\boldsymbol{\xi}}_1(t_1) = \nabla_p H(x_1, p_1)$. Since they solve the Euler–Lagrange equation, $\boldsymbol{\xi}_1$ and $\boldsymbol{x}_1$ must coincide on $[t_0, t_1]$.

Now, consider for a fixed arbitrary time $t \in (t_0, t_1]$ the problem of minimizing

$$E(\boldsymbol{\xi}; t_0, t) = S(t_0, \boldsymbol{\xi}(t_0)) + \int_{t_0}^{t} L(\boldsymbol{\xi}(s), \dot{\boldsymbol{\xi}}(s))\, ds - S(t, \boldsymbol{\xi}(t)) \tag{20}$$

over all Lipschitz arcs $\boldsymbol{\xi}$ subject to the requirement that $\boldsymbol{\xi}(t_0) = x_0$ and with $\boldsymbol{\xi}(t)$ free. Still, the solution $\boldsymbol{\xi}$ is unique and satisfies $\boldsymbol{\xi}(s) = \boldsymbol{x}_1(s)$ when $s \in [t_0, t]$; in particular, the endpoint $\boldsymbol{\xi}(t)$ coincides with $\boldsymbol{x}_1(t)$. After all, $E(\boldsymbol{x}_1; t_0, t_1) = 0$ and hence $E(\boldsymbol{x}_1; t_0, s) = 0$ when $s \in [t_0, t_1]$ since $E(\boldsymbol{x}_1; t_0, s) \geq 0$ is nondecreasing in $s \in [t_0, t_1]$; see Proposition 1. Furthermore, if $\boldsymbol{\eta}\colon [t_0, t] \to \mathbb{R}^n$, $\boldsymbol{\eta}(t_0) = x_0$, also minimizes (20) subject to $\boldsymbol{\xi}(t_0) = x_0$, then $E(\boldsymbol{\eta}; t_0, t) = E(\boldsymbol{x}_1; t_0, t) = 0$ and, hence, $\boldsymbol{\eta}$ can be extended to a minimizing arc, still denoted by $\boldsymbol{\eta}$, for $S(t, \boldsymbol{\eta}(t))$ according to part (iv) of Proposition 1. It follows that $\boldsymbol{\eta} = \boldsymbol{x}_1$ on $[0, t]$ as $x_0 = \boldsymbol{x}_1(t_0) = \boldsymbol{\eta}(t_0)$. Indeed, two distinct minimizing arcs can intersect only at their endpoints (see Lemma 2). This implies that the intrinsic characteristic $\boldsymbol{x}$, emanating from (t_0, x_0), is well-defined and agrees with $\boldsymbol{x}_1$ at time t, i.e., $\boldsymbol{x}(t) = \boldsymbol{x}_1(t)$. We conclude that $\boldsymbol{x}$ and $\boldsymbol{x}_1$ coincide on $[t_0, t_1]$. □

Proposition 3 *Suppose that the intrinsic characteristic $\boldsymbol{x}\colon [t_0, t_1] \to \mathbb{R}^n$ emanating from a given point (t_0, x_0) is well-defined for certain $0 < t_0 < t_1 < \infty$.*

(i) *Assume that $t^\circ \in (t_0, t_1]$ and $(t^\circ, \boldsymbol{x}(t^\circ)) \notin \Sigma$. Then, $\boldsymbol{x}(\cdot)$ coincides in $[t_0, t^\circ]$ with the unique minimizing curve for $S(t^\circ, \boldsymbol{x}(t^\circ))$. In particular, $(t, \boldsymbol{x}(t)) \notin \Sigma$ when $t \in [t_0, t^\circ]$. If $S_0 \in C^2$ or $(t^\circ, \boldsymbol{x}(t^\circ)) \notin \overline{\Sigma}$, then $(t, \boldsymbol{x}(t)) \notin \overline{\Sigma}$ when $t \in [t_0, t^\circ)$.*

(ii) *If $t^\bullet \in [t_0, t_1)$ and $(t^\bullet, \boldsymbol{x}(t^\bullet)) \in \Sigma(S)$, then $(t, \boldsymbol{x}(t)) \in \Sigma(S)$ for all $t \in [t^\bullet, t_1]$.*

Proof (i) Let $\boldsymbol{x}^\circ$ be the unique minimizing arc for $S(t^\circ, \boldsymbol{x}(t^\circ))$. Owing to Proposition 2, $\boldsymbol{x}(t) = \boldsymbol{x}^\circ(t)$ when $t \in [t_0, t^\circ]$. As $(s, \boldsymbol{x}^\circ(s)) \notin \Sigma(S)$ when $s \in (0, t^\circ)$, we conclude that the point $(t, \boldsymbol{x}(t)) \notin \Sigma$ for every $t \in [t_0, t^\circ]$; see Lemma 2. The remaining assertion is covered by Lemmas 9–10 in the appendix.

Part (ii) follows from (i) by contraposition. □

Assertion (ii) of Proposition 3 confirms that singularities propagate along intrinsic characteristics.

3.2 The role of uniform concavity

We next turn to statements of pivotal technical importance. For their proofs we recall that a sequence of functions $f_j\colon \mathbb{R}^n \to (-\infty, \infty]$ is said to *epi-converge* to f [6, 26] if the following conditions are fulfilled for each $x \in \mathbb{R}^n$:

$$f(x) \leq \liminf_{j\to\infty} f_j(x_j) \quad \text{for every sequence} \quad x_j \to x; \tag{21}$$

and

$$f(x) = \lim_{j\to\infty} f_j(z_j) \quad \text{for some sequence} \quad z_j \to x. \tag{22}$$

Furthermore, g_j *hypo-converges* to g if $-g_j$ epi-converges to $-g$.

Proposition 4 *Assume that for certain $0 < a < b < \infty$ there exists $\varepsilon > 0$ such that*

$$S(t, x) - A_{t-s}(z, x) + \frac{\varepsilon|x|^2}{2} \tag{23}$$

is a concave function of $x \in \mathbb{R}^n$ whenever $a \le s < t \le b$ and $z \in \mathbb{R}^n$. Then the following assertions hold true.

(i) *The intrinsic characteristic $\boldsymbol{x}(t) = \boldsymbol{x}(t, t_0, x_0)$ emanating from any point $(t_0, x_0) \in [a, b] \times \mathbb{R}^n$ is well-defined for $a \le t_0 \le t \le b$.*
(ii) *$(t, t_0, x_0) \mapsto \boldsymbol{x}(t, t_0, x_0)$ is continuous.*
(iii) *For any $t_0 \in [a, b)$, the right derivative $\dot{\boldsymbol{x}}_+(t_0)$ exists and*

$$\dot{\boldsymbol{x}}_+(t_0) = \nabla_p H(x_0, p^\sharp(t_0, x_0))$$

where $p^\sharp$ is defined by (7). In particular, $\dot{\boldsymbol{x}}_+(t_0) \neq 0$ if

$$0 \notin \nabla_p H(x_0, \partial_x^+ S(t_0, x_0)).$$

Proof Assertion (i) is correct by virtue of the concavity hypothesis on (23).

(ii) Let $a \le s_j \le t_j \le b$ and $x_j \in \mathbb{R}^n$ be such that $(t_j, s_j, x_j) \to (\hat{t}, \hat{s}, \hat{x})$ as $j \to \infty$. We shall demonstrate that $\xi_j := \boldsymbol{x}(t_j, s_j, x_j) \to \boldsymbol{x}(\hat{t}, \hat{s}, \hat{x}) =: \hat{\xi}$. We distinguish between two cases: (a) $\hat{t} > \hat{s}$ and (b) $\hat{t} = \hat{s}$.

Case (a). We can assume that $s_j < t_j$ for all j. In setting

$$F_j(x) = S(t_j, x) - A_{t_j - s_j}(x_j, x) \quad \text{and} \quad F(x) = S(\hat{t}, x) - A_{\hat{t}-\hat{s}}(\hat{x}, x),$$

we have $\{\xi_j\} = \arg\max F_j$ and $\{\hat{\xi}\} = \arg\max F$. The idea is to reduce the maximization problem to convergence of convex minimization problems and apply resolvent convergence. To this end, the functions

$$G_j(x) = -F_j(x) - \frac{\varepsilon}{2}|x|^2 \quad \text{and} \quad G(x) = -F(x) - \frac{\varepsilon}{2}|x|^2$$

are convex and

$$\xi_j = (\partial G_j + \varepsilon I)^{-1}(0) \quad \text{and} \quad \hat{\xi} = (\partial G + \varepsilon I)^{-1}(0), \tag{24}$$

where ∂G denotes the subdifferential of G in the sense of convex analysis. It is elementary to verify that G_j epi- as well as hypo-converges to G, i.e.,

$$G(x) = \lim_{j\to\infty} G_j(y_j) \quad \text{whenever} \quad y_j \to x.$$

In the context of Moreau–Yosida approximation of convex functions, it is well known that $G_j \to G$ in the sense of epi-convergence implies that the resolvents of the subdifferential operators converge according to

$$\lim_{j\to\infty} (I + \lambda \partial G_j)^{-1}(x) = (I + \lambda \partial G)^{-1}(x) \tag{25}$$

for all $\lambda > 0$ and all $x \in \mathbb{R}^n$. (For a general statement connecting, in certain reflexive Banach spaces, Mosco convergence $G_j \to G$ to (25) we refer to [6, Thm. 3.26].) Since (25) can be recast as

$$\lim_{j\to\infty} \left(\lambda^{-1} I + \partial G_j\right)^{-1} (\lambda^{-1} x) = \left(\lambda^{-1} I + \partial G\right)^{-1} (\lambda^{-1} x),$$

it follows from (24) that $\xi_j \to \hat{\xi}$ as $j \to \infty$ thus completing the proof of case (a).

