

**FOUNDATIONS OF  
NONSTANDARD ANALYSIS  
WITHOUT URELEMENTS**

**by**

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## Preface

The purpose of this text is twofold.

The first purpose is to develop basic nonstandard analysis by an axiomatic approach. The axiom system we use is a modified version of an axiom system introduced by Henson (See Henson, [1], pp. 1-49). We use this (modified) axiom system to prove the most fundamental properties of nonstandard extensions of superstructures, e.g. the transfer principle. We also prove that nonstandard extensions (defined by these axioms) exist. We also introduce *saturation*, and we prove that every superstructure has a  $\kappa$ -saturated extension, for every transfinite cardinal  $\kappa$ . All this means that our definition of nonstandard extension, gives rise to the "right" concept, i.e. essentially the same concept as nonstandard analysts usually work with. We do all this in full technical detail, not leaving many details to the reader.

The second purpose is to define and use our superstructures *within ZFC, without urelements*. To do this, we introduce a general type of hierarchy of sets, of which superstructures as well as the von Neumann cumulative hierarchy are special cases. We develop a little theory about such hierarchies, and we define a superstructure as such a hierarchy, satisfying some extra conditions. It is these superstructures we use to define nonstandard extensions of, using axioms. This is also done in full technical detail.

*A note on how to read the proofs.* The theorems and propositions in this text often consist of many results grouped together. For example, Proposition 3.16 consists of 10 parts, denoted by (1)—(10). When referring to different parts of different theorems or propositions, confusion may arise about which theorem or proposition is referred to. For example, suppose that theorems A and B consists of ten parts each, and that we now are proving part 10 of Theorem B. If it then says, e.g. "by (4) and (5) of Theorem A", does it then mean "by (4) of Theorem A and (5) of Theorem A", or does it mean "by (4) of Theorem B (which is supposed to be proved already) and (5) of Theorem A"? I have tried to resolve ambiguities of this type by a systematic usage of commas and the word "and" in a specific way.

I give some examples, to clarify this systematic usage.

"by (4) and (5) of Theorem A" means "by (4) of Theorem A and (5) of Theorem A".

"by (4), and (5) of Theorem A" means "by (4) of the present theorem and (5) of Theorem A".

"by (3), (4), and (5) of Theorem A" means "by (3) of Theorem A, (4) of Theorem A, and (5) of Theorem A".

"by (3), and (4) and (5) of Theorem A" means "by (3) of the present theorem, (4) of Theorem A, and (5) of Theorem A".

"by (3) and (4), and (5) of Theorem A", means "by (3) of the present theorem, (4) of the present theorem, and (5) of Theorem A".

"by (5) of Theorem A, and (6)" means "by (5) of Theorem A and (6) of the present theorem".

This list is by no means complete, and I do not claim to have resolved all possible ambiguities, but hopefully, the reader will now be able to read the proofs without confusion.

The proofs are very detailed. All earlier results needed for a proof are mentioned in the proof, except that definitions are not always mentioned, and I fully understand those readers who think that it is too much reiteration of many results, for example, of (5) of Proposition 3.6 and (3) of Proposition 4.8, which after a while might become so obvious for the reader that he/she might have difficulties to see exactly how they apply in the given situation. Still, for completeness, I have included all this information. But the reader should not feel obliged to look up exactly how all these earlier results mentioned apply, if he/she clearly sees that the conclusion of the given step in the proof follows anyway.

There is one exception of this given completeness rule: Proposition 4.3 is given once and for all and is then never referred to after that. Without this exception, the amount of reiteration would simply become unbearable. The reader could then go back and check the 23 parts of this proposition, whenever he/she finds an argument difficult to follow. Some of these 23 parts could perhaps apply.

I would like to thank Leif Arkeryd, Nigel Cutland, and Ward Henson, who were organizers of the July 1996 NATO Advanced Study Institute in Nonstandard Analysis, which was held at the University of Edinburgh, and which I got the privilege to attend. It was at this conference I learned most of what I now know about nonstandard analysis. In particular, it was there Henson presented his axioms for the first time. I also would like to thank the other lecturers and attendants at this conference; in particular, Renling Jin and Peter Loeb.

Also, I would like to thank Jan-Christoph Puchta in Freiburg, Germany, who, in correspondence in the Usenet group sci.math, gave me deep insights in hierarchies of sets. In fact, the main theorem about extensions of sets, Theorem 3.27, is based upon his idea. It was his ideas who gave me the impulse to develop this theory of hierarchies of sets.

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## CHAPTER 1

### Introduction

When Newton and Leibniz developed Calculus in the 17th century, they imagined infinitely small quantities, which Leibniz called *infinitesimals*, and which he denoted by  $dx$ ,  $dy$ , etc. These (if positive) were considered to be smaller than any positive real number, and yet not 0! An integral such as  $\int f(x) dx$ , for example, was considered as an infinite sum of infinitesimals  $f(x) dx$ . This approach turned out to be very fruitful, and this *infinitesimal calculus* led to an amazing development of mathematics and applied sciences during the 18th century. But Newton, Leibniz, and their successors all failed to develop a consistent theory about those infinitesimals. Their reasoning was based more on intuition than rigor, and, as was eloquently pointed out by Berkeley, the infinitesimals were actually used in a contradictory manner (sometimes they were considered to be nonzero, sometimes zero).<sup>1</sup> Still, because of their great intuitive skill, their results were essentially correct.

In the 19th century, however, there was a need to find a rigorous foundation of analysis. This led to the replacement of infinitesimals by *limits*, who were introduced by Cauchy and given a rigorous definition by Weierstrass, the so called  $\varepsilon$ - $\delta$ -definition. Thus, from the mid-19th-century on, infinitesimals were banned from all serious mathematics. Analysis is now completely rigorous, but, as many mathematics students can affirm, the  $\varepsilon$ - $\delta$ -definition is rather difficult to grasp and cumbersome to use.

In 1960, however, Abraham Robinson realized that one can give a rigorous foundation of the infinitesimals, based upon 20th century developments in mathematical logic and model theory. He called this theory *Nonstandard Analysis* [8], which we in this text abbreviate as NSA. ("Standard Analysis" would be analysis based upon the  $\varepsilon$ - $\delta$ -definition.) The basic idea is that the field of real numbers,  $\mathbb{R}$ , is extended to a larger field  ${}^*\mathbb{R}$ , whose elements are called *hyperreal numbers*, in such a way that every statement about real numbers can be "transferred" to a corresponding statement about hyperreal numbers, which is true if and only if the original statement is. The precise formulation of this is the *transfer principle*, of which there is a version in this text (Theorem 4.16). In  ${}^*\mathbb{R}$ , there are positive hyperreals, called *infinitesimals*, who are smaller than every positive (standard) real, as well as there are *infinite* hyperreals greater than all standard reals. It turns out that one can define limits and derivatives in terms of infinitesimals, and that these definitions are much simpler than the standard definitions with  $\varepsilon$  and  $\delta$ . Many theorems in analysis can be given nonstandard proofs (i.e. proofs using NSA), much simpler than the standard proofs. There even exists undergraduate Calculus textbooks based upon nonstandard analysis (See Keisler, [4]).

Similar extensions can be applied to other sets than  $\mathbb{R}$ , for example to topological spaces and measure spaces. Indeed, nonstandard methods have been successfully applied in many fields, such as real analysis, functional analysis, topology, measure theory, probability theory, differential equations, and applied fields such as mathematical physics and mathematical finance. NSA there gives us a powerful tool for obtaining results that are much harder to obtain and verify using standard meth-

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<sup>1</sup>A nice book about the history of Calculus, where Berkeley's criticisms can be found, is Edwards: *The historical development of Calculus* [2]

ods. However, this text does not cover applications of NSA. For this, we refer the reader to the literature.

Instead, we concentrate upon the *foundations* of NSA. For this, Robinson used set theory, logic, and model theory, and this is how it is still most commonly done. This follows the main trend of contemporary mathematics, to consider every field of mathematics (after reformulation, if necessary) as a subdiscipline of set theory, using some of the common existing axiomatizations of set theory, such as ZFC (Zermelo-Fraenkel with the Axiom of Choice) or NBG (von Neumann-Bernays-Gödel). A slightly different approach was given by Edward Nelson in his *Internal Set Theory* [7], where he extended standard set theory with a new primitive predicate: "standard", adding some new axioms for this predicate. This internal set theory, however, is difficult to combine with some aspects NSA where one mixes standard and nonstandard methods, such as Loeb Measure Theory (See Ross, [1], pp. 91-120). In this text, we do not use this internal set theory, but we use a more standard (Robinsonian) approach: founding NSA in ZFC.

However, as was pointed out by Ward Henson, most existing introductions to NSA are less available to many mathematicians (not to talk about applied scientists). Quoting Henson:

"All of the existing introductions [to NSA] have one or more of the following features: (A) heavy use of logical formalism right from the start; (B) early introduction of set theoretical apparatus in excess of what is needed for most applications; (C) dependence on an explicit construction of the nonstandard model, usually by means of the ultrapower construction." ([1], p. 1)

To avoid these drawbacks, Henson, in the same text, gives a "gentle introduction" to NSA. He defines a *nonstandard extension* by giving an *axiom system*, in which neither formal logic nor advanced set theory is used. Later, he introduces simple logical formulas, sufficient for formulating and proving the transfer principle (which itself is taken as the main axiom of NSA in many texts). He does all this first with objects of only one kind, e.g. real numbers. Then he introduces objects of several kinds (e.g. scalars and vectors in a vector space). He then continues to consider sets of objects, and finally he introduces *superstructures*, where one freely can form subsets, unions, power sets etc, of given sets (see below). For this, he gives some extra axioms, making a total of seven (or eight, depending on how one counts). He also treats, briefly, an important property of some nonstandard extensions:  $\kappa$ -*saturation* (see Definitions 7.10), where  $\kappa$  is a transfinite cardinal. It is desirable that a nonstandard extension has this property; the greater the cardinal  $\kappa$  is, the better.

The most common way of constructing a nonstandard extension is the *ultrapower construction*. Henson mentions this in his article, but he never carries it out, nor does he prove that the extension so constructed satisfies his axioms. There is, of course, good reasons for that, since the article only is intended as a rather brief introduction to NSA<sup>2</sup>, and a too many technical details would probably be misdirected for that purpose. Nevertheless, there is, in my opinion, a need to carry out those technical details, for the sake of completeness. To do this is one of the two purposes with this text. We prove, using a modified version of Henson's axiom

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<sup>2</sup>Henson's article was originally lecture notes used at a "NATO Advanced Study Institute" in NSA, held at the University of Edinburgh, Scotland, in July, 1996.

system, that for any transfinite cardinal  $\kappa$  there exists a  $\kappa$ -saturated extension of any given superstructure (Theorem 7.13).

To explain the second purpose of this text, we need to define "superstructure". To motivate this definition, suppose that we start from the real numbers,  $\mathbb{R}$ . In real analysis, it is not always sufficient to just talk about real numbers, but we need also talk about sets, relations, functions, and also families of sets, relations, functions etc. on real numbers, and all these types objects should have their counterparts for  ${}^*\mathbb{R}$ . For this purpose, we define<sup>3</sup> the *superstructure* over  $\mathbb{R}$  as the set  $S = \cup_{n=0}^{\infty} S_n$ , where  $S_n$  is defined recursively by:

$$S_0 = \mathbb{R}, \quad S_{n+1} = S_n \cup \mathcal{P}(S_n),$$

where  $\mathcal{P}(S_n)$  denotes the *power set* of  $S_n$ . The *rank*,  $r(x)$ , of an object  $x \in S$  is defined as the smallest number  $n$  such that  $x \in S_n$ . Thus, we start from real numbers, then we take sets of real numbers, then sets of sets of real numbers, etc. getting sets of successively greater complexity. The point is that, for sets in  $S$ , we can take subsets, unions, power sets, etc. without coming outside  $S$ . The set  $S$  is then sufficiently large for our needs in the vast majority of all applications. (One could continue and define  $S_\omega = S$ ,  $S_{\omega+1} = S_\omega \cup \mathcal{P}(S_\omega)$ , etc. but this is seldom necessary in applications.)

One generally assumes that the elements with rank 0, in this case the real numbers, are *urelements*.<sup>4</sup> This means that they are *not* sets, but that they are considered as "basic objects" or "atoms". They have no elements themselves, but still, they are distinct from each other. It is not difficult to see why one makes this assumption: Without it, the real numbers might be elements of each other, and equal to sets with higher rank. For example, if we use the most common set theoretic definition of the natural numbers (which are also real numbers), we have  $0 = \emptyset$ ,  $1 = \{0\}$ ,  $2 = \{0, 1\}$ ,  $3 = \{0, 1, 2\}$ , etc., what is then the rank of the set  $\{0, 1, 2\}$ ? It should be 1, since it is a set of elements of rank 0, but it is actually 0, since this set equals the number 3. In this situation, the rank function breaks down and loses its meaning. Now, if we just were concerned with standard objects, this would be no big deal. After all, we *can* do real analysis based upon this definition of the natural numbers. But if we consider nonstandard objects (e.g. hyperreal numbers and certain sets etc. of those) this becomes fatal. It is a fundamental property of a nonstandard extension that it preserves rank. Without that property, nonstandard analysis on superstructures would not be very useful. Therefore, we need an intact rank function on the standard objects. The simplest way to achieve this is to assume that the objects with rank 0 are urelements.

However, in ZFC, which is the most commonly used axiomatic set theory today, *there are no urelements*. All objects in ZFC are sets!<sup>5</sup> By the axiom of extensionality, two objects in ZFC are equal if they have the same elements. Thus, there can be only one object with no elements: the empty set  $\emptyset$  (see Chapter 8). All of ZFC is actually built up from the empty set. Starting from  $\emptyset$ , we can generate  $\{\emptyset\}$ ,

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<sup>3</sup> Actually, we will modify this definition slightly in this text, but for the moment, we use the standard definition.

<sup>4</sup> This is a German term, which has no established English translation. Actually, the correct German plural form is "Urelementen".

<sup>5</sup> In NBG, there are no urelements either, but there are objects who are not sets, namely, (proper) classes.

$\{\emptyset, \{\emptyset\}\}$ , etc., and this, using all the axioms of ZFC, turns out to be sufficient for basing almost all contemporary mathematics in ZFC. Therefore, there is no need to complicate the theory by introducing urelements.

In NSA, though, we seem to need urelements. The most common solution to this problem is to work, not in the true ZFC, but in some modified version of ZFC in which urelements exist.

Does that mean that it is impossible to do NSA in the true ZFC, without urelements? No, there are ways of going round the lack of urelements in ZFC. One possibility would be to "redefine" the membership relation,  $\in$ , and construct a *model* within ZFC of a superstructure over a set whose elements are urelements with respect to this "new" membership relation. While this would work, it will require some model theory, and the result would in practice be a set theory without urelements anyway. So, it seems that one could equally well use a set theory with urelements from the beginning.