Case (b). The task is to demonstrate that $\xi_j = \boldsymbol{x}(t_j, s_j, x_j) \to \hat{x}$ because $\boldsymbol{x}(\hat{s}, \hat{s}, \hat{x}) = \hat{x}$. Without loss of generality, we can assume that $s_j < t_j$ for all j since $\xi_j = x_j$ if $s_j = t_j$. As noted in the proof of case (a), it suffices to demonstrate that G_j epi-converges to G where, now, $G(x) = \infty$ if $x \neq \hat{x}$ while

$$G(\hat{x}) = -S(\hat{t}, \hat{x}) - \frac{\varepsilon|\hat{x}|^2}{2}.$$

To this end, let first $y_j \to y \neq \hat{x}$; then, using (10),

$$G_j(y_j) \geq (t_j - s_j)\ell_0\left(\frac{y_j - x_j}{t_j - s_j}\right) - S(t_j, y_j) - \frac{\varepsilon|y_j|^2}{2} \tag{26}$$

Since $0 < t_j - s_j \to 0$ and $y_j - x_j \to y - \hat{x} \neq 0$, the first term in the right-hand side of (26) tends to ∞ as $j \to \infty$, while the other terms converge to real numbers, yielding

$$\liminf_{j\to\infty} G_j(y_j) \geq \infty = G(y). \tag{27}$$

Now, let instead $y_j \to \hat{x}$. As $\min \ell_0 = -\ell_0^*(0)$, we find that

$$\liminf_{j\to\infty} G_j(y_j) \geq \liminf_{j\to\infty} \left(-(t_j - s_j)\ell_0^*(0) - S(t_j, y_j) - \frac{\varepsilon|y_j|^2}{2}\right) = G(\hat{x}). \tag{28}$$

Moreover, as

$$(t_j - s_j)\ell_0(0) \leq A_{t_j - s_j}(x_j, x_j) \leq (t_j - s_j)L(x_j, 0)$$

we have $\lim_{j\to\infty} A_{t_j - s_j}(x_j, x_j) = 0$ and, hence,

$$G_j(x_j) = A_{t_j - s_j}(x_j, x_j) - S(t_j, x_j) - \frac{\varepsilon|x_j|^2}{2} \to G(\hat{x}). \tag{29}$$

On account of (27), (28) and (29), $G_j \to G$ in the sense of epi-convergence; see (21)–(22). Observing that

$$\arg\min_{x\in\mathbb{R}^n}\left(G(x)+\frac{\varepsilon|x|^2}{2}\right)=\{\hat{x}\},$$

(24) is satisfied for $\hat{\xi}=\hat{x}$, which ensures that $\xi_j\to\hat{x}$. This concludes the proof of continuity.

Part (iii) is an extension from time-independent $S(x)$ to our evolutionary setting. It can be obtained with virtually the same proof as in [11, Prop. 3.4]. To this end, pick any sequence $a\le t_0<t_k\le b$ for $k=1,2,\dots$ such that $t_k\to t_0$ as $k\to\infty$ and consider the sequence of average velocities

$$v_k=\frac{\boldsymbol{x}(t_k)-x_0}{t_k-t_0}.$$

By part (ii), $\boldsymbol{x}(t_k)\to x_0$ and hence we can choose $\delta>0$ such that $(t_k,\boldsymbol{x}(t_k))\in\mathcal{C}=[t_0,b]\times\overline{B}(x_0,\delta)$ for all $k\ge 1$. The function $S(t,x)$ is Lipschitz continuous and semiconcave in $\mathcal{C}$. Let $\boldsymbol{\xi}_k(\cdot)\colon[t_0,t_k]\to\mathbb{R}^n$ be an arc that achieves the minimum of the excess action functional yielding $\boldsymbol{x}(t_k)$, i.e., $\boldsymbol{\xi}_k(t_0)=x_0$ and

$$E(\boldsymbol{\xi}_k;t_0,t_k)\le E(\boldsymbol{\xi};t_0,t_k) \tag{30}$$

for all Lipschitz arcs $\boldsymbol{\xi}(\cdot)$ such that $\boldsymbol{\xi}(t_0)=x_0$; see (14). Owing to Lemma 3.3 and Proposition 3.4 in [11], we may assume that v_k converges and, in fact, that

$$v_0=\lim_{k\to\infty}v_k=\lim_{k\to\infty}\dot{\boldsymbol{\xi}}_k(t_k) \tag{31}$$

for some $v_0\in\mathbb{R}^n$. By (30) and the definition of E, remembering that $\boldsymbol{\xi}_k(t_k)=\boldsymbol{x}(t_k)$, we note that the transversality condition

$$p_k:=\nabla_v L(\boldsymbol{x}(t_k),\dot{\boldsymbol{\xi}}_k(t_k))\in\partial_x^+ S(t_k,\boldsymbol{x}(t_k))$$

holds as well; cf. (15). By (31),

$$\lim_{k\to\infty}p_k=\nabla_v L(x_0,v_0)=:p_0. \tag{32}$$

Owing to the semiconcavity of S, as confirmed in Lemma 3 following this proof, there exists $\omega_k\in\mathbb{R}$ such that $(\omega_k,p_k)\in\partial^+S(t_k,\boldsymbol{x}(t_k))$. By extracting a further subsequence, we may assume that $(\omega_k,p_k)\to(\omega_0,p_0)$ as $k\to\infty$ for some $\omega_0\in\mathbb{R}$. It follows that

$$(\omega_0,p_0)\in\partial^+S(t_0,x_0) \tag{33}$$

for the semiconcavity of S implies that $\partial^+ S$ is outer semicontinuous. Furthermore, for some constant C of semiconcavity of $S|_{\mathcal{C}}$, we have, for any $(\omega, p) \in \partial^+ S(t_0, x_0)$,

$$(\omega_k - \omega)(t_k - t_0) + \langle p_k - p, \boldsymbol{x}(t_k) - x_0 \rangle \leq C((t_k - t_0)^2 + |\boldsymbol{x}(t_k) - x_0|^2) \quad (34)$$

as $(\omega_k, p_k) \in \partial^+ S(t_k, \boldsymbol{x}(t_k))$. Dividing through in (34) by $t_k - t_0 > 0$, and then passing to the limit, yields

$$\omega_0 - \omega + \langle p_0 - p, v_0 \rangle \leq 0. \quad (35)$$

Finally, (32) and (35) imply

$$\begin{aligned} \omega_0 + H(x_0, p_0) &= \omega_0 + \langle p_0, v_0 \rangle - L(x_0, v_0) \\ &\leq \omega + \langle p, v_0 \rangle - L(x_0, v_0) \leq \omega + H(x_0, p), \end{aligned}$$

revealing that $(\omega_0, p_0) = (\omega^\sharp(t_0, x_0), p^\sharp(t_0, x_0))$ as (ω, p) is an arbitrary element of $\partial^+ S(t_0, x_0)$ and (33) holds. Finally, again taking into account (32), we deduce that the initial velocity of $\boldsymbol{x}(\cdot)$ is given by

$$\dot{\boldsymbol{x}}_+(t_0) = v_0 = \nabla_p H(x_0, p^\sharp(t_0, x_0)),$$

completing the proof. □

Lemma 3 *If $(t_0, x_0) \in Q$ and $p_0 \in \partial_x^+ S(t_0, x_0)$, then $(\omega_0, p_0) \in \partial^+ S(t_0, x_0)$ for some $\omega_0 \in \mathbb{R}$.*

Proof Choose $\varrho > 0$ and $c \geq 0$ such that $\overline{B}_\varrho(t_0, x_0) \subset Q$ and such that the function defined by

$$f(\tau, \xi) = c(\tau^2 + |\xi|^2) - S(t_0 + \tau, x_0 + \xi) \quad \text{when} \quad (\tau, \xi) \in \overline{B}_\varrho(0, 0),$$

while $f(\tau, \xi) = \infty$ otherwise, is a convex and lower semicontinuous function defined on $\mathbb{R}^{1+n}$. By our assumptions, $-p_0 \in \partial_\xi f(0, 0)$, where $\partial_\xi f$ and later ∂f are subdifferentials in the sense of convex analysis. The Hahn–Banach theorem ensures the existence of $\omega_0 \in \mathbb{R}$ such that $(-\omega_0, -p_0) \in \partial f(0, 0)$ and hence $(\omega_0, p_0) \in \partial^+ S(t_0, x_0)$. Consult, e.g., Section 1 in [40]. □

The following relaxed concavity condition on (23) is pertinent for globally Lipschitz continuous initial data; see Proposition 6 and Theorem 3.

Proposition 5 *The conclusions of Proposition 4 remain fulfilled if there exists $\lambda > 0$ such that (23) is concave in the open ball $B(z, \lambda(t - s))$ and, in addition,*

$$\emptyset \neq \arg\max_{x \in \mathbb{R}^n} (S(t, x) - A_{t-s}(z, x)) \subset B(z, \lambda(t - s))$$

whenever $a \leq s < t \leq b$ and $z \in \mathbb{R}^n$.

Proof We adapt the proof of case (a) for assertion (ii) of Proposition 4 to the present conditions. To this end, we define $\mathcal{G}_j$ by $\mathcal{G}_j(x) = G_j(x)$ if $x \in \overline{B}(x_j, \lambda(t_j - s_j))$ and by $\mathcal{G}_j(x) = \infty$ otherwise; similarly, $\mathcal{G}(x) = G(x)$ if $x \in \overline{B}(\hat{x}, \lambda(\hat{t} - \hat{s}))$ and $\mathcal{G}(x) = \infty$ otherwise. The functions $\mathcal{G}_j$ and $\mathcal{G}$ are convex and lower semicontinuous. We claim that $\mathcal{G}_j$ epi-converges to $\mathcal{G}$; cf. (21)–(22). Let $y_j \to y$. We shall demonstrate that $\mathcal{G}(y) \le \liminf \mathcal{G}_j(y_j)$. It is enough to obtain this inequality assuming $y_j \in \overline{B}(x_j, \lambda(t_j - s_j))$ for all j, from which it follows that $y \in \overline{B}(\hat{x}, \lambda(\hat{t} - \hat{s}))$ and

$$\liminf_{j\to\infty} \mathcal{G}_j(y_j) = \liminf_{j\to\infty} G_j(y_j) = G(y) = \mathcal{G}(y).$$

Let $z \in \overline{B}(\hat{x}, \lambda(\hat{t} - \hat{s}))$. Then, a sequence $z_j \in \overline{B}(x_j, \lambda(t_j - s_j))$ can be chosen such that $z_j \to z$. For this sequence, $\mathcal{G}_j(z_j) \to \mathcal{G}(z)$. If instead $z \notin \overline{B}(\hat{x}, \lambda(\hat{t} - \hat{s}))$ then $\mathcal{G}_j(z) = \infty$ eventually and so $\mathcal{G}_j(z) \to \infty = \mathcal{G}(z)$.

We omit the proofs of the other parts since they follow by the same arguments as in Proposition 4. □

4 Extension

In this section we take steps to devise curves that carry singularities at all later times after a singularity has come into existence.

Definition 6 For a given Tonelli Lagrangian L and an initial datum S_0, suppose that there exists an increasing sequence of times

$$0 < t_0 < t_1 < t_2 < \cdots \quad \text{with} \quad \lim_{k\to\infty} t_k = \infty$$

such that

$$x \mapsto S(t, x) - A_{t-t_k}(z, x)$$

has a unique maximizer for each $k \in \mathbb{N}$, every $t \in (t_k, t_{k+1}]$ and every $z \in \mathbb{R}^n$. Then, for a given initial point $(t_0, x_0) \in (0, \infty) \times \mathbb{R}^n$, the *extended intrinsic characteristic* $\boldsymbol{x}\colon [t_0, \infty) \to \mathbb{R}^n$ is defined by setting $\boldsymbol{x}(t_0) = x_0$ and, proceeding recursively, by

$$\{\boldsymbol{x}(t)\} = \operatorname*{arg\,max}_{x\in\mathbb{R}^n} (S(t, x) - A_{t-t_k}(\boldsymbol{x}(t_k), x)) \quad \text{when} \quad t \in (t_k, t_{k+1}]$$

for each $k = 0, 1, 2, \ldots$.