There is, however, another possibility, and it is the second purpose of this text to develop a theory for this. The idea is construct sets whose elements are sufficiently "urelement-like" in relation to each other, so that the superstructure over such a set has an intact rank function. We call such a set  $\omega$ -grounded (see Definitions 3.12). Adding some extra conditions such as *intransitivity* (Definition 3.20), we obtain what we call an  $\omega$ -extendable set (see Definition 3.29). We prove, not only that there exists  $\omega$ -extendable sets with arbitrarily great cardinality, but also that *any*  $\omega$ -extendable set (which can be arbitrarily large) can be extended to a larger  $\omega$ -extendable set with arbitrarily great cardinality (Theorem 3.35). This property is then used in the construction of nonstandard extensions (See Remarks 6.12 ff.). We emphasize, though, that  $\omega$ -extendability is a *sufficient* condition to achieve this goal, we do not claim that this condition is *necessary* for that.

Furthermore, when we extend these  $\omega$ -extendable sets, in Theorem 3.35, we use the well known hierarchically constructed class of sets called the *von Neumann cumulative hierarchy*, as well as modified versions of this hierarchy. The von Neumann hierarchy can be defined by transfinite recursion<sup>6</sup> thus:

For all ordinals  $\alpha$ , we define the set  $H_\alpha$  as  $\cup_{\beta < \alpha} \mathcal{P}(H_\beta)$ , where the union ranges over all ordinals  $\beta$  less than  $\alpha$ .<sup>7</sup> (Compare Definitions 3.4. See also Krivine, [5]).

The Axiom of Regularity in ZFC (see 3.3.) is equivalent to the property that every set belongs to some  $H_\alpha$ . We include a proof of this well known result (Theorem 3.17).

However, a superstructure is also a hierarchy of sets, with some features in common with the von Neumann hierarchy. For example, rank is similarly defined in both. Now, it turns out that *it is possible to define a general type of hierarchy, of which superstructures, the von Neumann hierarchy, and the hierarchies related to von Neumann's used in the extension of  $\omega$ -extendable sets, all are special cases.* (See definitions 3.4 and 4.1). (We must admit, though, that in order to fit into this true generalization, we have slightly altered the ordinary definition of rank for a superstructure (see Definitions 3.8). This is only a minor inconvenience, though.

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<sup>6</sup>Most authors would call this a "definition with transfinite *induction*". I prefer to say "recursion" instead of "induction", because in my opinion, *induction* is a method of *proof*, while *recursion* is a method of definition or computation.

<sup>7</sup>It is not necessary to have separate clause for  $\alpha = 0$ , because the union over an empty family of sets is empty, by stipulation. Thus,  $H_0 = \emptyset$ .

Alternatively, we can define the hierarchies so that we have to alter the definition of rank in the von Neumann hierarchy instead. But since the von Neumann hierarchy is more established than superstructures, we choose the other alternative.)

We thus get a unified approach to both superstructures and extension of  $\omega$ -extendable sets. Actually, Theorem 3.35 is not formulated for  $\omega$ -extendable sets, but, more generally, for  $\alpha$ -extendable sets, where  $\alpha$  is an arbitrary ordinal. This allows us to use these hierarchies also for doing NSA on generalized superstructures, where transfinite ranks are allowed, should we feel that need. In this text however, we restrict ourselves to ordinary superstructures, where only finite ranks occur. All NSA results in this text are formulated in terms of these superstructures and proved by using results about these.

Let us now give a brief summary of the contents in this text.

In Chapter 2, we briefly give the necessary definitions of relations and functions. These definitions differ from how relations and functions usually are defined. But the definitions given here are suitable for NSA. They are designed to fit in logical formulas upon which the transfer principle can be applied.

In Chapter 3, we introduce the hierarchies mentioned above, and we prove the most basic properties of these. Then, we introduce the concepts *groundedness*, *intransitivity*, and *extendability* used for defining "urelement-like" sets, and for obtaining the extensions of sets mentioned above. Then, we use this to prove that such extensions exist.

In Chapter 4, we define *proper nonstandard extension* by giving twelve axioms, of which seven are essentially taken from Henson. Three of the other guarantee that we have a true proper extension onto a superstructure over a set which is not "unnecessarily large" (Definition 4.4). Unlike Henson, we do not consider objects of different kinds, but we consider NSA on superstructures only. Starting from these axioms, we prove the most basic properties about nonstandard extensions, including the transfer principle and the internal definition principle (Theorems 4.16 and 4.27). We also briefly consider hypernatural numbers and hyperfinite sets, including the Spillover Principle for hypernatural numbers (Proposition 4.36). Hyperfinite sets are necessary for defining *enlargement* in Chapter 7.

In Chapter 5, we briefly give the most basic properties of filters and ultrafilters, necessary in chapters 6 and 7.

In Chapter 6, we prove that any superstructure has a proper nonstandard extension (we never proved that in Chapter 2), by using the *ultrapower construction*. We construct the extension, and then we prove that the twelve axioms are satisfied.

In Chapter 7, we prove that we can compose proper nonstandard extensions to obtain new proper nonstandard extensions. We also define *limiting extensions*, the result of composing an infinite number of extensions without a "last" one. Even here, the main work is to verify that these extensions satisfy the axioms. We then define  $\kappa$ -saturated extensions (Definitions 7.10), and construct such extensions by using the results just mentioned. This is called the *ultralimit construction*. We then give some important applications of saturation, such as *comprehension* (Theorem 7.15), and the existence of *enlargements* (Definition 7.17 and Corollary 7.20).

Chapter 8, finally, is an appendix where we mention those set-theoretical concepts and results that are used in the text. We use ZFC throughout, and we present its axioms. We mention the difference between sets and classes, introduce different types of relations, consider functional relations and functions (not conforming

with the relations and functions in Chapter 2). We introduce and give the most fundamental properties of ordinals and cardinals. We also give the most important equivalents of the axiom of choice. In this appendix, we include no proofs, except a proof of the Schröder-Bernstein theorem.

If we now return to Henson's three drawbacks of NSA-introductions above, have we avoided them here?

Well (A) is certainly avoided. No logical formulas are introduced until a bit into Chapter 4, where they are needed in the transfer principle and the internal definition principle. A small amount also occurs in Chapter 7. It is not more than it should be tolerated and understood by most mathematicians, applied scientists, and advanced students. No model theory at all is included.

(B), however, is hardly avoided. There is a lot of set theory, in particular in Chapter 3. But no major set theoretic knowledge is necessary for the reader. Just some basic knowledge about ordinals and cardinals, and the reader should know something about the Axiom of Choice and its most important equivalents. All set theory which is needed is covered in the appendix (Chapter 8), which is probably more exhaustive than it has to be. A reader who despise this is uncomfortable with the amount of set theory in this book, may actually just read definitions 3.4 and 3.8 in Chapter 3, then jump directly to Chapter 4, not bother about the definition of  $\omega$ -extendability, but instead consider an  $\omega$ -extendable set (which contains an  $\aleph_1$ -copy, see Definitions 2.2) as set which is characterized by the properties in Proposition 4.3. No other properties of  $\omega$ -extendable sets than these are used in Chapter 4. Then, this reader can read Chapter 4 and then jump to the definition of saturation in Chapter 7, skip the proof that  $\kappa$ -saturated extensions exist, and then finish Chapter 7. Such a person must then accept without proof that arbitrarily large  $\omega$ -extendable sets and  $\kappa$ -saturated extensions of any superstructures exist.

(C) is avoided in the same way as Henson avoided it: by an axiomatic approach. We do cover the ultrapower and ultralimit constructions, but only in order to prove that proper nonstandard extensions in general and  $\kappa$ -saturated extensions in particular exist. The axioms define what a proper nonstandard extension is, and they must be verified in order to prove that a certain construction gives such an extension.

Finally, we must clarify this, about the usage of ZFC axioms:

*We will nowhere assume that the axiom of regularity holds. The axiom of choice will not be assumed to hold except when explicitly stated. In particular, the axiom of choice is assumed to hold in all of Chapter 6.*

*The other axioms of ZFC, we use freely.*

Some results are formulated with assumptions that certain sets can be well ordered (for example, Proposition 7.14). If the axiom of choice holds, then every set can be well ordered (see Chapter 8), so a reader who is a firm believer in the axiom of choice can disregard these assumptions.

## CHAPTER 2

### Relations and Functions

2.1. REMARKS. In this chapter, we will define  $m$ -tuples, relations, and functions. These definitions are different from the usual definitions of these concepts, but they are convenient for NSA. The definitions given here are designed to be useful for the type of logical formulas used in the transfer principle (Theorem 4.16) and the internal definition principle (Theorem 4.27).

We also give the usual definitions of relations and functions in the appendix (Chapter 8). Such relations and functions are occasionally used in the text. For example, a nonstandard extension  $*$ , is a function of the usual type (see Definition 4.4).

2.2. DEFINITIONS. Given  $a_1, a_2, \dots, a_m$  ( $m \geq 0$ ). We define  $\langle a_1, a_2, \dots, a_m \rangle$  as  $\{\{\{1\}, \{1, a_1\}\}, \{\{2\}, \{2, a_2\}\}, \dots, \{\{m\}, \{m, a_m\}\}\}$ . Any set of this form is called an  $m$ -tuple. We write  $\langle \rangle = \emptyset$ , and regard this as a 0-tuple, the only one.

A class of  $m$ -tuples is called an  $m$ -ary relation. Usually, we say *unary*, *binary*, and *ternary*, instead of 1-ary, 2-ary, and 3-ary.

2.3. REMARKS. According to Definitions 2.2, natural numbers occur as elements in elements in elements of  $m$ -tuples, but what is a natural number? Various sophisticated set theoretic definitions exist (see Chapter 8), but in nonstandard analysis, most of these definitions are inconvenient, because they imply that natural numbers can be elements of other natural numbers. As was explained in the introduction, this not desirable. However, in Definitions 2.2, the nature of a natural number is unimportant, and any countably infinite set  $X$  can play the role of the set of natural numbers,  $\mathbb{N}$ , where we just identify one element of  $X$  with 0, another with 1, yet another with 2, etc. exhausting all of  $X$ . The set  $X$  can then be chosen to have the properties we desire. (See Chapter 3). We will often do so, and we will then assume that when we talk about  $m$ -tuples and  $m$ -ary relations, the natural numbers related to these are elements of this set  $X$ , and we will also write  $\mathbb{N}$  instead of  $X$ . We express this by saying that  $X$  is an  $\mathbb{N}$ -copy. When we, in any context, use such an  $\mathbb{N}$ -copy, we must keep it fixed, and not use two different  $\mathbb{N}$ -copies in the same context, unless otherwise is explicitly stated.

In all definitions and propositions in this chapter, we may without problems assume that we have any such  $\mathbb{N}$ -copy as our  $\mathbb{N}$ .

2.4. DEFINITION. For each  $k \in \mathbb{N}$ , we put  $\mathbb{N}_k = \{j \in \mathbb{N} \mid 1 \leq j \leq k\}$ .

In particular,  $\mathbb{N}_0 = \emptyset$ .

2.5. PROPOSITION. For  $m, k \geq 0$ : If  $\langle a_1, a_2, \dots, a_m \rangle = \langle b_1, b_2, \dots, b_k \rangle$ , then  $m = k$ , and  $a_i = b_i$  for all  $i$  ( $1 \leq i \leq m = k$ ).

PROOF. Without loss of generality, we assume that  $m \geq k$ . If  $m = 0$ , then the conclusion is obvious. Otherwise: For every  $i$  ( $1 \leq i \leq m$ ), there is a  $j$  ( $1 \leq j \leq k$ ), such that  $\{\{i\}, \{i, a_i\}\} = \{\{j\}, \{j, b_j\}\}$ . Then, either  $\{i\} = \{j\}$  or  $\{i\} = \{j, b_j\}$ . In both cases we must have  $j = i$ . In particular, if  $i = m$ , then  $j = m$ , whence  $m \leq k$ , and so  $m = k$ .

In general, though, we have  $\{\{i\}, \{i, a_i\}\} = \{\{i\}, \{i, b_i\}\}$ . If  $i = a_i$ , then  $\{\{i\}\} = \{\{i\}, \{i, b_i\}\}$ . Hence  $b_i = i = a_i$ . If, instead,  $i \neq a_i$ , then  $\{i, a_i\} = \{i, b_i\}$ , and  $a_i = b_i$ .

Thus, in both cases,  $a_i = b_i$ . This holds for all  $i$ :  $1 \leq i \leq m = k$ . ■

2.6. DEFINITIONS. For any  $m \geq 0$ , any  $m$ -tuple  $\langle a_1, a_2, \dots, a_m \rangle$ , and any  $i$ : ( $1 \leq i \leq m$ ), the  $i$ :th coordinate of  $\langle a_1, a_2, \dots, a_m \rangle$  is  $a_i$ . (Its uniqueness is guaranteed by Proposition 2.5)

We write

$$\pi_i^m(a_1, a_2, \dots, a_m) = a_i, \quad \text{and}$$

$$\pi_i^m(a_1, a_2, \dots, a_m) = \langle a_1, a_2, \dots, a_{i-1}, a_{i+1}, \dots, a_m \rangle.$$

These are called the *projection onto the  $i$ -th coordinate*, and the *projection omitting the  $i$ -th coordinate*, respectively.

For every  $m$ -ary relation  $A$  we also write

$$\pi_i^m[A] = \{\pi_i^m(a_1, a_2, \dots, a_m) \mid \langle a_1, a_2, \dots, a_m \rangle \in A\}$$

and

$$\pi_i^m[A] = \{\pi_i^m(a_1, a_2, \dots, a_m) \mid \langle a_1, a_2, \dots, a_m \rangle \in A\}.$$

We also put, for every  $m$ -ary relation  $A$ :

$$\mathcal{C}(A) = \bigcup_{\langle a_1, a_2, \dots, a_m \rangle \in A} \{a_1, a_2, \dots, a_m\}.$$

In other words:  $\mathcal{C}(A)$  is the set of all coordinates of all  $m$ -tuples in  $A$ .

2.7. DEFINITIONS. For any class  $A$  and any  $m \geq 0$ , we put

$$A^m = \{\langle a_1, a_2, \dots, a_m \rangle \mid a_1, a_2, \dots, a_m \in A\}.$$

If  $X$  is an  $m$ -ary relation and  $Y$  is an  $k$ -ary relation, ( $m, k \geq 0$ ) then the *cartesian product* of  $X$  and  $Y$  is the class

$$X \times Y = \{\langle a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_k \rangle \mid \langle a_1, a_2, \dots, a_m \rangle \in X, \langle b_1, b_2, \dots, b_k \rangle \in Y\}.$$

We use left association: If  $X_k$  is a  $k$ -ary relation for each  $k \geq 1$ , we define, by recursion,

$$\times_{i=1}^0 X_i = \{\emptyset\} = \{\langle \rangle\}, \quad \times_{i=1}^{k+1} X_i = (\times_{i=1}^k X_i) \times X_{k+1},$$

and we usually write  $X_1 \times X_2 \times \dots \times X_k$  instead of  $\times_{i=1}^k X_i$ , if  $k > 0$ .

We say that  $A$  is an  $m$ -ary relation *on the class  $X$* , if  $A \subseteq X^m$ .

If  $A$  and  $B$  are  $m$ -ary relations, then the *restriction* of  $A$  to  $B$  is the  $m$ -ary relation  $A \cap B$ . If  $B = C^m$  for some  $C$ , we say *restriction to  $C$*  instead of restriction to  $C^m$ .

We give special names to the following important relations:

2.8. DEFINITIONS. We put  $\text{diag } A = \{\langle a, a \rangle \mid a \in A\}$  (for "diagonal"), and  $\text{memb } A = \{\langle a, b \rangle \in A^2 \mid a \in b\}$  (for "membership relation", restricted to  $A$ ).

The following is obvious:

2.9. PROPOSITION. For all  $m$ -ary relations  $A$  ( $m \geq 1$ ) and all classes  $B$ :  $A \subseteq B^m$  if and only if  $\mathcal{C}(A) \subseteq B$ , and  $\mathcal{C}(B^m) = B$ .

2.10. DEFINITIONS. Let the sets  $A$  and  $B$  be an  $m$ -ary and  $k$ -ary relations, respectively, for some  $m, k \geq 0$ .

A function  $f : A \rightarrow B$  is a triple  $\langle A, G_f, B \rangle$ , where  $G_f \subseteq A \times B$ , and for every  $\langle a_1, a_2, \dots, a_m \rangle \in A$  there exists a unique  $\langle b_1, b_2, \dots, b_k \rangle \in B$  such that  $\langle a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_k \rangle \in G_f$ .  $A$  is called the *domain* of  $f$ , and we write  $D_f = A$ .  $B$  is called the *codomain* of  $f$ .  $G_f$  is called the *graph* of  $f$ .

For  $\langle a_1, a_2, \dots, a_m \rangle \in A$ , we write  $f(a_1, a_2, \dots, a_m) = \langle b_1, b_2, \dots, b_k \rangle$  where  $\langle b_1, b_2, \dots, b_k \rangle$  is the unique  $k$ -tuple in  $B$  such that  $\langle a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_k \rangle \in G_f$ .

If  $C \subseteq A$ , then the *image* of  $C$  under  $f$  is the set

$$f[C] = \{f(a_1, a_2, \dots, a_m) \mid \langle a_1, a_2, \dots, a_m \rangle \in C\}.$$

The range of  $f$  is the set  $R_f = f[A]$ .  $f$  is *injective* or an *injection* if, for all  $\langle a_1, a_2, \dots, a_m \rangle, \langle b_1, b_2, \dots, b_m \rangle \in A$ ,  $f(a_1, a_2, \dots, a_m) = f(b_1, b_2, \dots, b_m)$  implies  $\langle a_1, a_2, \dots, a_m \rangle = \langle b_1, b_2, \dots, b_m \rangle$ .  $f$  is *surjective* or a *surjection* if  $R_f = B$ .  $f$  is *bijective* or a *bijection* if  $f$  is both injective and surjective.

If  $f : A \rightarrow B$  and  $g : B \rightarrow C$  are functions, then  $g \circ f : A \rightarrow C$  is the (unique) function with domain  $A$  and codomain  $C$ , such that for all  $\langle a_1, a_2, \dots, a_m \rangle \in A$ , there exists  $\langle b_1, b_2, \dots, b_k \rangle \in B$  such that  $f(a_1, a_2, \dots, a_m) = \langle b_1, b_2, \dots, b_k \rangle$  and  $g(b_1, b_2, \dots, b_k) = (g \circ f)(a_1, a_2, \dots, a_m)$ .

Such a function  $g \circ f$  is called a *composite function*.

For every  $m$ -ary relation  $A$  ( $m \geq 0$ ), the *identity function on  $A$* , is the function  $\text{id}_A : A \rightarrow A$  which satisfies  $\text{id}_A(a_1, a_2, \dots, a_m) = \langle a_1, a_2, \dots, a_m \rangle$ , for all  $\langle a_1, a_2, \dots, a_m \rangle \in A$ .

If (and only if)  $f : A \rightarrow B$  is bijective, then there exists a function  $f^{-1} : B \rightarrow A$ , called the *inverse* of  $f$ , such that  $f^{-1}(b_1, b_2, \dots, b_k) = \langle a_1, a_2, \dots, a_m \rangle$  if and only if  $f(a_1, a_2, \dots, a_m) = \langle b_1, b_2, \dots, b_k \rangle$ , for all  $\langle a_1, a_2, \dots, a_m \rangle \in A$  and all  $\langle b_1, b_2, \dots, b_k \rangle \in B$ . This inverse is unique, if it exists.

If  $f = \langle A, G_f, B \rangle$  is a function, and if  $C \subseteq A$  and  $f[C] \subseteq D \subseteq B$ , then the *restriction* of  $f$  to  $C$  and  $D$  is the function  $g = \langle C, G_g, D \rangle$ , where  $G_g$  is the restriction of  $G_f$  to  $C \times D$ . Often, we say just "restriction to  $C$ ", since the codomain  $D$  is unimportant in many cases, as long as  $f[C] \subseteq D$ .

The following are obvious.

2.11 PROPOSITION. Let  $A, B, C \in H_\omega^{S'}$  be  $m$ -ary,  $k$ -ary, and  $l$ -ary relations, respectively, for some  $m, k, l \geq 0$ , and let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be functions. Then:

- (1)  $f \circ \text{id}_A = f = \text{id}_B \circ f$ .
- (2)  $\text{id}_A$  has an inverse, namely  $\text{id}_A^{-1} = \text{id}_A$ .
- (3) If  $f$  and  $g$  have inverses  $f^{-1}$  and  $g^{-1}$ , respectively, then  $g \circ f$  has an inverse, namely  $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ .
- (4) If  $f$  has an inverse  $f^{-1}$ , then  $f^{-1} \circ f = \text{id}_A$  and  $f \circ f^{-1} = \text{id}_B$ .
- (5) If  $f$  has an inverse  $f^{-1}$ , then  $f^{-1}$  has an inverse, namely  $(f^{-1})^{-1} = f$ .

2.12. REMARKS. We will now investigate "degenerate" relations and functions, i.e. when the relations involved are empty or 0-ary.

This is something we will come back to (for example, in Remarks 3.26).

It may seem peculiar to bother about these cases, but it is, in fact, important. For example, in Definitions 4.14, we define a *sentence* as a formula with 0 free variables. The class of objects satisfying a sentence is then a 0-ary relation. To obtain a good understanding of this, one therefore needs to know what a 0-ary relation is.

By Definitions 2.2, there is only one 0-tuple,  $\langle \rangle = \emptyset$ . Hence, by Definitions 2.7, there are exactly two 0-ary relations on any class:  $\{\emptyset\}$  and  $\emptyset$ . In fact,  $\emptyset$  is an  $m$ -ary relation on any class, for every natural number  $n$ . Hence,  $B^0 = \{\langle \rangle\} = \{\emptyset\}$  for any class  $B$ . In particular,  $\emptyset^0 = \{\emptyset\}$ , while  $\emptyset^m = \emptyset$  if  $m \geq 1$ .

For any  $m \geq 0$ , if  $A$  is an  $m$ -ary relation, then  $A \times \{\emptyset\} = \{\emptyset\} \times A = A$ , and  $A \times \emptyset = \emptyset \times A = \emptyset$ .

For a function  $f : \emptyset \rightarrow B$ ,  $G_f = \emptyset$  and  $f = \langle \emptyset, \emptyset, B \rangle$ . If  $A$  is a nonempty  $m$ -ary relation, then there exists no function  $f : A \rightarrow \emptyset$ , but  $f = \langle \emptyset, \emptyset, \emptyset \rangle$  is a function from  $\emptyset$  to  $\emptyset$ , with graph  $\emptyset$ . If  $B$  is a nonempty  $m$ -ary relation with  $m \geq 1$ , then the functions  $f : \{\emptyset\} \rightarrow B$  are triples of the type  $\langle \{\emptyset\}, \{\langle b_1, b_2, \dots, b_m \rangle\}, B \rangle$ , where  $\langle b_1, b_2, \dots, b_m \rangle \in B$ . These functions can therefore in a natural way be identified with the elements in  $B$ . If  $B = \{\langle \rangle\}$  and  $A$  is a nonempty  $m$ -ary relation with  $m \geq 1$ , then, for  $f : A \rightarrow B$  we have  $f = \langle A, A, B \rangle$ , that is  $G_f = A$ .

If  $A = B = \{\langle \rangle\} = \{\emptyset\}$ , then  $f = \langle \{\emptyset\}, \{\emptyset\}, \{\emptyset\} \rangle$ .

Since the defining condition is vacuously satisfied,  $\text{id}_{\emptyset} = \{\emptyset, \emptyset, \emptyset\}$ . Also,  $\text{id}_{\{\langle \rangle\}} = \langle \{\emptyset\}, \{\emptyset\}, \{\emptyset\} \rangle$ .

## CHAPTER 3

### Hierarchies of Sets

3.1. REMARK. We use the convention that small Greek letters with and without subscripts and superscripts,  $\alpha, \beta, \gamma, \alpha_1, \beta'$ , etc. denote ordinals.

3.2. REMARK. In ZF and ZFC, one usually assumes that the axiom of regularity holds. As we said in the introduction, we do not assume this here.

Nevertheless, it plays a role in this chapter, so let us state it:

3.3. AXIOM OF REGULARITY. Every nonempty set  $x$  has an element  $y \in x$  such that  $x \cap y = \emptyset$ .

This axiom is sometimes also called the *Axiom of Restriction*, or the *Axiom of Foundation*.

Now, we will define a general type of hierarchy of sets, of which superstructures, the von Neumann hierarchy, and hierarchies related to the von Neumann hierarchy used for extending sets, all are special cases.

3.4. DEFINITIONS. Let  $Y$  be a set such that  $Y \setminus \{Y\} \neq \emptyset$ . Then, the sets  $H_\alpha^Y$  are defined with transfinite recursion as

$$H_\alpha^Y = \bigcup_{\beta < \alpha} ((\mathcal{P}(H_\beta^Y) \setminus \{\emptyset\}) \cup Y),$$

where the union ranges over all ordinals  $\beta$  less than  $\alpha$ .

$H^Y$  is the class of all  $x$  for which there is an ordinal  $\alpha$  such that  $x \in H_\alpha^Y$ . For  $Y = \{\emptyset\}$ , we write  $H_\alpha$  and  $H$  instead of  $H_\alpha^{\{\emptyset\}}$  and  $H^{\{\emptyset\}}$ , respectively.

3.5. REMARKS. The condition that  $Y \setminus \{Y\} \neq \emptyset$  could also be expressed as  $Y \neq \emptyset$  and  $Y \neq \{Y\}$ .

As we will see,  $Y \neq \{Y\}$  will hold for all sets  $Y$  if the axiom of regularity holds. (See Proposition 3.16 and Theorem 3.17.) In this case, then, every nonempty set  $Y$  satisfies  $Y \setminus \{Y\} \neq \emptyset$ .

The reason that we impose this condition upon  $Y$  is that without it, we would have either  $H_\alpha^Y = \emptyset$  for all  $\alpha$  (if  $Y = \emptyset$ ), or  $H_\alpha^Y = Y = \{Y\}$  for all  $\alpha > 0$  (if  $Y = \{Y\}$ ), and that would not be very useful. Imposing this condition is sufficient for getting (5) of Proposition 3.6 below to hold.

$H$  is actually the von Neumann-hierarchy.

The reason that we remove  $\emptyset$  from  $\mathcal{P}(H_\beta^Y)$  in the definition is that hierarchies not containing  $\emptyset$  will be used when we extend sets in theorems 3.27 and 3.35 below. In the cases of superstructures, we will have  $\emptyset \in Y$ , and in this case, the definition could be written as  $H_\alpha^Y = \bigcup_{\beta < \alpha} (\mathcal{P}(H_\beta^Y) \cup Y)$ .

This is also true for the von Neumann-hierarchy  $H$ , where  $Y = \{\emptyset\}$ .

The elements in  $Y \setminus \{\emptyset\}$  should be thought of as "urelements". We need to impose conditions on  $Y$  in order to get these urelements to behave properly. For such a suitably chosen  $Y$ ,  $H_\omega^Y$  will be a superstructure (see Definition 4.1).

First, however, we investigate the most general properties of the hierarchy  $H^Y$ . Some of these properties are simple, while other are more involved, and it may seem unjustified to state those latter, but they are needed later on.

3.6. PROPOSITION. For any set  $Y$  such that  $Y \setminus \{Y\} \neq \emptyset$ , and any ordinals  $\alpha$  and  $\alpha'$ :

- (1)  $H_0^Y = \emptyset$ .
- (2)  $H_1^Y = Y$ .
- (3)  $H_{\alpha+1}^Y = (\mathcal{P}(H_\alpha^Y) \setminus \{\emptyset\}) \cup Y$ .
- (4) If  $\alpha$  is a limit ordinal, then  $H_\alpha^Y = \cup_{\beta < \alpha} H_\beta^Y$ .
- (5) If  $\alpha < \alpha'$ , then  $H_\alpha^Y \subset H_{\alpha'}^Y$  (proper inclusion).
- (6)  $\emptyset \in H^Y$  if and only if  $\emptyset \in Y$ .
- (7)  $H_\alpha^Y \in H^Y$  if and only if  $\alpha > 0$  or  $\emptyset \in Y$ .
- (8) If  $H_\alpha^Y \in H^Y$ , then  $H_\alpha^Y \in H_{\alpha+1}^Y$ .
- (9) If  $x \in H_\alpha^Y \setminus (Y \setminus \{\emptyset\})$ , then  $x \subseteq H_\beta^Y$ , for some  $\beta < \alpha$ .
- (10) If  $x \subseteq H_\alpha^Y$ , and if either  $x \neq \emptyset$  or  $\emptyset \in Y$ , then  $x \in H_{\alpha+1}^Y$ .
- (11) If  $A, B \in H_\alpha^Y \setminus (Y \setminus \{\emptyset\})$ , then  $A \cup B \in H_\alpha^Y$ .
- (12) If  $A \in H_\alpha^Y \setminus (Y \setminus \{\emptyset\})$ ,  $B \subseteq A$ , and if either  $B \neq \emptyset$  or  $\emptyset \in Y$ , then  $B \in H_\alpha^Y$ .
- (13) If  $A \in H_\alpha^Y \setminus (Y \setminus \{\emptyset\})$  and  $\emptyset \in Y$ , then  $\mathcal{P}(A) \in H_{\alpha+1}^Y$ .
- (14) There is a subset  $X \subseteq H_\alpha^Y$  such that  $\text{card } X = \text{card } \alpha$ .
- (15) For every  $x \in H^Y$ , there exists a finite sequence  $\{x_k\}_{k=0}^n$  ( $n \geq 0$ ) of elements in  $H^Y$ , such that  $x_0 \in Y$ ,  $x_n = x$ , and  $x_{k-1} \in x_k$  for all  $k$  ( $1 \leq k \leq n$ ).
- (16)  $H^Y$  is not a set.
- (17) If  $Y' \setminus \{Y'\} \neq \emptyset$ , and  $Y' \subseteq Y$ , then  $H_\alpha^{Y'} \subseteq H_\alpha^Y$  for all  $\alpha$ , and  $H^{Y'} \subseteq H^Y$ .
- (18) If  $\{x_k\}_{k=0}^n$  is a finite sequence of sets such that  $x_n \in H^Y$ , and such that  $x_{k-1} \in x_k$  for all  $k$  ( $1 \leq k \leq n$ ), then, either
  - (i)  $x_k \in H^Y \setminus Y$  for all  $k$ , ( $0 \leq k \leq n$ ), or
  - (ii) there is an  $m$  such that  $0 \leq m \leq n$ ,  $x_m \in Y$ , and  $x_k \in H^Y \setminus Y$ , for all  $k$  such that  $m < k \leq n$ .