We shall impose conditions ensuring that the objective function (17) becomes a uniformly concave function on $\mathbb{R}^n$ (or on some sufficiently large ball) uniformly in t for times, e.g.,

- ($\mathcal{A}$) $t \in (t_0, t_0 + \tau]$ for some constant $\tau > 0$ which is independent of the initial point (t_0, x_0); or
- ($\mathcal{G}$) $t/t_0 \in (1, \theta]$ for some constant $\theta > 1$ which is independent of the initial point (t_0, x_0).

Assuming $(\mathcal{A})$, we may proceed by defining $t_{k+1} = t_k + \tau$ for each $k \in \mathbb{N}$, which means that $t_k = t_0 + k\tau$. If $(\mathcal{G})$ is assumed, then $t_{k+1} = \theta t_k$ for all $k \in \mathbb{N}$, i.e., $t_k = \theta^k t_0$. The labels "$(\mathcal{A})$" and "$(\mathcal{G})$" refer to, respectively, the arithmetic and geometric progression of the generated sequence $(t_k)_{k=0}^{\infty}$. In either of these clear-cut cases, the extended intrinsic characteristic $\boldsymbol{x} : [t_0, \infty) \to \mathbb{R}^n$ emanating from a general point x_0 is well-defined. However, we shall in fact adopt a more flexible partition-based construction. We assume that the time interval is divided into bounded subintervals. In case $(\mathcal{A})$ we could employ the partition

$$(0, \infty) = \bigcup_{j=0}^{\infty} (j\tau, (j+1)\tau], \tag{36}$$

while we could use the following one when $(\mathcal{G})$ is assumed:

$$(0, \infty) = \bigcup_{j=-\infty}^{\infty} (\theta^j, \theta^{j+1}]. \tag{37}$$

Either of these setups, (36) or (37), induces a decomposition of Q as a union of disjoint time strips. For instance,

$$Q = \bigcup_{j \in \mathbb{Z}} (\theta^j, \theta^{j+1}] \times \mathbb{R}^n$$

when $(\mathcal{G})$ is assumed.

In general, we shall select an increasing sequence $(T_j)_{j \in \mathcal{J}}$, where the index set $\mathcal{J}$ is either $\mathbb{N}$ or $\mathbb{Z}$, such that

$$\bigcup_{j \in \mathcal{J}} (T_j, T_{j+1}] = (0, \infty). \tag{38}$$

This allows us to construct the characteristic piecewise over time intervals where concavity holds uniformly. For a given initial time $t_0 > 0$ we pick $j_0 \in \mathcal{J}$ such that $t_0 \in [T_{j_0}, T_{j_0+1})$ and set $t_k = T_{j_0+k}$ for all positive integers k. If the intrinsic characteristic $\boldsymbol{x}(t)$ generated by the sequence $(t_k)_{k=0}^{\infty}$ is well-defined for any $(t_0, x_0) \in Q$, a mapping

$$\mathcal{X} : \{(t, s, x) \in \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n : t \geq s > 0\} \to \mathbb{R}^n$$

is defined by $\mathcal{X}(t, t_0, x_0) = \boldsymbol{x}(t)$.

We are now ready to state a basic technical result about extended intrinsic characteristics. Part (ii) of the following theorem expresses perpetual propagation of singularities. We remind the reader that $p^\sharp(t, x)$ is defined by (7).

Theorem 2 *Let (A), (B) and (C) be met. Assume that an increasing nonnegative sequence $(T_j)_{j \in \mathcal{J}}$ fulfilling (38) exists, along with a positive sequence $(\varepsilon_j)_{j \in \mathcal{J}}$, such that, for each $j \in \mathcal{J}$,*

- *either*

$$x \mapsto S(t,x) - A_{t-s}(z,x) + \frac{\varepsilon_j |x|^2}{2} \tag{39}$$

  is concave in $\mathbb{R}^n$,
- *or there exists* $\lambda > 0$ *such that the function in (39) is concave in the open ball* $B(z, \lambda(t-s))$ *and*

$$\emptyset \neq \arg\max_{x \in \mathbb{R}^n} (S(t,x) - A_{t-s}(z,x)) \subset B(z, \lambda(t-s)),$$

whenever $s > 0$, $T_j \leq s < t \leq T_{j+1}$ *and* $z \in \mathbb{R}^n$. *Then, the mapping* $\mathcal{X}$ *is well-defined and continuous. Moreover, the following assertions hold true for any point* $(t_0, x_0) \in Q$ *and the curve* $\boldsymbol{x}(t) = \mathcal{X}(t, t_0, x_0)$.

(i) *If* $(t^\circ, \boldsymbol{x}(t^\circ)) \notin \Sigma$ *for a certain* $t^\circ \in (t_0, \infty)$, *then* $\boldsymbol{x}(\cdot)$ *coincides in* $[t_0, t^\circ]$ *with the unique minimizing curve for* $S(t^\circ, \boldsymbol{x}(t^\circ))$.
(ii) *If* $t^\bullet \in [t_0, \infty)$ *and* $(t^\bullet, \boldsymbol{x}(t^\bullet)) \in \Sigma$, *then* $(t, \boldsymbol{x}(t)) \in \Sigma$ *for all* $t \in [t^\bullet, \infty)$.
(iii) *The initial velocity is given by* $\dot{\boldsymbol{x}}_+(t_0) = \nabla_p H(x_0, p^\sharp(t_0, x_0))$. *In fact,*

$$\dot{\boldsymbol{x}}_+(t_k) = \nabla_p H(\boldsymbol{x}(t_k), p^\sharp(t_k, \boldsymbol{x}(t_k))) \quad \text{for all} \quad k = 0, 1, 2, \ldots.$$

Proof A straightforward induction proof can be constructed on the basis of Propositions 3–5. □

Therefore, the objective of establishing perpetual propagation of singularities is accomplished as soon as the assumptions of Theorem 2 are satisfied.

5 Lipschitz continuity, semiconcavity and uniform convexity

In order to apply Theorem 2, we need to investigate the semiconcavity properties of $S(t,\cdot)$ and the uniform convexity properties of $A_t(y,\cdot)$ since the key mechanism behind the construction of intrinsic characteristics is the uniform concavity of $S(t,\cdot) - A_{t-s}(z,\cdot)$. However, we first examine, in Subsection 5.1, how global Lipschitz continuity can facilitate the analysis. After that we turn to the semiconcavity of $S(t,\cdot)$ in Subsection 5.2 and to the uniform convexity of $A_t(y,\cdot)$ in Subsection 5.3. We recall that conditions (A), (B) and (C) are assumed (see Subsection 2.1 for their formulation).

5.1 Lipschitz continuity

Using in particular condition (A) on the Lagrangian it is possible to show that if S_0 is globally Lipschitz continuous then so is S.

Proposition 6 *Let* $S_0 \colon \mathbb{R}^n \to \mathbb{R}$ *be globally Lipschitz continuous. Then the following assertions are true.*

(i) S is globally Lipschitz continuous in $\overline{Q} = [0, \infty) \times \mathbb{R}^n$.

(ii) There exists a constant $\lambda > 0$ *such that*

$$\emptyset \neq \arg\max_{x \in \mathbb{R}^n} \big(S(t,x) - A_{t-t_0}(x_0,x)\big) \subset B(x_0, \lambda(t-t_0))$$

for every $(t_0, x_0) \in Q$ *and every* $t \in (t_0, \infty)$.

(iii) There exists a constant $\lambda > 0$ *such that* $|\dot{\boldsymbol{\xi}}(s)| < \lambda$ *when* $s \in [0,t]$ *for every minimizing arc* $\boldsymbol{\xi}$ *for* $S(t,x)$ *for any point* $(t,x) \in Q$.

Proof (i) Let S_0 be Lipschitz continuous with constant C_0. We first prove the assertion under the provisional condition that $S_0 \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$. Let us denote by $(\boldsymbol{x}(t,y), \boldsymbol{p}(t,y))$ the solution of Hamilton's system

$$\begin{aligned} \dot{\boldsymbol{x}} &= \nabla_p H(\boldsymbol{x}, \boldsymbol{p}), \\ \dot{\boldsymbol{p}} &= -\nabla_x H(\boldsymbol{x}, \boldsymbol{p}), \end{aligned}$$

that satisfies the initial condition $(\boldsymbol{x}(0), \boldsymbol{p}(0)) = (y, \nabla S_0(y))$ for an arbitrary $y \in \mathbb{R}^n$. The Hamiltonian is conserved along the bicharacteristics $(\boldsymbol{x}, \boldsymbol{p})$, i.e.,

$$H(\boldsymbol{x}(s,y), \boldsymbol{p}(s,y)) = H(y, \nabla S_0(y)) \tag{40}$$

for all $s > 0$. Taking into account condition (A) and Remark 3, we infer from (40) the bounds

$$\ell_1^*(\boldsymbol{p}(s,y)) \leq H(\boldsymbol{x}(s,y), \boldsymbol{p}(s,y)) \leq \ell_0^*(\nabla S_0(y)) \leq C_1,$$

and so $|\boldsymbol{p}(s,y)| \leq C$. Here,

$$C_1 = \sup_{|p| \leq C_0} \ell_0^*(p) < \infty$$

and C is the smallest constant such that $\ell_1^*(p) \leq C_1$ implies $|p| \leq C$. Clearly, C depends only on C_0, ℓ_0 and ℓ_1. Now, take any nonsingular point $(t,x) \in Q$. The minimization problem for $S(t,x)$ has a unique solution given by $\boldsymbol{x}(s,y)$ when $s \in [0,t]$ for a certain unique initial point y such that $\boldsymbol{x}(t,y) = x$. Moreover,

$$\nabla S(t,x) = \nabla S(t, \boldsymbol{x}(t,y)) = \boldsymbol{p}(t,y)$$

and, hence, $|\nabla S(t,x)| \leq C$. Since Σ has vanishing Lebesgue measure, $|\nabla S| \leq C$ almost everywhere.