PROOF. (1) follows immediately from the definition, recalling that the union over an empty family of sets is empty.

(2) follows from the definition and (1), since  $\mathcal{P}(\emptyset) = \{\emptyset\}$ .

Before we prove (3) and (4), we notice that the following weaker version of (5) follows directly from the definition:

(5') If  $\alpha < \alpha'$ , then  $H_\alpha^Y \subseteq H_{\alpha'}^Y$ .

Now, (3) follows immediately from the definition and (5').

To obtain (4), let  $\alpha$  be a limit ordinal. (5') gives  $\cup_{\beta < \alpha} H_\beta^Y \subseteq H_\alpha^Y$ . Conversely, if  $x \in H_\alpha^Y$ , then  $x \in (\mathcal{P}(H_\beta^Y) \setminus \{\emptyset\}) \cup Y$  for some  $\beta < \alpha$ . Then  $x \in H_{\beta+1}^Y$ , by definition. Since  $\alpha$  is a limit ordinal,  $\beta + 1 < \alpha$ , and  $x \in \cup_{\beta < \alpha} H_\beta^Y$ . Since  $x$  was an arbitrary element in  $H_\alpha^Y$ , this and the previous case give  $H_\alpha^Y = \cup_{\beta < \alpha} H_\beta^Y$ . Hence, (4) holds.

To prove (5), assume that (5) fails. Since (5') holds, this means that there are ordinals  $\alpha$  and  $\alpha'$  such that  $\alpha < \alpha'$ , but  $H_\alpha^Y = H_{\alpha'}^Y$ . Then, (5') gives that  $H_\alpha^Y = H_\beta^Y$  for all ordinals  $\beta$  such that  $\alpha \leq \beta \leq \alpha'$ . In particular,  $H_\alpha^Y = H_{\alpha+1}^Y$ . Repeated use of (3) then gives  $H_\alpha^Y = H_{\alpha+1}^Y = H_{\alpha+2}^Y = H_{\alpha+3}^Y$ .

Now, since  $Y \neq \emptyset$ , (2) and (3) give  $Y \in H_2^Y$ , and, again by (3),  $\{Y\} \in H_3^Y$ . Hence, by (5'),  $\{Y\} \in H_{\alpha+3}^Y = H_\alpha^Y$ .

Put  $S = \{x \in H_\alpha^Y \mid x \notin x\}$ . Since  $Y \neq \{Y\}$ ,  $\{Y\} \in S$ , so  $S \neq \emptyset$ . Now, it follows from (3) that  $S \in \mathcal{P}(H_\alpha^Y) \setminus \{\emptyset\} \subseteq H_{\alpha+1}^Y = H_\alpha^Y$ .

Then, it follows from the definition of  $S$  that  $S \in S$  if and only if  $S \notin S$ , a contradiction.

Therefore, our assumption that (5) fails is false. Thus, (5) holds.

To obtain (6): If  $\emptyset \in H^Y$ , then  $\emptyset \in H_\alpha^Y$  for some ordinal  $\alpha$ , by definition. Hence, for some ordinal  $\beta < \alpha$ ,  $\emptyset \in (\mathcal{P}(H_\beta^Y) \setminus \{\emptyset\}) \cup Y$ , from which it follows that  $\emptyset \in Y$ . The converse is immediate from the definition. Hence, (6) holds.

To obtain (7) and (8): If  $\alpha = 0$  and  $\emptyset \notin Y$ , then  $H_\alpha^Y = \emptyset$ , by (1), whence  $H_\alpha^Y \notin H^Y$ , by (6).

Thus: If  $H_\alpha^Y \in H^Y$ , then either  $\alpha > 0$  or  $\emptyset \in Y$ , that is, the direct part of (7) holds.

Assume now that either  $\alpha > 0$  or  $\emptyset \in Y$  holds. If  $\alpha > 0$ , then (2) and (5) give  $Y \subseteq H_\alpha^Y$ , whence  $H_\alpha^Y \neq \emptyset$ , from which it follows that  $H_\alpha^Y \in \mathcal{P}(H_\alpha^Y) \setminus \{\emptyset\} \subseteq H_{\alpha+1}^Y$ , by (3). If  $\emptyset \in Y$ , then  $H_0^Y \in H_1^Y$ , by (1) and (2). Thus, in this case,  $H_\alpha^Y \in H_{\alpha+1}^Y$  holds also for  $\alpha = 0$ . Thus, in both cases,  $H_\alpha^Y \in H_{\alpha+1}^Y$ . Together with the definition, this gives the converse part of (7), and, together with the direct part of (7), which is already proved, this gives (8).

Thus, (7) and (8) both hold.

To obtain (9), assume that  $x \in H_\alpha^Y \setminus (Y \setminus \{\emptyset\})$ . Then  $\alpha > 0$ , by (1). If  $x = \emptyset$ , then, obviously,  $x \subseteq H_\beta^Y$  for any  $\beta < \alpha$ . Otherwise,  $x \in H_\alpha^Y \setminus Y$ . Then, by definition,  $x \in \mathcal{P}(H_\beta^Y)$  for some ordinal  $\beta < \alpha$ . Hence,  $x \subseteq H_\beta^Y$ .

Hence, (9) holds.

To, prove (10), let  $x \subseteq H_\alpha^Y$ . If  $x \neq \emptyset$ , then  $x \in \mathcal{P}(H_\alpha^Y) \setminus \{\emptyset\}$ , and, by (3),  $x \in H_{\alpha+1}^Y$ .

If  $x = \emptyset$  and  $\emptyset \in Y$ , then  $x = \emptyset \in H_{\alpha+1}^Y$  follows from (2) and (5).

Thus, (10) holds.

To prove (11), assume that  $A, B \in H_\alpha^Y \setminus (Y \setminus \{\emptyset\})$ . If  $A = \emptyset$  or  $B = \emptyset$ , then the conclusion obviously holds. Assume therefore that  $A \neq \emptyset$  and  $B \neq \emptyset$ .

Then, by (9) there are ordinals  $\beta_1 < \alpha$  and  $\beta_2 < \alpha$  such that  $A \subseteq H_{\beta_1}^Y$  and  $B \subseteq H_{\beta_2}^Y$ . If  $\beta = \max\{\beta_1, \beta_2\}$ , then, by (5),  $A \subseteq H_\beta^Y$  and  $B \subseteq H_\beta^Y$ . Hence,  $A \cup B \subseteq H_\beta^Y$ , and, by (10) and (5),  $A \cup B \in H_{\beta+1}^Y \subseteq H_\alpha^Y$ .

Hence, (11) holds.

To prove (12), assume that  $A, B$ , and  $\alpha$  are as in the statement.

By (9),  $A \subseteq H_\beta^Y$  for some  $\beta < \alpha$ , and hence  $B \subseteq H_\beta^Y$ . Thus, by (10) and (5),  $B \in H_{\beta+1}^Y \subseteq H_\alpha^Y$ .

Thus, (12) holds.

To prove (13), take  $A \in H_\alpha^Y \setminus (Y \setminus \{\emptyset\})$  and assume that  $\emptyset \in Y$ . Then, for any  $B \subseteq A$ , (12) gives  $B \in H_\alpha^Y$ . Thus,  $\mathcal{P}(A) \subseteq H_\alpha^Y$ .

Thus, by (10),  $\mathcal{P}(A) \in H_{\alpha+1}$ , that is, (13) holds.

To obtain (14), we prove it first with ordinary induction for  $\alpha = n < \omega$ .

By (1),  $H_0^Y = \emptyset$ , so the statement is trivially true for  $n = 0$ .

Assume that the statement is true for  $n < \omega$ . Then, there is a subset  $X \subseteq H_n^Y$  such that  $\text{card } X = n$ . By (5),  $H_{n+1}^Y \setminus H_n^Y \neq \emptyset$ , so we can choose  $x \in H_{n+1}^Y \setminus H_n^Y$ . Now, put  $X' = X \cup \{x\}$ . Then, by (5),  $X' \subseteq H_{n+1}^Y$  and  $\text{card } X' = n + 1$ . Thus, the statement is true for  $n + 1$ .

By ordinary induction, the statement is true for all  $\alpha = n < \omega$ .

For  $\alpha \geq \omega$ , we define a function<sup>8</sup>  $f : \alpha \rightarrow H_\alpha^Y$ , recalling that  $\alpha = \{\beta \mid \beta < \alpha\}$  (see Chapter 8).

$$f(\beta) = \begin{cases} H_{\beta+1}^Y, & \text{if } \beta < \omega, \\ H_\beta^Y, & \text{if } \omega \leq \beta < \alpha. \end{cases}$$

It follows from (5), (7), and (8) that  $f$  is an injection from  $\alpha$  into  $H_\alpha^Y$ . Therefore the statement is satisfied if we put  $X$  as the range of  $f$ .

We have proved that the statement is true for all ordinals  $\alpha$ , that is, (14) holds.

(15) is proved by transfinite induction: We prove that for every ordinal  $\alpha$ , the statement is true for every  $x \in H_\alpha^Y$ .

Assume that the statement holds for all  $x' \in H_\beta^Y$ , for all  $\beta < \alpha$ , and let  $x \in H_\alpha^Y$ . Then, by definition,  $x \in Y$  or  $x \in \mathcal{P}(H_\beta^Y) \setminus \{\emptyset\}$  for some  $\beta < \alpha$ . If  $x \in Y$ , then the singleton sequence  $\{x\}_{k=0}^0$  will do.

In the second case, there is an element  $x' \in x \subseteq H_\beta^Y$ . Hence, the induction hypothesis gives a sequence  $\{x_k\}_{k=0}^n$  which satisfies the conditions with  $x'$  instead of  $x$ . Then, if we put  $x_{n+1} = x$ , the sequence  $\{x_k\}_{k=0}^{n+1}$  will do.

Thus, by transfinite induction, the statement is true for all  $x \in H_\alpha^Y$ , for all ordinals  $\alpha$ . Hence, (15) holds.

To obtain (16): By the definition and (14) it follows that for every ordinal  $\alpha$ ,  $H^Y$  has a subset  $X$  such that  $\text{card } X = \text{card } \alpha$ . But by Hartogs' Theorem, no set has this property (see Chapter 8). Hence,  $H^Y$  is not a set, that is, (16) holds.

(17) is easily obtained by transfinite induction and the definition: Assume that  $Y' \setminus \{Y'\} \neq \emptyset$ ,  $Y' \subseteq Y$ , and that  $H_\beta^{Y'} \subseteq H_\beta^Y$  is true for all  $\beta < \alpha$ . Then, for all  $\beta < \alpha$ ,  $(\mathcal{P}(H_\beta^{Y'}) \setminus \{\emptyset\}) \cup Y' \subseteq (\mathcal{P}(H_\beta^Y) \setminus \{\emptyset\}) \cup Y$ . Hence,  $H_\alpha^{Y'} \subseteq H_\alpha^Y$ . By transfinite induction, (17) holds.

To obtain (18), let  $\{x_k\}_{k=0}^n$  satisfy the condition, and assume that (i) fails. Let  $m$  be the greatest number ( $0 \leq m \leq n$ ) such that  $x_m \notin H^Y \setminus Y$ . If  $m = n$ , then  $x_m \in H^Y$ , whence  $x_m \in Y$ .

If  $m < n$ , then  $x_{m+1} \in H_\alpha^Y \setminus Y$ , for some  $\alpha$ , and  $x_m \in x_{m+1}$ . Then, (9) gives  $x_{m+1} \subseteq H_\alpha^Y$ . Thus  $x_m \in H^Y$ , which implies  $x_m \in Y$ .

Thus, in any case,  $x_m \in Y$ .

We have proved that (ii) holds if (i) fails. Hence, (18) holds. ■

In superstructures, it is important to handle  $m$ -tuples and relations, especially in the formulas occurring in the transfer principle (see Definitions 4.14).

Without imposing extra conditions upon  $Y$ , we cannot say much more than the following, for the moment:

**3.7. PROPOSITION.** *Assume that the set  $Y$  contains an  $\mathbb{N}$ -copy, (see Remarks 2.3). Then, for all  $m \geq 1$  and all ordinals  $\alpha$ :*

- (1) *If  $a_1, a_2, \dots, a_m \in H_\alpha^Y$ , then  $\langle a_1, a_2, \dots, a_m \rangle \in H_{\alpha+3}^Y$ .*
- (2)
  - (a) *If  $A \in H_\alpha^Y \setminus Y$  and  $A \neq \emptyset$ , then  $A^m \subseteq H_{\alpha+2}^Y$  and  $A^m \in H_{\alpha+3}^Y$ .*
  - (b) *If  $A \in H_{\alpha+1}^Y \setminus Y$ ,  $A \neq \emptyset$ , and  $\alpha$  is a limit ordinal, then  $A^m \subseteq H_\alpha^Y$  and  $A^m \in H_{\alpha+1}^Y$ .*

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<sup>8</sup>This is a function of the usual type, not one as in Definitions 2.10. See Remarks 2.1.

PROOF. First,  $Y \setminus \{Y\} \neq \emptyset$ , since  $Y$  is infinite, so each  $H_\alpha^Y$  is defined.

Assume that  $a_1, a_2, \dots, a_m \in H_\alpha^Y$ , for some  $\alpha$ . By (1) of Proposition 3.6,  $\alpha \geq 1$ . For every  $i$  ( $1 \leq i \leq m$ ), our assumptions give  $i \in \mathbb{N} \subseteq Y = H_1^Y$ , by (2) of Proposition 3.6. Then, by (10) and (5) of Proposition 3.6,  $\{i\} \in H_2^Y$ ,  $\{i, a_i\} \in H_{\alpha+1}^Y$ , and  $\{\{i\}, \{i, a_i\}\} \in H_{\alpha+2}^Y$ . Now, Definitions 2.2 and (10) of Proposition 3.6 give  $\langle a_1, a_2, \dots, a_m \rangle \in H_{\alpha+3}^Y$ , that is, (1) holds.

To prove (2), let  $A \in H_\alpha^Y \setminus Y$  and  $A \neq \emptyset$ . By (9) of Proposition 3.6,  $A \subseteq H_\beta^Y$ , for some  $\beta < \alpha$ . Hence, by Definitions 2.7 and (1),  $A^m \subseteq H_{\beta+3}^Y$  and  $A^m \neq \emptyset$ . Since  $\beta + 3 \leq \alpha + 2$ , (5) and (10) of Proposition 3.6 then give  $A^m \subseteq H_{\alpha+2}^Y$  and  $A^m \in H_{\alpha+3}^Y$ , that is, (a) holds.