Now we remove the assumption that $S_0 \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$. We approximate S_0 from above by means of the Lasry–Lions method [7, 36]. In particular, this scheme preserves the Lipschitz continuity. We get $S_0 \leq S_\varepsilon \in C^{1,1}(\mathbb{R}^n)$ such that, on the one hand, $S_\varepsilon \to S_0$ uniformly as $\varepsilon \downarrow 0$ and, on the other,

$$|S_\varepsilon(x_1) - S_\varepsilon(x_2)| \leq C_0 |x_1 - x_2| \tag{41}$$

for all $x_k \in \mathbb{R}^n$ and $\varepsilon > 0$. We proceed by defining

$$S^\varepsilon(t,x) = \inf_{y\in\mathbb{R}^n} (S_\varepsilon(y) + A_t(y,x)).$$

Since $\inf_{\varepsilon>0} S_\varepsilon(y) = S_0(y)$, interchanging infima,

$$S(t,x) = \inf_{\varepsilon>0} S^\varepsilon(t,x) = \lim_{\varepsilon\downarrow 0} S^\varepsilon(t,x).$$

By (41) and the first part of the proof,

$$|S^\varepsilon(t,x_1) - S^\varepsilon(t,x_2)| \le C|x_1 - x_2|. \tag{42}$$

Sending $\varepsilon \downarrow 0$ in (42) shows that $S(t,\cdot)$ is globally Lipschitz continuous with constant C for every $t > 0$. Finally, as the Hamilton–Jacobi equation (1) is satisfied at all nonsingular points, also $\partial S/\partial t$ is globally bounded almost everywhere in Q owing to the bounds $|\nabla S| \le C$ and

$$-\ell_0^*(\nabla S(t,x)) \le \frac{\partial S}{\partial t} \le -\ell_1^*(\nabla S(t,x)) \quad \text{almost everywhere,}$$

concluding the proof.

(ii) For a fixed arbitrary point $(t_0,x_0) \in Q$, and any $t > t_0$, we set

$$\Phi(x) = S(t,x) - A_{t-t_0}(x_0,x).$$

Taking into account (i), it follows that $\Phi(x) \to -\infty$ as $|x| \to \infty$ and so Φ has at least one maximizer. If x maximizes Φ, then using part (i) and (10),

$$\begin{aligned} 0 \ge \Phi(x_0) - \Phi(x) &= S(t,x_0) - S(t,x) + A_{t-t_0}(x_0,x) - A_{t-t_0}(x_0,x_0) \\ &\ge -C|x-x_0| + (t-t_0)\ell_0\left(\frac{x-x_0}{t-t_0}\right) - (t-t_0)\ell_1(0), \end{aligned}$$

and, hence,

$$C\frac{|x-x_0|}{t-t_0} - \ell_0\left(\frac{x-x_0}{t-t_0}\right) \ge -\ell_1(0). \tag{43}$$

Let $\lambda > 0$ be a constant such that

$$C|v| - \ell_0(v) \ge -\ell_1(0) \quad \Rightarrow \quad |v| < \lambda.$$

It follows from (43) that $|x - x_0| < \lambda(t - t_0)$.

(iii) We fix $0 < s < t$ and $x \in \mathbb{R}^n$. Let $\boldsymbol{\xi}$ be a minimizing arc for $S(t,x)$. By the principle of optimality, for any τ such that $s < s + \tau < t$,

$$S(s+\tau, \boldsymbol{\xi}(s+\tau)) = \inf_{z\in\mathbb{R}^n} \Psi(z) = \Psi(\boldsymbol{\xi}(s))$$

where

$$\Psi(z) = S(s,z) + A_\tau(z, \xi(s+\tau)).$$

Thus,

$$\begin{aligned} 0 &\geq \Psi(\xi(s)) - \Psi(\xi(s+\tau)) \\ &= S(s,\xi(s)) - S(s,\xi(s+\tau)) + A_\tau(\xi(s),\xi(s+\tau)) - A_\tau(\xi(s+\tau),\xi(s+\tau)) \\ &\geq -C|\xi(s+\tau)-\xi(s)| + \tau\ell_0\left(\frac{\xi(s+\tau)-\xi(s)}{\tau}\right) - \tau\ell_1(0), \end{aligned}$$

and, hence, $v_\tau = (\xi(s+\tau)-\xi(s))/\tau$ satisfies

$$C|v_\tau| - \ell_0(v_\tau) \geq -\ell_1(0). \tag{44}$$

Sending $\tau \downarrow 0$ in (44) yields $|\dot\xi(s)| < \lambda$. □

5.2 Semiconcavity

Under appropriate additional conditions, $S(t,x) - \frac{1}{2}c(t)|x|^2$ is a concave function of $x \in \mathbb{R}^n$ for some function $c(t)$. The viscosity solution $S(t,x)$ of (1)–(2) possesses this global semiconcavity property in x if suitable semiconcavity hypotheses are imposed on L or S_0. We recall some estimates from Section 6.4 in [18] and combine these with what we learned about Lipschitz continuity in the previous subsection.

Lemma 4 *Assume that there exist constants $\gamma \geq 0$ and $\alpha \geq 0$ such that $S_0 - \frac{1}{2}\gamma|\cdot|^2$ is a concave function while $x \mapsto L(x,v) - \frac{1}{2}\alpha|x|^2$ is concave for any $v \in \mathbb{R}^n$. Then*

$$x \mapsto S(t,x) - \frac{(\gamma+\alpha t)|x|^2}{2}$$

is concave in $\mathbb{R}^n$ for each $t \in (0,\infty)$.

Proof Let $\xi(\cdot)$ be a minimizing arc for $S(t,x)$ and set

$$\xi_\pm(s) = \xi(s) \pm h \quad \text{when} \quad s \in [0,t].$$

Then, since $\xi_\pm(t) = x \pm h$,

$$\begin{aligned} &S(t,x+h) + S(t,x-h) - 2S(t,x) \\ &\leq S_0(\xi(0)+h) + S_0(\xi(0)-h) - 2S_0(\xi(0)) \\ &+ \int_0^t \left[L(\xi(s)+h,\dot\xi(s)) + L(\xi(s)-h,\dot\xi(s)) - 2L(\xi(s),\dot\xi(s))\right] ds \\ &\leq \gamma|h|^2 + \int_0^t \alpha|h|^2\, ds = (\gamma+\alpha t)|h|^2. \end{aligned}$$

By virtue of Lemma 1, the proof is complete. □

In our next lemma S_0 is only assumed to satisfy condition (C).

Lemma 5 *Suppose that* $L(x,v) - \frac{1}{2}(\alpha|x|^2 + \beta|v|^2)$ *is a concave function of* $(x,v) \in \mathbb{R}^n \times \mathbb{R}^n$ *for some constants* $\alpha \geq 0$ *and* $\beta > 0$. *Then*

$$x \mapsto S(t,x) - \left(\frac{\alpha t}{6} + \frac{\beta}{2t}\right)|x|^2$$

is concave in $\mathbb{R}^n$ *for each* $t \in (0,\infty)$.

Proof Let $\boldsymbol{\xi}(\cdot)$ be a minimizing arc for $S(t,x)$ and set

$$\boldsymbol{\xi}_\pm(s) = \boldsymbol{\xi}(s) \pm \frac{s}{t}h \quad \text{when} \quad s \in [0,t].$$

Then, since $\boldsymbol{\xi}_\pm(0) = \boldsymbol{\xi}(0)$ and $\boldsymbol{\xi}_\pm(t) = x \pm h$,

$$\begin{aligned} &S(t,x+h) + S(t,x-h) - 2S(t,x) \\ &\leq \int_0^t \left[L(\boldsymbol{\xi}_+(s), \dot{\boldsymbol{\xi}}_+(s)) + L(\boldsymbol{\xi}_-(s), \dot{\boldsymbol{\xi}}_-(s)) - 2L(\boldsymbol{\xi}(s), \dot{\boldsymbol{\xi}}(s))\right] ds \\ &\leq \int_0^t \frac{(\alpha s^2 + \beta)|h|^2}{t^2}\, ds = \left(\frac{\alpha t}{3} + \frac{\beta}{t}\right)|h|^2. \end{aligned} \tag{45}$$

In light of Lemma 1, this completes the proof. □

If S_0 is globally Lipschitz continuous, then the hypothesis in Lemma 5 that L be globally semiconcave can be cancelled owing to Proposition 6 and hypothesis (B).

Lemma 6 *Let* S_0 *be globally Lipschitz continuous. Then there exist two constants* $\alpha \geq 0$ *and* $\beta > 0$ *such that*

$$x \mapsto S(t,x) - \left(\frac{\alpha t}{6} + \frac{\beta}{2t}\right)|x|^2$$

is concave in $\mathbb{R}^n$ *for each* $t \in (0,\infty)$.

Proof By virtue of Proposition 6, there is a constant $\lambda > 0$ such that every minimizing arc $\boldsymbol{\xi}(s)$ for the value function is Lipschitz continuous of rate λ. Referring to the proof of Lemma 5, fix $t > 0$ and restrict h to $h \in \overline{B}(0,t)$. Since

$$\dot{\boldsymbol{\xi}}_\pm(s) = \dot{\boldsymbol{\xi}}(s) \pm t^{-1}h$$

we have $|\dot{\boldsymbol{\xi}}_\pm(s)| \leq \lambda + 1$ for all $s \in [0,t]$. Moreover, condition (B) ensures that the Hessian $\nabla^2 L(x,v)$, a $2n \times 2n$ matrix, satisfies, for some positive constant c,

$$\nabla^2 L(x,v) \leq cI \quad \text{for all} \quad (x,v) \in \mathbb{R}^n \times \overline{B}(0,\lambda+1).$$

Thus $L(x,v) - \frac{1}{2}(\alpha|x|^2 + \beta|v|^2)$ is a concave function of $(x,v) \in \mathbb{R}^n \times \overline{B}(0,\lambda+1)$ for some constants $\alpha \geq 0$ and $\beta > 0$. It is readily verified that (45) remains valid. □

To give an application of Lemma 6, we recall that a continuous function $S(x)$ is called a *weak KAM solution* of

$$H(x, \nabla S(x)) = 0 \quad \text{in } \mathbb{R}^n \tag{46}$$

if S has the fixed point property that

$$S(x) = \inf_{y \in \mathbb{R}^n} (S(y) + A_t(y, x)) \tag{47}$$

for all $t \in (0, \infty)$ and all $x \in \mathbb{R}^n$. We can easily recover some known properties of $S(x)$ [11, 29].

Proposition 7 *Let $S(x)$ be a weak KAM solution of (46). Then S is globally Lipschitz continuous and globally semiconcave.*

Proof First we notice that the right-hand side of (47) is a locally Lipschitz continuous function of x. At every point x where S is differentiable we have

$$\ell_1^*(\nabla S(x)) \le H(x, \nabla S(x)) = 0;$$

hence, ∇S is bounded almost everywhere. Thus, S is in fact globally Lipschitz continuous. On the basis of Lemma 6, for some constants $\alpha \ge 0$ and $\beta > 0$, $S(x) - \frac{1}{2}\gamma|x|^2$ is a concave function for

$$\gamma = \inf_{t>0} \left(\frac{\alpha t}{3} + \frac{\beta}{t} \right),$$

which completes the proof. □

5.3 Uniform convexity of $A_t(y, \cdot)$

In the seminal paper [11] on intrinsic characteristics, the convexity properties of $A_t(y, x)$ were investigated in great detail. We shall make use of the following excerpt from [11, Prop. B.8].