To prove (b), assume that  $A \in H_{\alpha+1}^Y \setminus Y$ ,  $A \neq \emptyset$ , and that  $\alpha$  is a limit ordinal. By (3) of Proposition 3.6,  $A \subseteq H_\alpha^Y$ . Since  $\alpha$  is a limit ordinal, (4) and (5) of Proposition 3.6 give  $A = \cup_{\beta < \alpha} (A \cap H_\beta^Y)$  and  $A^m = \cup_{\beta < \alpha} (A \cap H_\beta^Y)^m$ . For each  $\beta < \alpha$ ,  $(A \cap H_\beta^Y)^m \subseteq H_{\beta+3}^Y \subseteq H_\alpha^Y$ , by (1), and (5) of Proposition 3.6, since  $\alpha$  is a limit ordinal.

It follows that  $A^m \subseteq H_\alpha^Y$ . Also,  $A^m \neq \emptyset$ , since  $A \neq \emptyset$ .  $A^m \in H_{\alpha+1}^Y$  then follows from (10) of Proposition 3.6. Thus (b) holds, and all of (2) is proved. ■

3.8. DEFINITION. For every  $x \in H^Y$ , where  $Y$  is a set such that  $Y \setminus \{Y\} \neq \emptyset$ , we define the  $Y$ -rank of  $x$ , denoted  $r^Y(x)$ , as the smallest ordinal  $\alpha$  for which  $x \in H_\alpha^Y$ . If  $Y = \{\emptyset\}$ , we say rank instead of  $\{\emptyset\}$ -rank, and write  $r(x)$  instead of  $r^{\{\emptyset\}}(x)$ , for all  $x \in H$ .

3.9. PROPOSITION. For all sets  $Y$  such that  $Y \setminus \{Y\} \neq \emptyset$ , and all ordinals  $\alpha$ :

- (1)  $r^Y(x)$  is a successor ordinal, for every  $x \in H^Y$ .
- (2) For every ordinal  $\alpha$ , there is an  $x \in H^Y$  such that  $r^Y(x) = \alpha + 1$ .
- (3)  $r^Y(y) = 1$  if and only if  $y \in Y$ .
- (4) If  $\emptyset \in H^Y$ , then  $r^Y(\emptyset) = 1$ .
- (5) If  $x \in H^Y \setminus Y$ , then  $x \subseteq H^Y$  and  $(\sup_{x' \in x} r^Y(x')) + 1 = r^Y(x)$ , and in particular,  $r^Y(x') < r^Y(x)$  for all  $x' \in x$ .
- (6) If  $x \in H^Y \setminus Y$ , then  $x \notin x$ .
- (7) If  $H_\alpha^Y \in H^Y \setminus Y$ , then  $r^Y(H_\alpha^Y) = \alpha + 1$ .
- (8) If  $x$  is a set such that  $x \neq \emptyset$  and  $x \subseteq H^Y$ , then  $x \in H^Y$ .
- (9) If  $Y'$  is a set such that  $Y' \setminus \{Y'\} \neq \emptyset$  and  $Y' \subseteq H^Y$ , then  $H^{Y'} \subseteq H^Y$ .
- (10) Let  $Y' \subseteq Y$ , with  $Y' \setminus \{Y'\} \neq \emptyset$ . Assume that  $x \in H_\alpha^Y$  is such that for any  $n \geq 0$ , any sequence  $\{x_k\}_{k=0}^n$  of the type described in (15) of Proposition 3.6, with  $x_n = x$ , satisfies  $x_0 \in Y'$ .

Then,  $x \in H_\alpha^{Y'}$ .

- (11) If  $\{x_k\}_{k=0}^\infty$  is an infinite sequence of sets such that  $x_0 \in H^Y$  and such that  $x_{n+1} \in x_n$  for all  $n \geq 0$ , then there exists an  $n \geq 0$  such that  $x_n \in Y$ .

PROOF. To obtain (1): Take  $x \in H^Y$ . The definition and (1) of Proposition 3.6 give  $r^Y(x) > 0$ . If  $\alpha$  is a limit ordinal and  $x \in H_\alpha^Y$ , then, by (4) of Proposition 3.6,  $x \in H_\beta^Y$  for some  $\beta < \alpha$ . Hence, by definition,  $r^Y(x) \leq \beta < \alpha$ . The only remaining possibility is that  $r^Y(x)$  is a successor ordinal. Hence, (1) holds.

(2) follows from the definition and (5) of Proposition 3.6; just take  $x \in H_{\alpha+1}^Y \setminus H_\alpha^Y$ .

(3) follows immediately from (1) and (2) of Proposition 3.6, and the definition.

(4) follows immediately from (6) of Proposition 3.6, and (3).

To obtain (5), assume that  $x \in H^Y \setminus Y$  with  $r^Y(x) = \alpha$ . Then,  $x \neq \emptyset$ , by (6) of Proposition 3.6. By (9) of Proposition 3.6,  $x \subseteq H_\beta^Y$  for some  $\beta$ , which we may choose as minimal. Thus,  $x \subseteq H^Y$ , and we must prove that  $\beta + 1 = \alpha$ , because, by our choice of  $\beta$ , and (5) of Proposition 3.6,  $\sup_{x' \in x} r^Y(x') = \beta$ .

We note that (1) implies that  $\alpha = \gamma + 1$ , for some ordinal  $\gamma$ . Since  $x \notin Y$ , we obtain from the definition and (3) of Proposition 3.6 that  $x \in \mathcal{P}(H_\gamma^Y)$ , that is,  $x \subseteq H_\gamma^Y$ . Hence,  $\beta \leq \gamma$ , and  $\beta + 1 \leq \gamma + 1 = \alpha$ .

On the other hand, since  $x \subseteq H_\beta^Y$  and  $x \neq \emptyset$ , (10) of Proposition 3.6 gives  $x \in H_{\beta+1}^Y$ , and hence that  $\beta + 1 \geq \alpha$ . It follows that  $\beta + 1 = \alpha$ .

Thus, (5) holds.

(6) follows from (5), since  $x \in x$  and  $x \in H^Y \setminus Y$  would imply  $r^Y(x) < r^Y(x)$ , a contradiction.

To obtain (7): If  $H_\alpha^Y \in H^Y \setminus Y$ , then, by (8) of Proposition 3.6,  $H_\alpha^Y \in H_{\alpha+1}^Y$ , so by definition,  $r^Y(H_\alpha^Y) \leq \alpha + 1$ .

Also, by (6),  $H_\alpha^Y \notin H_\alpha^Y$ . Hence, by (5) of Proposition 3.6,  $H_\alpha^Y \notin H_\beta^Y$ , for any ordinal  $\beta \leq \alpha$ . So, the definition gives  $r^Y(H_\alpha^Y) \geq \alpha + 1$ .

Hence,  $r^Y(H_\alpha^Y) = \alpha + 1$ , that is, (7) holds.

To prove (8), notice that  $r^Y(x') = \beta$  is a functional relation (see Chapter 8). Therefore, since  $x$  is a set, the range  $\{r^Y(x') \mid x' \in x\}$  is set too. Since a set of ordinals cannot be unbounded (see Chapter 8), there is an ordinal  $\alpha$  such that  $r^Y(x') \leq \alpha$ , for all  $x' \in x$ . Hence,  $x \subseteq H_\alpha^Y$ , by (5) of Proposition 3.6. Since  $x \neq \emptyset$ , (10) of Proposition 3.6 then gives  $x \in H_{\alpha+1}^Y$ . Thus,  $x \in H^Y$ , that is, (8) holds.

To obtain (9), we prove by transfinite induction that  $H_\alpha^{Y'} \subseteq H^Y$  for all ordinals  $\alpha$ .

Assume that  $H_\beta^{Y'} \subseteq H^Y$  for all  $\beta < \alpha$ . Pick  $x \in H_\alpha^{Y'}$ . Then, by definition,  $x \in (P(H_\beta^{Y'}) \setminus \{\emptyset\}) \cup Y'$ , for some  $\beta < \alpha$ . If  $x \in Y'$ , then  $x \in H^Y$ , by hypothesis. Otherwise,  $x \subseteq H_\beta^{Y'}$  and  $x \neq \emptyset$ . Since, by the induction hypothesis,  $H_\beta^{Y'} \subseteq H^Y$ ,  $x \subseteq H^Y$ . Then, by (8),  $x \in H^Y$ .

Thus, in any case  $x \in H^Y$ . This holds for all  $x \in H_\alpha^{Y'}$ , so  $H_\alpha^{Y'} \subseteq H^Y$ . By transfinite induction, this holds for all ordinals  $\alpha$ . Hence  $H^{Y'} \subseteq H^Y$ , that is, (9) holds.

We prove (10) by transfinite induction on  $\alpha$ . Assume that the statement is true for all  $\beta < \alpha$ . Let  $x$  satisfy the assumptions of (10), for  $\alpha$ . By (1),  $r^Y(x)$  is a successor ordinal. Hence,  $x \in H_{\gamma+1}^Y$  for some  $\gamma < \alpha$ . By (3) of Proposition 3.6,  $x \in (P(H_\gamma^Y) \setminus \{\emptyset\}) \cup Y$ .

If  $x \in Y$ , then the singleton sequence  $\{x\}_{k=0}^0$  satisfies the conditions. Hence,  $x \in Y'$ . Then, (2) and (5) of Proposition 3.6 give  $x \in H_\alpha^{Y'}$ .

Otherwise,  $x \subseteq H_\gamma^Y$ , and  $x \neq \emptyset$ . Now, pick an arbitrary  $x' \in x$ . Let  $\{x_k\}_{k=0}^m$  be any sequence satisfying the conditions described, with  $\gamma$  instead of  $\alpha$  and with  $x_m = x'$ . If we put  $x_{m+1} = x$ , then the sequence  $\{x_k\}_{k=0}^{m+1}$  now satisfies the conditions for  $\alpha$ . Hence, by hypothesis,  $x_0 \in Y'$ . Now, the induction hypothesis gives that  $x' \in H_\gamma^{Y'}$ . This holds for all  $x' \in x$ , so  $x \in \mathcal{P}(H_\gamma^{Y'}) \setminus \{\emptyset\} \subseteq H_{\gamma+1}^{Y'} \subseteq H_\alpha^{Y'}$ , by definition and (5) of Proposition 3.6.

Hence, in both cases,  $x \in H_\alpha^{Y'}$ . The conclusion now follows by transfinite induction. Thus, (10) holds.

To obtain (11), we prove by transfinite induction on  $\alpha$  that the statement is true for all such sequences with  $r^Y(x_0) = \alpha$ .

Assume that this holds for all ordinals  $\beta < \alpha$ , and assume that we have such a sequence with  $r^Y(x_0) = \alpha$ .

If  $x_0 \in Y$ , then the conclusion is true, with  $n = 0$ . Otherwise  $x_0 \in H^Y \setminus Y$ . Since  $x_1 \in x_0$ ,  $x_1 \in H^Y$  with  $r^Y(x_1) < \alpha$ , by (5). Now, the subsequence  $\{x_k\}_{k=1}^{\infty}$  satisfies the conditions, with  $r^Y(x_1) < \alpha$  (that the lowest index here is 1 instead of 0 is, of course, insignificant). Thus, by the induction hypothesis,  $x_n \in Y$  for some  $n \geq 1$ . But this  $x_n$  is also an element of the original sequence, so the conclusion is true for  $\alpha$ .

(11) now follows by transfinite induction. ■

3.10. REMARK. The rank function here differs slightly from the rank function usually defined on a superstructure. The "urelements", i.e. the elements in  $Y \setminus \{\emptyset\}$ , have  $Y$ -ranks 1 here, not 0, and  $\emptyset$ , which is always included in a superstructure, has the same  $Y$ -rank as the urelements, namely 1. (It has rank 1 by the usual definition too, but this is then not the same as the ranks of the urelements.) These are only minor differences, however.

3.11. REMARKS. We do not want that our rank function, on a superstructure  $H_{\omega}^Y$ , shall break down in the way as was described in Chapter 1. The cause of such a breakdown is that  $Y$  may have elements  $y \in Y$ , which themselves have elements that lie in  $H_{\omega}^Y$ . To avoid this, we make the next definition:

3.12 DEFINITIONS. We say that a set  $Y$  is  $\alpha$ -grounded if  $\alpha > 0$ ,  $Y \setminus \{Y\} \neq \emptyset$ , and  $H_{\alpha}^Y \cap (\cup_{y \in Y} y) = \emptyset$ .  $Y$  is said to be *completely grounded* if it is  $\alpha$ -grounded for all ordinals  $\alpha > 0$ .<sup>9</sup>

3.13. REMARKS. An equivalent way to express that  $Y$  is  $\alpha$ -grounded, if  $\alpha > 0$  and  $Y \setminus \{Y\} \neq \emptyset$ , is to say that there are no  $x$  and  $y$  such that  $x \in H_{\alpha}^Y$ ,  $y \in Y$ , and  $x \in y$ .

The reason that we exclude the case  $\alpha = 0$  in the definition is that otherwise, (4) of Proposition 3.14 below would not necessarily hold; it might happen that  $Y'$  there satisfies  $Y' = \{Y'\}$  if  $\alpha = 0$ . Since  $H_0^Y = \emptyset$ , by (1) of Proposition 3.6, every set such that  $Y \setminus \{Y\} \neq \emptyset$  would be "0-grounded" anyway, so it is no real loss to exclude this case.

Let us now state and prove some important properties of  $\alpha$ -grounded and completely grounded sets. Again, some may seem involved, but they are needed later.

3.14. PROPOSITION. *Let  $Y$  be a set such that  $Y \setminus \{Y\} \neq \emptyset$ , and let  $\alpha$  be an ordinal.*

*Then:*

- (1)  $Y$  is completely grounded if and only if  $H^Y \cap (\cup_{y \in Y} y) = \emptyset$ .
- (2) If  $Y$  is  $\alpha$ -grounded, then  $Y$  is  $\beta$ -grounded, for all  $\beta$  such that  $0 < \beta \leq \alpha$ .
- (3)  $Y$  is 1-grounded if and only if  $x \notin y$  for all  $x, y \in Y$ .

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<sup>9</sup>Intuitively, keeping (5) of Proposition 3.9 in mind, we "reach the ground", when we come down to an element  $y \in Y$ . We cannot come to another point in the hierarchy by taking an element  $z \in y$ . With an electric analogy, we may also say that if we have an element  $z \in y$  which lies in  $H_{\alpha}^Y$ , this produces a "short circuit". This is avoided if  $Y$  is "grounded". This motivates the choice of word  $\alpha$ -grounded.

- (4) If  $Y$  is  $\alpha$ -grounded (completely grounded) and if  $Y' \subseteq Y$  with  $Y' \neq \emptyset$ , then  $Y'$  is  $\alpha$ -grounded (completely grounded).
- (5) If  $Y$  is 1-grounded and  $H_\beta^Y \in H^Y$ , for some ordinal  $\beta$ , then  $r^Y(H_\beta^Y) = \beta + 1$ .
- (6) If  $Y$  is  $\alpha$ -grounded (completely grounded), and if  $x \in H^Y$  with  $x \cap H_\alpha^Y \neq \emptyset$  ( $x \cap H^Y \neq \emptyset$ ), then  $x \subseteq H^Y$ , and  $(\sup_{x' \in x} r^Y(x')) + 1 = r^Y(x)$ , and in particular,  $r^Y(x') < r^Y(x)$  for all  $x' \in x$ .
- (7) If  $Y$  is  $\alpha$ -grounded (completely grounded), then  $x \notin x$  for all  $x \in H_\alpha^Y$  ( $x \in H^Y$ ).
- (8) If  $Y$  is  $\alpha$ -grounded and  $\beta \leq \alpha$ , then  $(\mathcal{P}(H_\beta^Y) \setminus \{\emptyset\}) \cap Y = \emptyset$ .  
If  $Y$  is completely grounded, then this holds for all ordinals  $\beta$ .
- (9) If  $Y$  is  $\alpha$ -grounded, then  $H_\beta^Y \notin Y$ , for all ordinals  $\beta$  such that  $0 < \beta \leq \alpha$ .  
If  $Y$  is completely grounded, then this holds for all ordinals  $\beta > 0$ .
- (10) Assume that  $Y$  is completely grounded, and let  $Y'$  be a set such that  $Y' \setminus \{Y'\} \neq \emptyset$  and  $Y' \subseteq H^Y$ .  
Also, assume that  $r^Y(y) \geq \alpha$ , for all  $y \in Y'$ .  
Then,  $H^{Y'} \subseteq H^Y$ , with  $r^Y(x) \geq \alpha$  for all  $x \in H^{Y'}$ .

PROOF. (1) is immediate from the definitions. (2) follows from the definitions and (5) of Proposition 3.6. (3) follows from the definitions and (2) of Proposition 3.6.

To obtain (4), assume, to get a contradiction, that  $Y' \setminus \{Y'\} = \emptyset$ . Then  $Y' \in Y'$ , since  $Y' \neq \emptyset$ . Since  $Y' \subseteq Y$ ,  $Y' \in Y$ . Since  $\alpha > 0$ , (2) and (5) of Proposition 3.6 give  $Y \subseteq H_\alpha^Y \subseteq H^Y$ . Now,  $Y' \in H_\alpha^Y \subseteq H^Y$  and  $Y' \in Y' \in Y$ , contradicting the  $\alpha$ -groundedness of  $Y$  (or complete groundedness, using (1)).

Thus,  $Y' \setminus \{Y'\} \neq \emptyset$ . Now, since  $Y' \subseteq Y$ , the  $\alpha$ -groundedness (complete groundedness) of  $Y'$  follows from (17) of Proposition 3.6 and the corresponding property for  $Y$ .

Thus, (4) holds.

To obtain (5): If  $\beta = 0$ , then (1) of Proposition 3.6 implies that  $\emptyset = H_\beta^Y \in H^Y$ , and then (4) of Proposition 3.9 gives  $r(H_\beta^Y) = 1 = \beta + 1$ , so the statement is true for  $\beta = 0$ .

If  $\beta > 0$ , then (2) and (5) of Proposition 3.6 give  $Y \subseteq H_\beta^Y$ . Since  $Y \neq \emptyset$ , there is an  $x \in Y$ . Then,  $x \in H_\beta^Y$ . Then, if  $H_\beta^Y \in Y$ , this would contradict the 1-groundedness of  $Y$ , by (3). Thus  $H_\beta^Y \in H^Y \setminus Y$ .

Now, (7) of Proposition 3.9 gives  $r^Y(H_\beta^Y) = \beta + 1$  in this case too.

Thus, (5) holds.

To obtain (6), Let  $x \in H^Y$  with  $x \cap H_\alpha^Y \neq \emptyset$  ( $x \cap H^Y \neq \emptyset$ ). Then, there is a  $y$  such that  $y \in x$  and  $y \in H_\alpha^Y$ , ( $y \in H^Y$ ). Therefore, if  $Y$  is  $\alpha$ -grounded (or completely grounded, using (1)),  $x \notin Y$ .

The conclusion now follows from (5) of Proposition 3.9.

Thus, (6) holds.

(7) follows immediately from (6).

To prove (8), we first note that the second statement follows from the first, by definition, so let us prove the first statement:

Assume, to get a contradiction, that there is an  $x \in (\mathcal{P}(H_\beta^Y) \setminus \{\emptyset\}) \cap Y$ , where  $\beta \leq \alpha$ . This means, using (5) of Proposition 3.6, that  $x \subseteq H_\beta^Y \subseteq H_\alpha^Y$ ,  $x \neq \emptyset$ , and  $x \in Y$ .

Now, there is a  $y \in x \subseteq H_\beta^Y \subseteq H_\alpha^Y$ . So,  $x \in Y$ ,  $y \in H_\alpha^Y$ , and  $y \in x$ . This contradicts the  $\alpha$ -groundedness of  $Y$ .

Hence,  $(\mathcal{P}(H_\beta^Y) \setminus \{\emptyset\}) \cap Y = \emptyset$ , for all ordinals  $\beta \leq \alpha$ .

Thus, (8) holds.

To obtain (9), we note that the second statement follows from the first.

To prove the first statement, we see that with  $\alpha$  and  $\beta$  as stated,  $H_\beta^Y \in Y$  implies  $H_\beta^Y \in H_\beta^Y$  and  $H_\beta^Y \in H_\alpha^Y$  ( $H_\beta^Y \in H^Y$ ) by (2) and (5) of Proposition 3.6. This contradicts (7).

Hence, (9) holds.

To obtain (10), we first see that  $H^{Y'} \subseteq H^Y$ , by (9) of Proposition 3.9.

We prove by transfinite induction on  $\beta$  that for all ordinals  $\beta$ ,  $r^Y(x) \geq \alpha$  for all  $x \in H_\beta^{Y'}$ .

Assume that this holds for all  $\gamma < \beta$ , and let  $x \in H_\beta^{Y'}$ . Then,  $x \in H^Y$ . By definition,  $x \in (\mathcal{P}(H_\gamma^{Y'}) \setminus \{\emptyset\}) \cup Y'$ , for some  $\gamma < \beta$ . If  $x \in Y'$ , then the statement is true, by hypothesis.

Otherwise,  $x \subseteq H_\gamma^{Y'}$  and  $x \neq \emptyset$ . Then, if  $x' \in x$ , the induction hypothesis gives  $x' \in H^Y$  with  $r^Y(x') \geq \alpha$ . Since  $Y$  is completely grounded, (6) now gives  $r^Y(x) > r^Y(x') \geq \alpha$ . Since all this holds for all  $x \in H_\beta^{Y'}$ , the statement is true for  $\beta$ .

(10) now follows by transfinite induction. ■

The following lemma will be used in the proof of Proposition 3.33.

3.15. LEMMA. *Assume that  $Y$  contains an  $\mathbb{N}$ -copy (as in Remarks 2.3).*

*Also, assume that  $Y$  is  $\alpha$ -grounded, for some ordinal  $\alpha > 0$ .*

*Then, if  $A$  is an  $m$ -ary relation such that  $A \cap (H_\alpha^Y)^m \neq \emptyset$ , for some  $m \geq 1$ ,  $\mathcal{C}(A) \notin Y$  holds.*

*If  $Y$  is completely grounded, this holds for all sets  $A$  which are  $m$ -ary relations such that  $A \cap (H^Y)^m \neq \emptyset$ .*

PROOF. We have  $\emptyset \neq \mathcal{C}(A \cap (H_\alpha^Y)^m) \subseteq \mathcal{C}(A) \cap H_\alpha^Y$ . Hence,  $\mathcal{C}(A) \in Y$  would imply that every element in  $\mathcal{C}(A \cap (H_\alpha^Y)^m)$  would violate the  $\alpha$ -groundedness of  $Y$ .

Hence,  $\mathcal{C}(A) \notin Y$ .

If  $Y$  is completely grounded and  $A \cap (H^Y)^m \neq \emptyset$ , then (5) and (1) of Proposition 3.6 imply that  $A \cap (H_\alpha^Y)^m \neq \emptyset$  for some  $\alpha > 0$ . Also,  $Y$  is  $\alpha$ -grounded for all  $\alpha > 0$ , by definition. Hence, this reduces to the previous case. ■

For the von Neumann hierarchy,  $H$ , some of the statements in propositions 3.6, 3.9, and 3.14, can be improved:

3.16. PROPOSITION. *For all ordinals  $\alpha$ :*

- (1)  $H_\alpha = \cup_{\beta < \alpha} \mathcal{P}(H_\beta)$ .
- (2)  $H_{\alpha+1} = \mathcal{P}(H_\alpha)$ .
- (3)  $H_\alpha \in H_{\alpha+1} \subseteq H$ .
- (4)  $H_1 = \{\emptyset\} \in H$ .
- (5) *If  $x \in H$ , then  $x \subseteq H$ , and  $(\sup_{x' \in x} r(x')) + 1 = r(x)$ , and in particular,  $r(x') < r(x)$  for all  $x' \in x$ .*
- (6) *If  $x$  is a set such that  $x \subseteq H$ , then  $x \in H$ .*
- (7) *A set  $Y \in H$  such that  $\emptyset \in Y$  is completely grounded if and only if  $Y = \{\emptyset\}$ .*
- (8)  $r(H_\alpha) = \alpha + 1$ .
- (9) *If  $\{x_k\}_{k=0}^n$  is a finite sequence of sets such that  $x_n \in H$ , and such that  $x_{k-1} \in x_k$  for all  $k$  ( $1 \leq k \leq n$ ), then  $x_k \in H$  for all  $k$  ( $0 \leq k \leq n$ ).*

- (10)  $x \notin x$ , for all  $x \in H$ .  
 (11) There exists no infinite sequence  $\{x_n\}_{n=0}^{\infty}$  such that  $x_0 \in H$  and  $x_{n+1} \in x_n$  for all  $n \geq 0$ .

PROOF. (1) follows immediately from the definitions. (2) follows from (3) of Proposition 3.6. (3) follows from (7) and (8) of Proposition 3.6. (4) follows from (2) of Proposition 3.6, and (3). (5) follows from (5) of Proposition 3.9 if  $x \neq \emptyset$ . If  $x = \emptyset$ , then  $x \in H$ , by (4), and  $r(x) = 1$ , by (4) of Proposition 3.9. Since, by stipulation,  $\sup_{x' \in \emptyset} r(x') = 0$ , (5) holds in this case too. Thus, (5) holds in all cases.

Since  $\emptyset \in H$ , by (4), (6) follows from (8) of Proposition 3.9.

To prove (7): That  $\{\emptyset\}$  is completely grounded follows immediately from the definition. Conversely, assume that  $Y \in H$ ,  $\emptyset \in Y$ , and  $x \in Y$  with  $x \neq \emptyset$ . Then,  $Y \setminus \{Y\} \neq \emptyset$ . Since  $Y \in H$  and  $x \in Y$ , (5) gives  $x \in H$ . Since  $x \neq \emptyset$ , there is a  $y \in x$ , and, again by (5),  $y \in H$ . By (17) of Proposition 3.6,  $H \subseteq H^Y$ . Hence,  $y \in H^Y$ .

Now,  $x \in Y$ ,  $y \in H^Y$ , and  $y \in x$ . Thus, by (1) of Proposition 3.14,  $Y$  is not completely grounded.

Hence, (7) holds.

(8) now follows from (4) and (7), and (5) of Proposition 3.14.

(9) follows from (18) of Proposition 3.6: If (i) there holds, the conclusion is true. If (ii) holds, we must have  $x_m = \emptyset$ , whence  $m = 0$ . Since, by (4),  $\emptyset \in H_1 \subseteq H$ , the conclusion is true in this case too. Thus, in both cases, (9) holds.

Since  $x \in x$  and  $x \in H$  would imply  $r(x) < r(x)$ , by (5), (10) follows.

To obtain (11): If there were such a sequence, then, by (11) of Proposition 3.9,  $x_n = \emptyset$  for some  $n \geq 0$ . But then,  $x_{n+1} \in x_n = \emptyset$ , which is a contradiction.

Hence, no such sequence exists, that is, (11) holds. ■

Although it is somewhat outside the subject of this text, we include a proof of the following well known results, since they deal with concepts studied here, and since it could be interesting for a reader not very familiar with set theory to see the proof.

3.17. THEOREM. Consider the following statements:

- (1) The axiom of regularity holds.
- (2) Every set belongs to  $H$ .
- (3) There exists no infinite sequence  $\{x_n\}_{n=0}^{\infty}$  such that  $x_{n+1} \in x_n$  for all  $n \geq 0$ .<sup>10</sup>

Then, (1) and (2) are equivalent, and each of them implies (3). In addition, if the axiom of choice holds, then all three statements are equivalent.

PROOF. That (2) implies (3) follows immediately from (11) of Proposition 3.16.

We now prove simultaneously that (1) implies (2) and that the axiom of choice and (3) together imply (2).

To do this, assume that there is a set  $x$  such that  $x \notin H$ . We define the sets  $S_n$  ( $n \geq 0$ ), with ordinary recursion thus:<sup>11</sup>

$$S_0 = \{x\}, \quad S_{n+1} = \bigcup_{y \in S_n} y.$$

<sup>10</sup>This is called *Fundierungsaxiom*.

<sup>11</sup>The union of these sets  $S_n$  is called the *transitive closure* of  $x$ . It is the smallest transitive set (see Chapter 8) which contains  $x$ .

Next, we put  $S = (\cup_{n=0}^{\infty} S_n) \setminus H$ .  $x \in S$ , so  $S \neq \emptyset$ . Pick  $y \in S$ . Then,  $y \notin H$ , because otherwise, (6) of Proposition 3.16 would imply that  $y \in H$ , contradicting  $y \in S$ . Hence, there is a  $y' \in y$  with  $y' \notin H$ . Also  $y \in S_n$  for some  $n$ . Hence  $y' \in S_{n+1}$ . It follows that  $y' \in S$ , so that  $S \cap y \neq \emptyset$ . This holds for all  $y \in S$ . But this contradicts the axiom of regularity. Hence, (1) implies (2)

We saw above that to every  $y \in S$  there is an  $y' \in y \cap S$ . Using the axiom of choice, we can then by recursion define a sequence  $\{x_n\}_{n=0}^{\infty}$  such that  $x_0 = x$  and  $x_{n+1} \in x_n \cap S$  for all  $n \geq 0$ . But this sequence contradicts (3). It follows that the Axiom of Choice and (3) together imply (2).

Next, we prove that (2) implies (1):

Assume that  $x \in H$  for every set  $x$ . Let  $x \neq \emptyset$ . Then,  $r(y)$  is defined for all  $y \in x$ . The set of ordinals  $\{r(y) \mid y \in x\}$  has a smallest element  $\alpha$ , and  $r(y) = \alpha$  for some  $y \in x$ . Now, for all  $y' \in y$ ,  $r(y') < r(y)$ , by (5) of Proposition 3.16. If  $y' \in x$ , then  $x$  has an element with smaller rank than  $y$ , which contradicts the choice of  $y$ . Hence,  $x \cap y = \emptyset$ . Thus, every nonempty set  $x$  has an element  $y$  such that  $x \cap y = \emptyset$ . In other words, the axiom of regularity holds. Thus, (2) implies (1).

Putting these pieces together, the theorem is now proved. ■

3.18. REMARKS. Although we do not assume that the axiom of regularity holds, we nevertheless assume that an  $\alpha$ -extendable set lies in  $H$ , by Definitions 3.29 below. We do so, because we use von Neumann ranks  $r(x)$  in order to construct extensions of sets.