Lemma 7 *For any $\lambda > 0$, there exist $\tau_\lambda > 0$ and $c_\lambda > 0$ such that*

$$x \mapsto A_t(y, x) - \frac{c_\lambda}{2t}|x|^2$$

is convex in the ball $B(y, \lambda t)$ for every $t \in (0, \tau_\lambda]$ and every $y \in \mathbb{R}^n$.

Under more restrictive conditions on L it is possible to derive uniform convexity not only locally but globally by using some classical elements from the calculus of variations.

Lemma 8 *Suppose that $L(x, v) + \frac{1}{2}a|x|^2 - \frac{1}{2}b|v|^2$ is a convex function of $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$ for some constants $a \ge 0$ and $b > 0$. Then the following assertions are true.*

(i) *If $a = 0$, then*

$$\mathbb{R}^n \ni x \mapsto A_t(y, x) - \frac{b}{2t}|x|^2$$

is a convex function for any $t \in (0, \infty)$ and any $y \in \mathbb{R}^n$.

(ii) *If $a > 0$, then for some constants $c \in (0, b)$ and $\tau > 0$,*

$$\mathbb{R}^n \ni x \mapsto A_t(y, x) - \frac{c}{2t}|x|^2$$

is a convex function for any $t \in (0, \tau]$ and any $y \in \mathbb{R}^n$.

Proof Let $x_k \in \mathbb{R}^n$ along with $y \in \mathbb{R}^n$ and $t > 0$ be fixed and let $\boldsymbol{\xi}_k(s)$ be arcs such that $\boldsymbol{\xi}_k(0) = y$, $\boldsymbol{\xi}_k(t) = x_k$ and $A_t(y, x_k) = \int_0^t L(\boldsymbol{\xi}_k(s), \dot{\boldsymbol{\xi}}_k(s))\, ds$ for $k = 0, 1$. For any $\lambda \in (0, 1)$ we set

$$x_\lambda = (1 - \lambda)x_0 + \lambda x_1 \quad \text{and} \quad \boldsymbol{\xi}_\lambda(s) = (1 - \lambda)\boldsymbol{\xi}_0(s) + \lambda \boldsymbol{\xi}_1(s).$$

Then, as the curve $\boldsymbol{\xi}_\lambda$ is feasible for $A_t(y, x_\lambda)$, we have

$$A_t(y, x_\lambda) \le \int_0^t L(\boldsymbol{\xi}_\lambda(s), \dot{\boldsymbol{\xi}}_\lambda(s))\, ds$$

where the integrand satisfies the following inequality

$$\begin{aligned} L(\boldsymbol{\xi}_\lambda(s), \dot{\boldsymbol{\xi}}_\lambda(s)) \le{}& (1 - \lambda)L(\boldsymbol{\xi}_0(s), \dot{\boldsymbol{\xi}}_0(s)) + \lambda L(\boldsymbol{\xi}_1(s), \dot{\boldsymbol{\xi}}_1(s)) \\ &+ \lambda(1 - \lambda)\left(a|\boldsymbol{\xi}_1(s) - \boldsymbol{\xi}_0(s)|^2 - b|\dot{\boldsymbol{\xi}}_1(s) - \dot{\boldsymbol{\xi}}_0(s)|^2\right)/2 \end{aligned}$$

by virtue of the convexity hypothesis on L. Hence,

$$A_t(y, x_\lambda) \le (1 - \lambda)A_t(y, x_0) + \lambda A_t(y, x_1) - \lambda(1 - \lambda)\mathcal{F}(\boldsymbol{\xi}_1 - \boldsymbol{\xi}_0) \tag{48}$$

where $\mathcal{F}$ is the functional defined by

$$\mathcal{F}(\boldsymbol{\xi}) = \frac{1}{2}\int_0^t \left(b|\dot{\boldsymbol{\xi}}(s)|^2 - a|\boldsymbol{\xi}(s)|^2\right) ds$$

for arcs $\boldsymbol{\xi} \in \mathrm{Lip}([0, t]; \mathbb{R}^n)$. For the problem at hand, the relevant endpoint conditions for $\mathcal{F}(\boldsymbol{\xi})$ are $\boldsymbol{\xi}(0) = 0$ and $\boldsymbol{\xi}(t) = x_1 - x_0$. If $a = 0$ we find that

$$\mathcal{F}(\boldsymbol{\xi}) \ge \frac{b}{2t}|x_1 - x_0|^2.$$

In view of (48), this proves (i).

(ii) Suppose that $a > 0$. Then

$$\mathcal{F}(\xi) = \frac{b}{2}\int_0^t \left(|\dot{\xi}(s)|^2 - \omega^2|\xi(s)|^2\right) ds$$

for $\omega = \sqrt{a/b} > 0$. Under the additional assumption that $\omega t < \pi$, $\mathcal{F}$ has a unique extremal, namely

$$\xi(s) = \frac{\sin \omega s}{\sin \omega t}(x_1 - x_0) \quad \text{when} \quad s \in [0, t].$$

This extremal minimizes $\mathcal{F}$ under the endpoint conditions in question since it can be embedded in a field of extremals (see, e.g., Problem 47 in [31]). A computation results in

$$\min \mathcal{F} = \frac{\omega b \cot \omega t}{2}|x_1 - x_0|^2.$$

Returning to (48), we conclude that

$$x \mapsto A_t(y, x) - \frac{\omega b \cot \omega t}{2}|x|^2$$

is a convex function. To finish the proof, we choose $\tau > 0$ such that $\omega\tau = \theta < \pi/2$. It can be checked that $c \in (0, b)$ and $\omega b \cot \omega t \geq c/t$ when $t \in (0, \tau]$ if $c = b\theta \cot \theta$. □

6 Principal results

Our main results draw, in particular, on Theorem 2 and Section 5. To help the reader navigate the paper we recall that the standing conditions (A) and (B) on the Lagrangian are stated in Subsection 2.1 and that Section 5 contains relevant assertions on the semiconcavity properties of $S(t, \cdot)$ (Lemmas 4–6) and the uniform convexity properties of $A_t(z, \cdot)$ (Lemmas 7–8) which are crucial for Theorem 2. Each of the following three theorems provides an affirmative answer to the perpetual propagation problem (P). To help the reader understand the structure and ideas of proofs, and to clarify the role of these lemmas, we mention that each of these theorems combines one lemma about $S(t, \cdot)$ with one about $A_t(z, \cdot)$ according to the following:

- Theorem 3 builds on Proposition 6 together with Lemmas 6 and 7;
- Theorem 4 is based on Lemma 5 and Lemma 8; while
- Theorem 5 combines Lemma 4 and Lemma 8.

6.1 Singular propagation under global Lipschitz continuity

We utilize the simplifying effect of global Lipschitz continuity in this subsection.

Theorem 3 *Let conditions (A) and (B) be satisfied and let S_0 be globally Lipschitz continuous. Then, the hypotheses and conclusions of Theorem 2 are fulfilled.*

Proof On account of Lemma 6,

$$S(t,x)-\left(\frac{\alpha t}{6}+\frac{\beta}{2t}\right)|x|^2 \tag{49}$$

is a concave function of $x\in\mathbb{R}^n$ for some constants $\alpha>0$ and $\beta>0$ since S_0 is globally Lipschitz continuous. By Proposition 6, there exists $\lambda>0$ such that

$$\emptyset\neq\operatorname*{arg\,max}_{x\in\mathbb{R}^n}\left(S(t,x)-A_{t-s}(z,x)\right)\subset B(z,\lambda(t-s)) \tag{50}$$

for all $z\in\mathbb{R}^n$ and all $0<s<t$. Furthermore, for this value of λ, Lemma 7 furnishes $\tau>0$ and $c>0$ such that

$$x\mapsto\frac{c}{2(t-s)}|x|^2-A_{t-s}(z,x) \tag{51}$$

is concave in the ball $B(z,\lambda(t-s))$ for every $z\in\mathbb{R}^n$ provided $0<t-s\leq\tau$. We now investigate the sum of (49) and (51). However, we prefer to multiply this sum by t to find that

$$t(S(t,x)-A_{t-s}(z,x))+\frac{1}{6}\left(\frac{3ct}{t-s}-\alpha t^2-3\beta\right)|x|^2 \tag{52}$$

is concave in $x\in B(z,\lambda(t-s))$ when $0<t-s\leq\tau$. We note that the second expression within parentheses in (52) is a decreasing function of t when $t>s$. Next we choose $\varepsilon>0$ such that $\beta+\varepsilon>c$ and continue by introducing an increasing sequence $(T_j)_{j\in\mathbb{Z}}$ by fixing $T_0>0$ and requiring that

$$\frac{3cT_{j+1}}{T_{j+1}-T_j}-\alpha(T_{j+1})^2-3\beta=3\varepsilon\quad\text{for all}\quad j\in\mathbb{Z}. \tag{53}$$

This definition requires a brief justification. To this end, (53) expresses that $T_j=f(T_{j+1})$ for the function

$$f(t)=t-\frac{3ct}{\alpha t^2+3(\beta+\varepsilon)},\qquad t\in[0,\infty).$$

A straightforward calculation shows that $f'(t)>0$ holds for all $t\in[0,\infty)$ since $\beta+\varepsilon>c$. Moreover, $f(0)=0$ and $f(t)\to\infty$ as $t\to\infty$. Therefore, f is an auto-homeomorphism of $[0,\infty)$ confirming that $(T_j)_{j\in\mathbb{Z}}$ is a well-defined positive sequence defined recursively from $T_j=f(T_{j+1})$, or $T_{j+1}=f^{-1}(T_j)$, for our preselected $T_0>0$.

Furthermore, we have $T_{j+1} - T_j = g(T_{j+1}) > 0$ for $g(t) = t - f(t)$. Therefore, $(T_j)_{j\in\mathbb{Z}}$ is an increasing positive sequence. As $g(0) = 0$ and $g(t) \to 0$ as $t \to \infty$, g has a maximum value on $[0, \infty)$. A calculation yields

$$\max g = \frac{c}{2}\sqrt{\frac{3}{\alpha(\beta+\varepsilon)}}.$$

Thus $0 < T_{j+1} - T_j \le \tau$ for all $j \in \mathbb{Z}$ if the value of ε is increased, if necessary, to make $\max g \le \tau$. It is easily verified that $T_j \to 0$ as $j \to -\infty$ while $T_j \to \infty$ as $j \to \infty$. Indeed, if $T_j \to T$ as $j \to -\infty$ or $j \to \infty$ for some finite limit T, then $0 = g(T)$ and hence $T = 0$. By virtue of (52) and (53),

$$t(S(t,x) - A_{t-s}(z,x)) + \frac{\varepsilon|x|^2}{2}$$

is a concave function of $x \in B(z, \lambda(t-s))$ whenever $T_j \le s < t \le T_{j+1}$ and $z \in \mathbb{R}^n$ since

$$\frac{3ct}{t-s} - \alpha t^2 - 3\beta \ge \frac{3ct}{t-T_j} - \alpha t^2 - 3\beta \ge \frac{3cT_{j+1}}{T_{j+1}-T_j} - \alpha(T_{j+1})^2 - 3\beta = 3\varepsilon.$$

Moreover, (50) holds true. We have constructed the infrastructure for the mapping $\mathcal{X}$ that gives rise to an extended intrinsic characteristic emanating from a general point $(t_0, x_0) \in Q$. We conclude that Theorem 2 can be invoked. □

We next relax the hypothesis that S_0 be globally Lipschitz continuous.

Corollary 1 *Let $S(\tau, \cdot)$ be globally Lipschitz continuous in $\mathbb{R}^n$ for every $\tau > 0$. Then, the hypotheses and conclusions of Theorem 2 are still fulfilled.*

By Proposition 6, the hypothesis in the corollary is equivalent to asking that S be globally Lipschitz continuous in $[\tau, \infty) \times \mathbb{R}^n$ as soon as $\tau > 0$.

Proof First choose $\tau = 2^{-1}$ and construct $1 = T_0 < T_1 < \cdots$ such that $\lim_{j\to\infty} T_j = \infty$ as in Theorem 3 employing the time shift $t \mapsto t - \tau$. This first step gives rise to a subdivision of $[1, \infty)$ which meets the requirements of uniform concavity in Theorem 2.

We proceed by partitioning each of the subintervals $[2^{-m-1}, 2^{-m}]$ as m ranges over $\mathbb{N}$ respecting the hypotheses of Theorem 2. To this end, for the base case $m = 0$, we switch to $\tau = 2^{-2}$ and define $2^{-1} = T_{N_0} < T_{N_0+1} < \cdots < T_0 = 1$ in the following way. The proof of Theorem 3 allows us to define an increasing sequence $(T_j^0)_{j\in\mathbb{Z}}$ such that $T_0^0 = 1$ and $T_j^0 \to \tau = 2^{-2}$ as $j \to -\infty$. Let $N_0 < 0$ be that index j for which $2^{-1} \in [T_j^0, T_{j+1}^0)$. The requested partition of $[2^{-1}, 1]$ is chosen as $T_{N_0} = 2^{-1}$ and $T_j = T_j^0$ if $j = N_0 + 1, \ldots, 0$. By induction on $m \ge 1$, from an increasing sequence $(T_j^m)_{j\in\mathbb{Z}}$ which is obtained by utilizing the global Lipschitz continuity of

the restriction of S to $[\tau, \infty) \times \mathbb{R}^n$ for $\tau = 2^{-m-2}$, and requiring that $T^m_{N_{m-1}} = 2^{-m}$, we can extract a finite increasing sequence such that

$$2^{-m-1} = T_{N_m} < T_{N_m+1} < \cdots < T_{N_{m-1}} = 2^{-m}.$$

This completes the construction. □

It is an interesting known fact that the hypothesis in Corollary 1 is satisfied, in particular, if the initial function S_0 is *Lipschitz in the large* [14, 35, 52] in the sense that, for some constants K_0 and K_1,

$$|S_0(x) - S_0(y)| \leq K_0 + K_1|x - y| \quad \text{for all} \quad x, y \in \mathbb{R}^n. \tag{54}$$

The functions that are Lipschitz in the large form a real vector space, which encompasses, in particular, all bounded functions and all uniformly continuous functions. In fact, this vector space is spanned by the globally Lipschitz continuous functions along with the bounded functions, because a function $f : \mathbb{R}^n \to \mathbb{R}$ is Lipschitz in the large if and only if it can be expressed as the sum of a globally Lipschitz continuous function and a bounded function [28, Prop. 10.3]. In particular, if S_0 is lower semicontinuous and Lipschitz in the large, then (C) holds. As already mentioned in Remark 1, [14] answers (P) affirmatively under the hypothesis that S_0 be Lipschitz in the large. However, this requirement can be weakened as shown below. The infimum (4) is attained at a unique point $y(t, x)$ for each nonsingular point $(t, x) \in Q$ and we set

$$v(t, x) = \frac{x - y(t, x)}{t}.$$

Proposition 8 *Under (A), (B) and (C), the implications (i) ⇒ (ii) ⇒ (iii) ⇒ (iv) hold true between the following conditions.*

(i) *S_0 is Lipschitz in the large.*
(ii) *For every $\tau > 0$ there exists a constant $c_\tau < \infty$ such that $|v(t, x)| \leq c_\tau$ for all nonsingular points $(t, x) \in [\tau, \infty) \times \mathbb{R}^n$.*
(iii) *For every $\tau > 0$ there exists a constant $\varrho_\tau < \infty$ such that $|x - y(\tau, x)| \leq \varrho_\tau$ whenever $(\tau, x) \notin \Sigma$.*
(iv) *$S(\tau, \cdot)$ is globally Lipschitz continuous in $\mathbb{R}^n$ as soon as $\tau > 0$.*

Proof "(i) ⇒ (ii)": Assume (i). For each nonsingular point $(t, x) \in Q$,

$$S_0(y(t, x)) + A_t(y(t, x), x) = S(t, x) \leq S_0(x) + t\ell_1(0). \tag{55}$$

Taking into account (54) and (10) we deduce from (55) that

$$-K_0 - K_1|x - y(t, x)| + t\ell_0\left(\frac{x - y(t, x)}{t}\right) \leq t\ell_1(0).$$

Hence, $v(t, x)$ satisfies

$$\ell_0(v(t, x)) - K_1|v(t, x)| \leq \ell_1(0) + K_0/\tau$$

provided $t \geq \tau$. This inequality implies the bound $|v(t,x)| \leq c_\tau$ where the constant depends only on τ, K_j and ℓ_j.

If (ii) holds, then (iii) holds for $\varrho_\tau = c_\tau \tau$.

"(iii) $\Rightarrow$ (iv)": Assume (iii). On account of Corollary 4.25 of Fathi's paper [28], there exists a nondecreasing function $\theta\colon [0,\infty) \to [0,\infty)$ such that $|\nabla_x A_t(y,x)| \leq \theta(A_t(y,x)/t)$ whenever $\nabla_x A_t$ exists. We also use the fact that

$$\nabla_x S(t,x) = \nabla_x A_t(y,x)|_{y=y(t,x)}$$

in conjunction with (10) and $|v(t,x)| \leq \varrho_t/t$ to conclude that there exists another constant $C_t < \infty$ such that

$$|\nabla_x S(t,x)| \leq \theta(A_t(y(t,x),x)/t) \leq \theta(\ell_1(v(t,x))) \leq C_t$$

for almost all $x \in \mathbb{R}^n$. □

Incidentally, assertion (i) of Proposition 6 is recovered by this proof for $K_0 = 0$.

Example 1 We investigate briefly the relation between the conditions in Proposition 8 for $L(x,v) \equiv L(v)$. For every nonsingular point $(t,x) \in Q$ we have

$$\nabla_x S(t,x) = \nabla L\left(\frac{x - y(t,x)}{t}\right) = \nabla L(v(t,x)),$$

so conditions (ii), (iii) and (iv) are equivalent (since (iv) is equivalent to S being globally Lipschitz continuous in $[\tau,\infty)\times\mathbb{R}^n$ for each $\tau > 0$ by Proposition 6). To see that the implication (i) $\Rightarrow$ (ii) cannot be reversed, let S_0 be the lower semicontinuous function defined by $S_0(x) = 0$ when $x \in \mathbb{Z}^n$ and $S_0(x) = |x|$ otherwise. The confirmation that (i) fails yet (ii) is fulfilled is straightforward and therefore omitted.

6.2 A result for general initial data

We shall no longer ask that $S(t,\cdot)$ be globally Lipschitz continuous when $t > 0$. In fact, assuming that L is "nearly quadratic" we address general initial data in our next theorem.

Theorem 4 *Let (A) and (B) be met and suppose, in addition, that there exist constants $a \geq 0$, $\alpha \geq 0$ and $\beta \geq b > 0$ such that*

$$(x,v) \mapsto L(x,v) + \frac{1}{2}a|x|^2 - \frac{1}{2}b|v|^2 \quad \textit{is convex} \tag{56}$$

while

$$(x,v) \mapsto L(x,v) - \frac{1}{2}\alpha|x|^2 - \frac{1}{2}\beta|v|^2 \quad \textit{is concave.} \tag{57}$$

Then, the assumptions and conclusions of Theorem 2 hold true for any function $S_0\colon \mathbb{R}^n \to (-\infty,\infty]$ that satisfies (C).

Proof By (57) and Lemma 5, (49) is a concave function. On the basis of (56), Lemma 8 furnishes $\tau > 0$ and $c > 0$ such that

$$\mathbb{R}^n \ni x \mapsto \frac{c}{2(t-s)}|x|^2 - A_{t-s}(z,x) \tag{58}$$

is concave for every $z \in \mathbb{R}^n$ provided $0 < t - s \leq \tau$. It follows from the concavity of (49) and (58) that also

$$t(S(t,x) - A_{t-s}(z,x)) + \frac{1}{6}\left(\frac{3ct}{t-s} - \alpha t^2 - 3\beta\right)|x|^2 \tag{59}$$

is concave in $x \in \mathbb{R}^n$ as long as $0 < t - s \leq \tau$. Along the lines of the proof of Theorem 3, and with the same choices of T_j and ε, we deduce that

$$t(S(t,x) - A_{t-s}(z,x)) + \frac{\varepsilon|x|^2}{2}$$

is a concave function of $x \in \mathbb{R}^n$ when $T_j \leq s < t \leq T_{j+1}$ for any $j \in \mathbb{Z}$ and any $z \in \mathbb{R}^n$. Theorem 2 can now be applied. □

Remark 4 The hypothesis (56)–(57) on L in Theorem 4 is equivalent to the requirement that there exist constants $0 \leq c_1 < c_2$ such that the full gradient of $L(x,v) - \frac{1}{2}c_2|v|^2$ is globally Lipschitz continuous with constant c_1 in $\mathbb{R}^n \times \mathbb{R}^n$. Indeed, the latter condition is equivalent to $L(x,v) - \frac{1}{2}c_2|v|^2 + \frac{1}{2}c_1(|x|^2 + |v|^2)$ being convex and $L(x,v) - \frac{1}{2}c_2|v|^2 - \frac{1}{2}c_1(|x|^2 + |v|^2)$ being concave [33, 36]. In particular, ∇L is globally Lipschitz continuous in $\mathbb{R}^n \times \mathbb{R}^n$ under (56)–(57).

6.3 Propagation for globally semiconcave initial data

In our third main theorem we are concerned with globally semiconcave initial functions.