A reader, who is a firm believer in the axiom of regularity, may therefore, by Theorem 3.17, assume that all sets lie in  $H$ , and thus skip all verifications that particular sets lie in  $H$ . Such a reader may also replace the condition  $Y \setminus \{Y\} \neq \emptyset$  with  $Y \neq \emptyset$ , by (10) of Proposition 3.16 and Remarks 3.5.

3.19. REMARKS. It would be very nice if we had superstructure  $H_{\omega}^Y$ , where  $Y$  is completely grounded. Now, we want to have  $\emptyset \in H^Y$ , since it is hard to imagine meaningful NSA without the empty set. Then, we must have  $\emptyset \in Y$ , by (6) of Proposition 3.6. Also, we want to have infinitely many urelements.

Unfortunately, no set  $Y \in H$  satisfies all these conditions, by (7) of Proposition 3.16. Therefore, we must weaken some of these conditions. Now, the complete groundedness is actually an unnecessarily strong assumption. It suffices for our purposes that  $Y$  is  $\omega$ -grounded.

We therefore want an infinite  $\omega$ -grounded set  $Y$  such that  $\emptyset \in Y$ . Actually, we want such sets of arbitrarily great cardinalities, and we want to be able to extend any such set to an  $\omega$ -grounded set with arbitrarily great cardinality. If, for example, we want our proper nonstandard extension to be  $\kappa$ -saturated, where  $\kappa$  is an arbitrarily great transfinite cardinal (see Definitions 7.10), then we must be able to extend  $Y$  to an  $\omega$ -grounded set with cardinality at least  $\kappa$  (see Proposition 7.14).

To find such  $\omega$ -grounded sets, we introduce another concept, *intransitivity*:

3.20. DEFINITION. A set  $S$  is called *intransitive* if there is no finite sequence of sets  $\{x_k\}_{k=0}^n$  with  $n \geq 1$  such that  $x_0 \in S$ ,  $x_n \in S$ , and  $x_{k-1} \in x_k$  for all  $k$  ( $1 \leq k \leq n$ ).<sup>12</sup>

<sup>12</sup>Equivalently, we could say that no element in  $S$  should belong to the transitive closure (see the previous note) of any other element in  $S$ . This is why I chose the word "intransitive".

## 3.21. PROPOSITION.

- (1) *Every subset of an intransitive set is intransitive.*
- (2)  $\emptyset$  and  $\{\emptyset\}$  are intransitive.

PROOF. (1) is obvious. It is also obvious that  $\{\emptyset\}$  is intransitive. Thus, by (1),  $\emptyset$  is intransitive. Hence, (2) holds. ■

3.22. PROPOSITION. *Assume that  $Y \in H$ ,  $Y \neq \emptyset$ , and that  $Y$  is intransitive. Then:*

- (1)  *$Y$  is completely grounded.*
- (2) *If  $\alpha > 0$  is an ordinal such that  $r(y) > \alpha$  for all  $y \in \cup_{x \in Y} x$ , then  $Y \cup \{\emptyset\}$  is  $\alpha$ -grounded.*
- (3) *If  $\emptyset \in Y$ , then  $Y = \{\emptyset\}$ .*

PROOF. It follows from (5) of Proposition 3.16 that  $\cup_{x \in Y} x \subseteq H$ , so  $r(y)$  is defined for all  $y \in \cup_{x \in Y} x$ .

We prove (1) and (2) simultaneously. For (1), we put  $Y' = Y$  and assume that  $\beta > 0$  is an arbitrary ordinal. For (2), we put  $Y' = Y \cup \{\emptyset\}$  and we assume that  $\beta = \alpha$ .

By (4), (5), (6), and (10) of Proposition 3.16, and the assumptions that  $Y \in H$  and  $Y \neq \emptyset$ , it follows that  $Y' \in H$  and  $Y' \setminus \{Y'\} \neq \emptyset$ , in both cases.

Next, in both cases, assume that  $x \in Y'$ ,  $x' \in H_\beta^{Y'} \subseteq H^{Y'}$ , and  $x' \in x$ . Then  $x \neq \emptyset$ , so  $x \in Y$ .

By (15) of Proposition 3.6, there exists a finite sequence  $\{x_k\}_{k=0}^n$  such that  $x_0 \in Y'$ ,  $x_n = x'$ , and  $x_{k-1} \in x_k$  for all  $k$  ( $1 \leq k \leq n$ ). Put  $x_{n+1} = x$  and consider the sequence  $\{x_k\}_{k=0}^{n+1}$ . If  $x_0 \in Y$ , this contradicts the intransitivity of  $Y$ .

In the case (1), we have now reached a contradiction, and we conclude that  $Y$  is completely grounded.

In the case (2), we conclude that we must have  $x_0 = \emptyset$  for every such sequence  $\{x_k\}_{k=0}^n$ . Hence, by (10) of Proposition 3.9,  $x' \in H_\beta = H_\alpha$ . Thus,  $r(x') \leq \alpha$ . But  $x' \in \cup_{x \in Y'} x = \cup_{x \in Y} x$ , so  $r(x') > \alpha$ . We have now reached a contradiction in this case too, and we conclude that  $Y' = Y \cup \{\emptyset\}$  is  $\alpha$ -grounded.

To summarize, we have proved that both (1) and (2) hold.

(3) now follows from (1), and (7) of Proposition 3.16. ■

3.23. REMARKS. Our objective is now to find intransitive sets  $Y \in H$  with prescribed transfinite cardinalities such that the elements in the elements in  $Y$  satisfy the condition in (2) of Proposition 3.22, for  $\alpha = \omega$ . Because if we put  $Y' = Y \cup \{\emptyset\}$ , then  $Y'$  is  $\alpha$ -grounded, and we can then let  $Y$  be the set of "urelements" in our superstructure.

But let us first see what nice, and important, effects intransitivity has on  $m$ -tuples, relations, and functions. Proposition 3.7 is considerably strengthened if we assume that  $Y \setminus \{\emptyset\}$  is intransitive.

3.24. LEMMA. *Let  $Y$  be an intransitive set such that  $i \in Y$ , where  $i$  belongs to an  $\mathbb{N}$ -copy (see Remarks 2.3) and  $i > 0$ . Let  $\langle a_1, a_2, \dots, a_m \rangle$  be an  $n$ -tuple, where  $m \geq i$ , and let  $A$  be a set such that  $\langle a_1, a_2, \dots, a_m \rangle \in A$ .*

*Then, neither of the sets  $\{i, a_i\}$ ,  $\{\{i\}\}$ ,  $\{\{i, a_i\}\}$ ,  $\langle a_1, a_2, \dots, a_m \rangle$ , and  $A$  belongs to  $Y$ .*

Hence,  $Y$  contains no  $m$ -tuples.

PROOF. If any of these sets would belong to  $Y$ , then, some initial segment with at least two elements of the finite sequence

$$\{i, \{i, a_i\}, \{\{i\}, \{\{i, a_i\}\}\}, \langle a_1, a_2, \dots, a_m \rangle, A\}$$

would violate the intransitivity of  $Y$ . Hence, none of these sets belongs to  $Y$ . ■

3.25. PROPOSITION. Assume that  $Y$  is a set such that  $Y \setminus \{\emptyset\}$  is intransitive and such that it contains an  $\mathbb{N}$ -copy. Then, for  $m, k \geq 1$ :

- (1)  $Y$  contains no  $m$ -tuples, and no nonempty  $m$ -ary relations.
- (2) For all  $a_1, a_2, \dots, a_m: a_1, a_2, \dots, a_m \in H^Y$  if and only if  $\langle a_1, a_2, \dots, a_m \rangle \in H^Y$ .  
If these conditions hold, then  $(\max_{1 \leq i \leq m} r^Y(a_i)) + 3 = r^Y(\langle a_1, a_2, \dots, a_m \rangle)$ .
- (3) Let the set  $A$  be a nonempty  $m$ -ary relation:  
Then,  $A \subseteq (H^Y)^m$  if and only if  $A \in H^Y$ .  
If these conditions hold, then  $\sup_{a \in \mathcal{C}(A)} (r^Y(a) + 3) + 1 = r^Y(A)$ .
- (4) If  $\alpha$  is a limit ordinal, and  $A \neq \emptyset$  is an  $m$ -ary relation, then  $A \subseteq (H_\beta^Y)^m$  for some  $\beta < \alpha$  if and only if  $A \in H_\alpha^Y$ .
- (5) For all nonempty sets  $B$  such that  $B \notin Y: B \in H^Y$  if and only if  $B^m \in H^Y$ .  
If these conditions hold, then there is an ordinal  $\alpha > 0$  such that, either,
  - (a)  $\alpha$  is a successor ordinal,  $r^Y(B) = \alpha + 1$ , and  $r^Y(B^m) = \alpha + 4$ , or
  - (b)  $\alpha$  is a limit ordinal,  $r^Y(B) = \alpha + 1$ , and  $r^Y(B^m) = \alpha + 1$ .
- (6) Assume that the sets  $A$  and  $B$  are nonempty  $m$ -ary and  $k$ -ary relations, respectively.  
Then,  $A \in H^Y$  and  $B \in H^Y$  both hold, if and only if  $A \times B \in H^Y$ .  
If these conditions hold, then  $\max(r^Y(A), r^Y(B)) = r^Y(A \times B)$ .
- (7) Assume that the sets  $A$  and  $B$ , are  $m$ -ary and  $k$ -ary nonempty relations, respectively, and let  $f: A \rightarrow B$  be a function.  
Then,  $f \in H^Y$  if and only if  $A \in H^Y$  and  $B \in H^Y$  both hold.  
If these conditions hold then:
  - (a)  $G_f \in H^Y$ , with  $r^Y(A) \leq r^Y(G_f) \leq \max(r^Y(A), r^Y(B))$ ,
  - (b)  $r^Y(f) = \max(r^Y(A), r^Y(B)) + 3$ .

Also, if  $f$  is surjective, then the above conditions are equivalent to  $G_f \in H^Y$ , and if the conditions hold, then equality holds in the last step in (a).

PROOF. First of all:  $Y$  is infinite since it contains an  $\mathbb{N}$ -copy. Hence, certainly,  $Y \setminus \{Y\} \neq \emptyset$ . Therefore,  $H^Y$  is defined, and  $r^Y(y)$  is defined for all  $y \in H^Y$ .

(1) follows immediately from Lemma 3.24.

To obtain (2): If  $a_1, a_2, \dots, a_m \in H^Y$ , then, by (1) of Proposition 3.7 and (5) of Proposition 3.6,  $\langle a_1, a_2, \dots, a_m \rangle \in H^Y$ , with  $r^Y(\langle a_1, a_2, \dots, a_m \rangle) \leq (\max_{1 \leq i \leq m} r^Y(a_i)) + 3$ .

Conversely, assume that  $\langle a_1, a_2, \dots, a_m \rangle \in H^Y$ . For each  $i$  ( $1 \leq i \leq m$ ), consider the finite sequence:  $\{a_i, \{i, a_i\}, \{\{i\}, \{\{i, a_i\}\}\}, \langle a_1, a_2, \dots, a_m \rangle\}$ . By Lemma 3.24, no element in this sequence lies in  $Y$ , except possibly  $a_i$ . Then, by (5) of Proposition 3.9 and (18) of Proposition 3.6, all elements in this sequence lie in  $H^Y$ , and all elements except possibly  $a_i$  lie in  $H^Y \setminus Y$ .

Now,  $r^Y(a_i) + 3 \leq r^Y(\langle a_1, a_2, \dots, a_m \rangle)$  follows by repeated applications of (5) of Proposition 3.9. Since this holds for all  $i$  ( $1 \leq i \leq m$ ), this and the previous case give  $(\max_{1 \leq i \leq n} r^Y(a_i)) + 3 = r^Y(\langle a_1, a_2, \dots, a_n \rangle)$ .

Hence, (2) holds.

To obtain (3): It follows from (2) that  $A \subseteq (H^Y)^m$  if and only if  $A \subseteq H^Y$ , with  $\sup_{a \in \mathcal{C}(A)}(r^Y(a) + 3) = \sup_{x \in A} r^Y(x)$  in this case. But  $A \notin Y$ , by (1). Since also  $A \neq \emptyset$ , (8) and (5) of Proposition 3.9 give that  $A \subseteq H^Y$  if and only if  $A \in H^Y$ , with  $(\sup_{x \in A} r^Y(x)) + 1 = r^Y(A)$  in this case.

Putting these two pieces together, (3) follows.

To obtain (4): By Proposition 2.9,  $A \subseteq (H_\beta^Y)^m$  if and only if  $\mathcal{C}(A) \subseteq H_\beta^Y$ , for all ordinals  $\beta$ . Since  $r^Y(A)$  cannot be a limit ordinal, by (1) of Proposition 3.9, the conclusion follows from (3), and (5) of Proposition 3.6.

To obtain (5): Since  $B \notin Y$  and  $B \neq \emptyset$ , (8) and (5) of Proposition 3.9 give that  $B \subseteq H^Y$  if and only if  $B \in H^Y$ , with  $(\sup_{b \in B} r^Y(b)) + 1 = r^Y(B)$  in this case. Also, By (1) of Proposition 3.9,  $r^Y(b) \geq 1$  for all  $b \in B$ . Since also  $\mathcal{C}(B^m) = B$ , by Proposition 2.9, the desired conclusions now follow from (3), taking  $\alpha = \sup_{b \in B} r^Y(b)$  in both (a) and (b).

Thus, (5) holds.

To obtain (6): It follows from (3) that  $A \subseteq (H^Y)^m$  if and only if  $A \in H^Y$ , and that  $\sup_{a \in \mathcal{C}(A)}(r^Y(a) + 3) + 1 = r^Y(A)$  in this case. Similarly for  $B$  and  $A \times B$ . Clearly,  $A \subseteq (H^Y)^m$  and  $B \subseteq (H^Y)^k$  both hold if and only if  $A \times B \subseteq (H^Y)^{m+k}$ . Also,  $\mathcal{C}(A \times B) = \mathcal{C}(A) \cup \mathcal{C}(B)$ . Hence,  $\sup_{c \in \mathcal{C}(A \times B)}(r^Y(c) + 3) = \max(\sup_{a \in \mathcal{C}(A)}(r^Y(a) + 3), \sup_{b \in \mathcal{C}(B)}(r^Y(b) + 3))$ . Thus, (6) follows.

To obtain (7): Clearly,  $\mathcal{C}(A) \subseteq \mathcal{C}(G_f) \subseteq \mathcal{C}(A) \cup \mathcal{C}(B)$ , with equality in the last step if  $f$  is surjective. Also, by (1),  $A \notin Y$ ,  $B \notin Y$ , and  $G_f \notin Y$ . Since  $G_f \subseteq (H^Y)^{m+k}$  if and only if  $\mathcal{C}(G_f) \subseteq H^Y$ , by Proposition 2.9, and the corresponding conditions hold for  $A$  and  $B$ , (3) now gives that if  $A, B \in H^Y$ , then  $G_f \in H^Y$ , and that (a) holds. By the same argument,  $A, B \in H^Y$  if  $f$  is surjective and  $G_f \in H^Y$ , and then the claimed equality in (a) also holds.