Theorem 5 *Let (A), (B) and (C) be met and suppose, in addition, that there exist constants $a \geq 0$, $b > 0$ and $\alpha \geq 0$ such that*

$$(x,v) \mapsto L(x,v) + \frac{1}{2}a|x|^2 - \frac{1}{2}b|v|^2 \quad \textit{is convex}$$

while

$$x \mapsto L(x,v) - \frac{1}{2}\alpha|x|^2 \quad \textit{is concave for any } v \in \mathbb{R}^n.$$

Suppose $S_0 \colon \mathbb{R}^n \to \mathbb{R}$ is globally semiconcave. Then, the assumptions and conclusions of Theorem 2 hold.

Proof Owing to Lemma 4,

$$S(t,x) - \frac{(\gamma + \alpha t)|x|^2}{2} \tag{60}$$

is a concave function of $x \in \mathbb{R}^n$ for each $t > 0$, where $\gamma \geq 0$ is a constant of semiconcavity for S_0. In addition, (58) is still concave by Lemma 8. Adding (60) and (58) shows that

$$S(t,x) - A_{t-s}(z,x) + \frac{1}{2}\left(\frac{c}{t-s} - \gamma - \alpha t\right)|x|^2 \tag{61}$$

is a concave function of $x \in \mathbb{R}^n$ when $0 < t-s \leq \tau$. The expression within parentheses in (61) is a decreasing function of $t > s$. In order to devise a time partition adapted to (61), we first pick $T_1 \in (0, \tau]$ such that

$$\varepsilon := \frac{c}{T_1} - \gamma - \alpha T_1 > 0. \tag{62}$$

For this fixed $\varepsilon > 0$, we now define an increasing sequence $(T_j)_{j=0}^{\infty}$ by

$$T_0 = 0 \quad \text{and} \quad \frac{c}{T_{j+1} - T_j} - \gamma - \alpha T_{j+1} = \varepsilon \quad \text{for} \quad j = 0, 1, 2, \ldots. \tag{63}$$

We note that the value of T_1 given by (63) is consistent with (62). It is elementary to confirm that $T_j \to \infty$ as $j \to \infty$ and that $T_{j+1} - T_j$ is decreasing. Since $T_1 - T_0 = T_1 \leq \tau$, we certainly have $T_{j+1} - T_j \leq \tau$ for all $j \in \mathbb{N}$. On the basis of (61), in conjunction with our choice of $(T_j)_{j=0}^{\infty}$, we find that

$$S(t,x) - A_{t-s}(z,x) + \frac{\varepsilon|x|^2}{2}$$

is a concave function of $x \in \mathbb{R}^n$ whenever $s > 0$ and $T_j \leq s < t \leq T_{j+1}$ for any $j \in \mathbb{N}$ and any $z \in \mathbb{R}^n$. The hypotheses of Theorem 2 are now satisfied. □

6.4 A particular case

We apply Theorems 3–5, Corollary 1 and Proposition 8 to Lagrangians of the form

$$L(x,v) = K(v) - U(x). \tag{64}$$

Corollary 2 *Suppose, on the one hand, that K is a superlinear C^2 function such that $\nabla^2 K(v)$ is positive definite for every $v \in \mathbb{R}^n$ and, on the other, that U is a bounded C^2 function whose first- and second-order derivatives are all bounded. Let the Lagrangian L be defined by (64) and let S_0 satisfy (C). Then, the hypotheses and conclusions of Theorem 2 hold if one of the following additional conditions is met.*

(i) *For every $t > 0$ there exists a constant $\varrho_t < \infty$ such that $|x - y(t,x)| \leq \varrho_t$ whenever $(t,x) \notin \Sigma$, where $y(t,x)$ denotes the minimizer in (4).*
(ii) *There are constants $0 < b \leq \beta$ such that $bI \leq \nabla^2 K(v) \leq \beta I$ for all $v \in \mathbb{R}^n$.*
(iii) *S_0 is globally semiconcave and there exists a constant $b > 0$ such that $\nabla^2 K(v) \geq bI$ for all $v \in \mathbb{R}^n$.*

Proof Assertion (i) follows from Proposition 8 and Corollary 1, assertion (ii) from Theorem 4, and assertion (iii) from Theorem 5. □

7 The case where *H* depends on *p* only

This section is devoted to the following special case of (1)–(2):

$$\partial S/\partial t + H(\nabla S) = 0 \quad \text{in } Q, \tag{65}$$

$$S(0+, x) = S_0(x) \quad \text{in } \mathbb{R}^n. \tag{66}$$

The unique viscosity solution of (65)–(66) is given by the Hopf–Lax formula, i.e.,

$$S(t, x) = \inf_{y \in \mathbb{R}^n} (S_0(y) + A_t(y, x)) \quad \text{for all} \quad (t, x) \in Q,$$

where

$$A_t(y, x) = tL\left(\frac{x - y}{t}\right). \tag{67}$$

We return to the pivotal problem of maximizing the function (17), i.e.,

$$\mathbb{R}^n \ni x \mapsto S(t, x) - (t - t_0)L\left(\frac{x - x_0}{t - t_0}\right). \tag{68}$$

By virtue of Proposition 13 in the appendix, (68) tends to $-\infty$ as $|x| \to \infty$ for any $x_0 \in \mathbb{R}^n$ and any $0 < t_0 < t$. Owing to Lemma 5, global semiconcavity of $S(t, \cdot)$ can be derived for an arbitrary initial datum S_0 by assuming that L is globally semiconcave. Since L is convex, this hypothesis is equivalent to asking that L be differentiable with a globally Lipschitz continuous gradient ∇L, as we next recall.

Proposition 9 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a superlinear convex function and let α and β be positive constants such that $\alpha\beta = 1$. Then the following conditions are mutually equivalent.*

(i) *f is differentiable and ∇f is Lipschitz continuous of rate α.*
(ii) *$f - \frac{1}{2}\alpha|\cdot|^2$ is concave.*
(iii) *$f^* - \frac{1}{2}\beta|\cdot|^2$ is convex.*

Proof Condition (i) holds if and only if $f - \frac{1}{2}\alpha|\cdot|^2$ is concave and $f + \frac{1}{2}\alpha|\cdot|^2$ is convex [33, 36]. As f is assumed to be convex, we conclude that conditions (i) and (ii) are equivalent. We refer to Corollary 2.9 in [8] for a proof of the equivalence (ii) $\Leftrightarrow$ (iii). □

The following proposition is a special case of Theorems 3–4 but uses a geometric sequence $T_j = \theta^j$ for the time partition. We recall condition $(\mathcal{G})$ in Section 4.

Proposition 10 *Assume that one of the following conditions is met.*

(i) S_0 *is globally Lipschitz continuous,* $L:\mathbb{R}^n\to\mathbb{R}$ *is* C^2 *and* $\nabla^2 L(v)>0$ *for all* $v\in\mathbb{R}^n$.

(ii) S_0 *satisfies condition (C) while* L *and* H *are* C^2 *and uniformly convex.*

Then, the assumptions of Theorem 2 are fulfilled with a geometric sequence $(T_j)_{j\in\mathbb{Z}}$ *for the time partition.*

Proof We carry out the proof assuming (ii). Case (i) is analogous. By Lemma 5 in combination with Proposition 9, owing to the uniform convexity of H, there exists a constant $\beta>0$ such that

$$\mathbb{R}^n\ni x\mapsto S(t,x)-\frac{\beta|x|^2}{2t}\quad\text{is concave}\tag{69}$$

for every $t>0$. The uniform convexity of L amounts to the existence of a constant $b>0$ such that

$$L(v)-\frac{b|v|^2}{2}\quad\text{is a convex function of}\quad v\in\mathbb{R}^n.$$

Hence, by part (i) of Lemma 8, or directly from (67),

$$-A_{t-s}(z,x)+\frac{b|x|^2}{2(t-s)}\tag{70}$$

is a concave function of $x\in\mathbb{R}^n$ whenever $0<s<t$. Consequently, the sum of (69) and (70) is concave in $x\in\mathbb{R}^n$ and, hence, setting $\mu=t/s>1$,

$$x\mapsto t(S(t,x)-A_{t-s}(z,x))+\frac{1}{2}\left(\frac{b}{\mu-1}+b-\beta\right)|x|^2\tag{71}$$

is also concave. We claim that condition $(\mathcal{G})$ in Section 4 is satisfied if $\theta>1$ is chosen so as to satisfy

$$(\beta-b)\theta<\beta.\tag{72}$$

To verify this assertion, we note that if $\mu\in(1,\theta]$, taking into account (72), the second expression within parentheses in (71) satisfies

$$\frac{b}{\mu-1}+b-\beta\geq\frac{b}{\theta-1}+b-\beta=\frac{\beta-(\beta-b)\theta}{\theta-1}=:\varepsilon>0.$$

This inequality, in conjunction with (71), ensures that

$$S(t,x)-A_{t-s}(z,x)+\frac{\varepsilon|x|^2}{2t}$$

is a concave function of $x\in\mathbb{R}^n$ as long as $t/s\in(1,\theta]$, regardless of the particular choice of $(s,z)\in Q$, confirming the claim. In light of Theorem 2, the mapping $\mathcal{X}$ is well-defined if the geometric sequence $(T_j=\theta^j)_{j\in\mathbb{Z}}$ is employed. □

The next proposition is a particular case of Theorem 5. This time arithmetic progression is suitable for the time partition.

Proposition 11 *Let S_0 be globally semiconcave and such that $S_0(x) \geq -C(1+|x|)$ and let $L \in C^2(\mathbb{R}^n)$ be uniformly convex. Then, the assumptions of Theorem 2 hold true with an arithmetic sequence $(T_j)_{j\in\mathbb{N}}$.*

Proof Owing to Lemma 4, for some constant $\gamma > 0$,

$$S(t,x) - \frac{\gamma|x|^2}{2} \tag{73}$$

is a concave function of $x \in \mathbb{R}^n$ for each $t > 0$. Since (70) remains concave for some constant $b > 0$, the sum of (73) and (70), i.e.,

$$S(t,x) - A_{t-s}(z,x) + \frac{1}{2}\left(\frac{b}{t-s} - \gamma\right)|x|^2,$$

is a concave function of $x \in \mathbb{R}^n$. In conclusion, we find that $(\mathcal{A})$ of Section 4 holds for any constant $0 < \tau < b/\gamma$. Hence, using the arithmetic sequence $(T_j = j\tau)_{j\in\mathbb{N}}$, the hypotheses and conclusions of Theorem 2 hold true. □

Appendix A. Action and value function

The appendix reviews the existence and uniqueness of a viscosity solution of (1)–(2) under weaker conditions than formulated in (A) and (B). The value function V is defined by (11)–(12) or, equivalently,

$$V(t,x) = \inf_{y\in\mathbb{R}^n} (S_0(y) + A_t(y,x)) \quad \text{for all} \quad (t,x) \in Q.$$

It is known since the early work on viscosity solution theory that V is a viscosity solution of the Hamilton–Jacobi equation [39].

Proposition 12 *Let $L(x,v)$ be a continuous function such that $v \mapsto L(x,v)$ is convex for every $x \in \mathbb{R}^n$ and $L(x,v) \geq \ell_0(v)$ where $\ell_0(v)/|v| \to \infty$ as $|v| \to \infty$. Let $S_0 \colon \mathbb{R}^n \to (-\infty,\infty]$ satisfy (C). Then, on the one hand, each minimizing arc for the value function is Lipschitz continuous and, on the other, V is locally Lipschitz continuous in Q and a viscosity solution of (1)–(2) satisfying the initial condition $\lim_{t\downarrow 0} V(t,\cdot) = S_0(\cdot)$ pointwise and in the sense of epi-convergence. Moreover, V is locally semiconcave in Q if L is locally semiconcave in $\mathbb{R}^n \times \mathbb{R}^n$.*

Proof Since L is autonomous, by the very general results of [19, 20, 27, 39], V is a locally Lipschitz continuous viscosity solution. The assertion about semiconcavity is proved in Section 6.4 in [18]. □

In particular, V is locally semiconcave away from $t = 0$ if $L \in C^2$. We point out that local semiconcavity of V cannot be expected if the smoothness condition on L is relaxed to, say, $L \in C^1$.

Example 2 Let $L(v)$ be a convex superlinear function on $\mathbb{R}^n$ and consider $S_0(x) = L(x)$. Then V is the jointly convex function given by

$$V(t,x) = (1+t)L\left(\frac{x}{1+t}\right) \quad \text{for all} \quad (t,x) \in Q.$$

It follows that V is locally semiconcave if and only if $L \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)$.

Under the hypothesis on L formulated in the first sentence in Proposition 12, we go back to the Hamilton–Jacobi equation

$$\partial S/\partial t + H(x, \nabla S) = 0, \qquad (t,x) \in Q_T = (0,T) \times \mathbb{R}^n, \tag{74}$$

for a finite and positive T. It is easy to see that a lower semicontinuous function $S_0 \colon \mathbb{R}^n \to (-\infty, \infty]$ can be *defined* as the pointwise limit $S_0(x) := \lim_{t \downarrow 0} S(t,x)$ for any viscosity solution of (74) alone. For the value function V associated with L and the limit function S_0, it is well-known that $S \le V$ in Q_T and that $S \ne V$ cannot be ruled out [44], in general. However, the equality $S = V$ in Q_T is ensured, in particular, by the following condition.

(D) For some $\tau > 0$ and every choice of $x \in \mathbb{R}^n$ and $\varepsilon \in (0, \tau)$,

$$\lim_{|y| \to \infty} (A_t(y,x) - A_{t+\varepsilon}(y,x)) = \infty$$

uniformly with respect to $t \in (0, \tau - \varepsilon)$.

Theorem 6 *Let $L(x,v)$ be a continuous function such that $v \mapsto L(x,v)$ is convex for every $x \in \mathbb{R}^n$ and $L(x,v) \ge \ell_0(v)$ where $\ell_0(v)/|v| \to \infty$ as $|v| \to \infty$. Let $S \in C(Q_T)$ be a viscosity solution of (74). Then, S coincides in Q_T with the value function V associated with $S_0(\cdot) := \lim_{t \downarrow 0} S(t,\cdot)$ and the Lagrangian L, i.e., S is given by (11)–(12) or*

$$S(t,x) = \inf_{y \in \mathbb{R}^n} (S_0(y) + A_t(y,x)) \quad \textit{for all} \quad (t,x) \in Q_T,$$

if condition (D) is fulfilled or if $S(t,x) \ge -C_T(1+|x|)$ for all $(t,x) \in Q_T$. Furthermore, (D) is met if one of the following conditions holds true.

(i) *$L(x,v) \equiv L(v)$ is independent of x.*

(ii) *$x \mapsto L(x,0)$ is bounded from above and there exists a nondecreasing function $\delta \colon [0,\infty) \to [0,\infty)$ with $\liminf_{s \to \infty} \delta(s)/s > 0$ such that*

$$L(x, \lambda v) \le (1-\lambda)L(x,0) + \lambda L(x,v) - \lambda(1-\lambda)\delta(|v|)$$

for all $\lambda \in [0,1]$, $x \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$.

(iii) *There exist constants* $C > 0$ *and* $\varepsilon > 0$ *such that, on the one hand,* $L(x, 0) \leq C(1 + |x|^2)$ *for all* $x \in \mathbb{R}^n$ *and, on the other, the epigraph* $\mathcal{E}(x)$ *of* $v \mapsto L(x, v) - \varepsilon|v|^2$*, i.e.,*

$$\mathcal{E}(x) = \left\{(v, \alpha) \in \mathbb{R}^n \times \mathbb{R} \colon L(x, v) - \varepsilon|v|^2 \leq \alpha\right\},$$

is star-shaped with respect to $(0, L(x, 0))$ *for every* $x \in \mathbb{R}^n$.

Proof We refer to [42, Thm. 4] and specifically to [43] for (i) and to [44, Thm. 2] for (ii) and (iii). □

The sharpness of conditions (ii) and (iii) in Theorem 6 is discussed in [44].

Remark 5 It is an elementary fact that if $L_0(x, v)$ satisfies condition (D) and $B(x, v)$ is a bounded function, then also $L = L_0 + B$ satisfies (D). In particular, by Theorem 6(i), a Lagrangian $L(x, v)$ satisfies (D) if there exists a convex superlinear function $L_0 \colon \mathbb{R}^n \to \mathbb{R}$ such that $L(x, v) - L_0(v)$ is bounded on $\mathbb{R}^n \times \mathbb{R}^n$. This observation applies to Corollary 2.

The existence of maximizers of (17) is also ensured by a variant of condition (D).

Proposition 13 *The function (17) approaches* $-\infty$ *as* $|x| \to \infty$ *if*

$$\lim_{|x|\to\infty} (A_{t-t_0}(x_0, x) - A_{t-\varepsilon}(x_0, x)) = \infty \tag{75}$$

for some $\varepsilon \in (0, t_0)$*. Condition (75) holds in particular if one of conditions (i), (ii) and (iii) of Theorem 6 is met.*

Proof Since we do not exclude the possibility that $S_0(x_0) = \infty$, we use the identity

$$S(t, x) = \inf_{z\in\mathbb{R}^n} (S(\varepsilon, z) + A_{t-\varepsilon}(z, x))$$

for times $t > t_0 > \varepsilon$ in combination with (75) to find that

$$S(t, x) - A_{t-t_0}(x_0, x) \leq S(\varepsilon, x_0) + A_{t-\varepsilon}(x_0, x) - A_{t-t_0}(x_0, x) \to -\infty$$

as $|x| \to \infty$. We observe that $A_t(y, x) = A_t^L(y, x)$ for the Lagrangian L is identical to $A_t^{L_r}(x, y)$ for the reflected Lagrangian $L_r(x, v) = L(x, -v)$. It follows that (75) is fulfilled if at least one of conditions (i), (ii) and (iii) of Theorem 6 is fulfilled. □

We close the paper by recalling some cases where the points along a minimizing arc for (11)–(12), except perhaps for the endpoints, lie in the complement of $\overline{\Sigma}$.

The following is a result of Albano [1, Thm. 2.3].

Lemma 9 *Given* $(t, x) \in Q \setminus \overline{\Sigma}$*, let* $\boldsymbol{\xi}$ *be the unique minimizer of (12) for the terminal point* (t, x) *under conditions (A), (B) and (C). Then* $(s, \boldsymbol{\xi}(s)) \notin \overline{\Sigma}$ *for all* $s \in (0, t)$.

Assuming $S_0 \in C^2$, let us denote by $(\boldsymbol{x}(t, y), \boldsymbol{p}(t, y))$ the solution of Hamilton's system that satisfies $(\boldsymbol{x}(0), \boldsymbol{p}(0)) = (y, \nabla S_0(y))$. For any terminal point $(t, x) \in Q$, each minimizer for the variational problem (11)–(12) has the form $\boldsymbol{x}(\cdot, y)$ for some $y \in \mathbb{R}^n$ such that $\boldsymbol{x}(t, y) = x$. The point (t, x) is called *conjugate* if there exists a point $y \in \mathbb{R}^n$ such that $\boldsymbol{x}(\cdot, y)$ is a minimizer of (11)–(12) with $\det \nabla_y \boldsymbol{x}(t, y) = 0$. The set of conjugate points is denoted by Γ.

Lemma 10 *Assume (A), (B), (C) and $S_0 \in C^2$. Then $\overline{\Sigma} = \Sigma \cup \Gamma$ and $S \in C^2(Q \setminus \overline{\Sigma})$. Moreover, if $\boldsymbol{\xi}$ is a minimizing arc for (11)–(12) for some terminal point $(t, x) \in Q$, then $(s, \boldsymbol{\xi}(s)) \notin \overline{\Sigma}$ for every $s \in (0, t)$.*

Proof Consult [30] or Theorems 6.3.6, 6.4.11 and Corollary 6.5.2 of [18]. □

These lemmas were called upon in the proof of Proposition 3. We remark that the singular set is generically dense if S_0 is merely Lipschitz continuous [37, 51].

Remark 6 The singular set Σ_F of a closed nonempty set F consists of those points in $\complement F = \mathbb{R}^n \setminus F$ that have more than one nearest point in F. It is an intriguing fact that Σ_F is dense in $\complement F$, or even in $\mathbb{R}^n$, for most compact sets (in the sense of Baire category) F [51]. Let $S_0(x) = \frac{1}{2} d_F^2(x)$ and $L(v) = \frac{1}{2}|v|^2$. Then $S_0 - \frac{1}{2}|\cdot|^2$ is a concave function and the unique viscosity solution S of (1)–(2) is given by

$$S(t, x) = V(t, x) = \frac{d_F^2(x)}{2(1 + t)} \quad \text{for all} \quad (t, x) \in Q.$$

Thus, $\Sigma(S) = (0, \infty) \times \Sigma_F$ is dense in $(0, \infty) \times \complement F$ for such compact sets F. If instead $S_0(x) = d_F(x)$, then S_0 is globally Lipschitz continuous, with constant 1, and

$$S(t, x) = \begin{cases} d_F(x) - t/2 & \text{if} \quad t \le d_F(x), \\ d_F^2(x)/2t & \text{if} \quad t > d_F(x). \end{cases}$$

Still, $\Sigma(S) = (0, \infty) \times \Sigma_F$.

Santilli [41] proved that the singular set $\Sigma_{\complement\Omega}$ is dense in Ω for most open convex sets $\Omega \subset \mathbb{R}^n$ with C^1 boundary. Furthermore, under the assumptions that $n = 1$, $H(x, p) \equiv H(p)$ and $H \in C^2(\mathbb{R})$ is uniformly convex, Li [38] studied the set $\mathcal{D}$ of all globally Lipschitz continuous $S_0 \in C^1(\mathbb{R})$ such that almost all points on the real axis are original points of singular curves. Li was able to demonstrate that $\mathcal{D}$ is a dense subset of $C^1 \cap W^{1,\infty}$.

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