Since  $f = \langle A, G_f, B \rangle$ , it follows from (2) that  $A, G_f, B \in H^Y$  if and only if  $f \in H^Y$ . Together with the above, this gives that  $A, B \in H^Y$  if and only if  $f \in H^Y$ , and that these conditions hold if and only if  $G_f \in H^Y$  if  $f$  is surjective. Also, (2) and (a) give that (b) holds then.

Thus, (7) holds. ■

3.26. REMARKS. We will now investigate what happens in the degenerate cases when  $m = 0$  or  $k = 0$ , or when we have empty relations or sets in Proposition 3.25. We will rely upon Remarks 2.12.

First, we assume that  $\emptyset \in H^Y$ , which is equivalent to  $\emptyset \in Y$ , by (6) of Proposition 3.6. Without this assumption, the sides of the equations and inequalities might not be defined. In this case,  $r^Y(\emptyset) = 1$ , by (4) of Proposition 3.9. We also make the additional assumption that  $\{\emptyset\} \notin Y$ . This will be satisfied in the most interesting cases.

Then, by (8) and (5) of Proposition 3.9,  $\{\emptyset\} \in H^Y$  with  $r^Y(\{\emptyset\}) = 2$ .

Now, (1) fails for the 0-tuple  $\langle \rangle = \emptyset$ , which lies in  $Y$ . It also fails for any empty relation (of arbitrary arity). It holds, however, for the nonempty 0-ary relation  $\{\emptyset\}$ , which does not lie in  $Y$ .

For, (2) the condition  $a_i \in H^Y$  for all  $i$  ( $1 \leq i \leq 0$ ) is vacuously satisfied. It is also true that  $\langle \rangle = \emptyset \in H^Y$ , so the equivalence holds.

The equation fails in this case, though. Since we stipulate that  $\max_{1 \leq i \leq 0} r^Y(a_i) = 0$ , the correct equation should be  $\max_{1 \leq i \leq 0} r^Y(a_i) + 1 = r^Y(\emptyset)$ .

For (3), the equivalence holds if  $A$  is the empty relation, for all  $m$ . Since  $\mathcal{C}(\emptyset) = \emptyset$ , the equation holds in this case, too:  $\sup_{a \in \mathcal{C}(A)} (r^Y(a) + 3) + 1 = 0 + 1 = 1 = r^Y(A)$ .

If  $A$  is the nonempty 0-ary relation  $\{\emptyset\}$ , then,  $(H^Y)^0 = \{\emptyset\} = A$ , and  $\{\emptyset\} \in H^Y$ , so the equivalence holds. But the equation fails, since  $\mathcal{C}(A) = \emptyset$ . The correct equation should be  $(\sup_{a \in \mathcal{C}(A)} r^Y(a)) + 2 = r^Y(A) = 2$ .

Since  $r^Y(\emptyset) = 1$  and  $r^Y(\{\emptyset\}) = 2$ , (5) of Proposition 3.6 gives that these sets lie in  $H_\alpha^Y$  if  $\alpha$  is a limit ordinal. Also, for all ordinals  $\beta$ ,  $\emptyset \subseteq (H_\beta^Y)^m$  for every  $m \geq 0$ , and  $\{\emptyset\} = (H_\beta^Y)^0$ , so (4) holds if either  $A$  is the empty relation, for any  $m \geq 0$ , or if  $A$  is the 0-ary relation  $\{\emptyset\}$ .

Since  $\emptyset \in Y$ , (5) does not apply if  $B = \emptyset$ . In principle, therefore, (5) holds vacuously in this case. If  $B \neq \emptyset$  and  $B \notin Y$ , but  $m = 0$ , then, since  $B^0 = \{\emptyset\} \in H^Y$ , the equivalence holds if and only if  $B \in H^Y$ . Assuming this, (a) and (b) fail utterly, because  $r^Y(B^0) = 2$ . No matter what  $r^Y(B)$  is, it can never satisfy (a) or (b) if  $r^Y(B^m) = 2$ .

For (6): If  $A = \emptyset$  and  $B \neq \emptyset$ , then  $A \times B = \emptyset$ , independently of what  $B$  is. Since  $\emptyset \in H^Y$ , the equivalence holds if and only if  $B \in H^Y$ . The equation fails utterly in this case, because the right side will be  $r^Y(\emptyset) = 1$ , while the left side will be  $r^Y(B)$ , by (1) of Proposition 3.9, and since  $B \notin Y$ , by (1) of Proposition 3.25 (this holds here also if  $k = 0$ ), the equation cannot work in this case, by (3) of Proposition 3.9. Similarly if  $A \neq \emptyset$  and  $B = \emptyset$ . If  $A = B = \emptyset$ , then  $A \times B = \emptyset$ , and the equivalence and the equation hold in this case.

If  $A$  is the 0-ary relation  $\{\emptyset\}$ , and if  $B \neq \emptyset$  then  $A \times B = B$ . Since  $\{\emptyset\} \in H^Y$ , the equivalence holds in this case.

Assume that  $B \in H^Y$ . Since  $r^Y(A) = 2$  and  $B \notin Y$ , by (1) of Proposition 3.25 (this is also true if  $k = 0$ ), it follows from (1) and (3) of Proposition 3.9 that  $\max(r^Y(A), r^Y(B)) = r^Y(B)$ . Hence, the equations hold in this case, even when also  $B = \{\emptyset\}$ . Similarly if we interchange the roles of  $A$  and  $B$ . The case when  $A = B = \{\emptyset\}$  is already covered.

In (7) we cannot have  $A \neq \emptyset$  and  $B = \emptyset$ . Assume that  $A = \emptyset$ . Then  $G_f = \emptyset$ , and it follows from (2) that the equivalence holds. Since  $r^Y(B) \geq 1$ , by (1) of Proposition 3.9, (a) holds if the conditions hold, with equality if and only if  $B = \emptyset$ , since  $B \notin Y$  for all other relations, using (1), (3), and (4) of Proposition 3.9, and (1) of Proposition 3.25. Clearly, in this case,  $f$  is surjective if and only if  $B = \emptyset$ , so even the surjectivity clause holds in this case. Also, it follows from (2) that (b) holds in this case.

If  $A$  is the 0-ary relation  $\{\emptyset\}$  and  $B \neq \emptyset$ , then  $A \in H^Y$ , and  $G_f$  is a singleton subset of  $B$ . Then  $G_f \notin Y$ , by (1) of Proposition 3.25 (this holds also if  $k = 0$ ). By the same reason,  $B \notin Y$ . By (5) and (8) of Proposition 3.9,  $G_f \in H^Y$  if  $B \in H^Y$ . It follows from (2) of Proposition 3.25 that the equivalence holds in this case.

Assume that the conditions hold. Since  $B \notin Y$ , (12) and (5) of Proposition 3.6, and (1) and (3) of Proposition 3.9, give  $2 = r^Y(\{\emptyset\}) \leq r^Y(G_f) \leq r^Y(B)$ . Hence, (a) holds. It is also clear from (2) of Proposition 3.25 that (b) holds.

Surjectivity holds if and only if  $B$  is singleton, and then  $G_f = B$ , so the surjectivity clause holds.

Thus, all of (7) is true in this case, even when also  $B = \{\emptyset\}$ .

Finally, assume that  $m \geq 1$ ,  $A \neq \emptyset$ , and  $B = \{\emptyset\}$ . Then,  $G_f = A$ , and

$f$  is surjective. Clearly, the equivalence holds in this case, by (2) of Proposition 3.25. By (1) of Proposition 3.25,  $A \notin Y$ , so (1) and (3) of Proposition 3.9 give  $r^Y(G_f) = r^Y(A) \geq 2 = r^Y(B)$ . Thus (a) holds with equality, and (b) holds by (2) of Proposition 3.25.

The next theorem is the main result in this chapter. It gives sufficient conditions for when an intransitive set  $X \in H$ , such that the elements in its elements satisfy a given rank condition, can be extended to a larger intransitive set with the same property.

After proving this theorem, all we need to do is to use it to construct the intransitive sets and the extensions we need. This is done in the more directly useful Theorem 3.35.

Note that in of Theorem 3.27, we must have  $\emptyset \notin Y$ , unless  $Y = \{\emptyset\}$  (but  $Y \neq \{\emptyset\}$  when we apply this in Theorem 3.35, if  $b > 1$ ), by (1) of Proposition 3.21 and (3) of Proposition 3.22. This where we need to use a hierarchy  $H^Y$  with  $\emptyset \notin Y$ .

**3.27. THEOREM.** *Let  $X \in H$  and  $Y \in H$ , with  $Y \neq \emptyset$  and  $X \cap Y = \emptyset$ , and assume that  $X \cup Y$  is intransitive. Assume also that  $\alpha$  is an ordinal such that  $r(y) > \alpha$  for all  $y \in (\cup_{x \in X} x) \cup Y$ .*

*Let  $Z \subseteq \{\{z\} \mid z \in H_{\beta+1}^Y \setminus H_{\beta}^Y\}$ , for some ordinal  $\beta$ .*

*Then:*

- (1)  $Z \subseteq H^Y$ , and  $r^Y(y) = \beta + 2$ , for all  $y \in Z$ .
- (2)  $X \cup Z \in H$ .
- (3)  $X \cap Z = \emptyset$ .
- (4)  $r(y) > \alpha$  for all  $y \in (\cup_{x \in X \cup Z} x) \cup Z$ .
- (5)  $X \cup Z$  is intransitive.

**PROOF.** First, since  $X \in H$  and  $Y \in H$ ,  $X \subseteq H$  and  $Y \subseteq H$ , by (5) of Proposition 3.16. Since  $Y \neq \emptyset$ , (10) of Proposition 3.16 gives  $Y \setminus \{Y\} \neq \emptyset$ . Therefore, by Definitions 3.4 and 3.8, expressions such as  $H_{\alpha}^Y$ ,  $H^Y$ , and  $r^Y(x)$  makes sense.

Now, to prove (1), pick  $z \in H_{\beta+1}^Y \setminus H_{\beta}^Y$ . Then,  $r^Y(z) = \beta + 1$ , by (5) of Proposition 3.6. Also,  $\{z\} \in \mathcal{P}(H_{\beta+1}^Y) \setminus \{\emptyset\}$ , and so  $\{z\} \in H_{\beta+2}^Y$ , by (3) of Proposition 3.6. This holds for all  $z \in H_{\beta+1}^Y \setminus H_{\beta}^Y$ , so  $Z \subseteq H_{\beta+2}^Y \subseteq H^Y$ .

Next,  $Y$  is completely grounded, by (1) of Proposition 3.21 and (1) of Proposition 3.22. Then, (6) of Proposition 3.14 gives  $r^Y(y) = \beta + 2$  for all  $y \in Z$ .

Thus, (1) holds, and we also obtained that  $Y$  is completely grounded.

Next, we prove (2). We saw above that  $X \subseteq H$  and  $Y \subseteq H$ . Since also  $Y \setminus \{Y\} \neq \emptyset$ , (9) of Proposition 3.9 gives that  $H^Y \subseteq H$ . (1) then gives  $Z \subseteq H$ . Thus,  $X \cup Z \subseteq H$ . Then, (6) of Proposition 3.16 gives that  $X \cup Z \in H$ , that is, (2) holds.

We now prove that  $X \cap H^Y = \emptyset$ .

To do this, let  $x \in X \cap H^Y$ . Then, there is a finite sequence  $\{x_k\}_{k=0}^n$  as in (11) in Proposition 3.6, with  $x_0 \in Y$  and  $x_n = x$ . If  $n = 0$ , then  $x \in Y$ , contradicting  $X \cap Y = \emptyset$ . If  $n > 0$ , then this sequence violates the intransitivity of  $X \cup Y$ . Thus, we reach a contradiction in any case.

It follows that  $X \cap H^Y = \emptyset$ .

From this and (1), (3) follows.

Next, since, by (4) and (7) of Proposition 3.16,  $\{\emptyset\}$  is completely grounded, and since  $Y \subseteq H$ , (10) of Proposition 3.14 gives that  $r(y) > \alpha$  for all  $z \in H^Y \subseteq H$ . Hence,  $r(y) > \alpha$  for all  $y \in H_{\beta+1}^Y \setminus H_\beta^Y$ . Also, since  $Z \subseteq H^Y$ , by (1),  $r(y) > \alpha$  for all  $y \in Z$ . Hence,  $r(y) > \alpha$  for all  $y \in (\cup_{x \in X \cup Z} x) \cup Z$ , that is, (4) holds.

To prove (5), assume that there exists a finite sequence  $\{x_k\}_{k=0}^n$ , with  $n \geq 1$ , such that  $x_0 \in X \cup Z$ ,  $x_n \in X \cup Z$ , and  $x_{k-1} \in x_k$  for all  $k$  ( $1 \leq k \leq n$ ).

Now,  $x_0 \in X$  and  $x_n \in X$  cannot both hold, since  $X \cup Y$  is intransitive. Assume that  $x_0 \in Z$  and  $x_n \in X$ . Then, since  $x_0 \in H^Y$ , by (1), (15) of Proposition 3.6 gives a finite sequence  $\{y_j\}_{j=0}^m$  such that  $y_0 \in Y$ ,  $y_m = x_0$ , and  $y_{j-1} \in y_j$  for all  $j$  ( $1 \leq j \leq m$ ). If we put  $y_{m+k} = x_k$  for all  $k$  ( $1 \leq k \leq m$ ), then the sequence  $\{y_i\}_{i=0}^{m+n}$  violates the intransitivity of  $X \cup Y$ .

It follows that we must have  $x_n \in Z$ . Hence  $x_n \in H^Y$ , by (1). Then, by (18) of Proposition 3.6, we must have either (i)  $x_k \in H^Y \setminus Y$ , for all  $k$  ( $0 \leq k \leq n$ ), or (ii)  $x_m \in Y$  for some  $m$  ( $0 \leq m \leq n$ ), and  $x_k \in H^Y \setminus Y$  for all  $k$  such that  $m < k \leq n$ .

In case (i), repeated applications of (5) of Proposition 3.9 give  $r^Y(x_{k-1}) < r^Y(x_k)$  for all  $k$  ( $1 \leq k \leq n$ ). It follows that  $r^Y(x_0) < r^Y(x_n) = \beta + 2$ , by (1), which, again by (1), rules out the possibility  $x_0 \in Z$ . Hence,  $x_0 \in X \cap H^Y$ . But this is a contradiction, since we proved above that  $X \cap H^Y = \emptyset$ .

Hence, case (ii) must hold:  $x_m \in Y$  for some  $m$  ( $0 \leq m \leq n$ ), and  $x_k \in H^Y \setminus Y$  for all  $k$  with  $m < k \leq n$ . If then  $x_0 \in X$ ,  $X \cap Y = \emptyset$  implies  $m \geq 1$  and the subsequence  $\{x_k\}_{k=0}^m$  violates the intransitivity of  $X \cup Y$ . Therefore,  $x_0 \in Z$ . Again, (15) of Proposition 3.6 gives a finite sequence  $\{y_j\}_{j=0}^l$  such that  $y_0 \in Y$ ,  $y_l = x_0$ , and  $y_{j-1} \in y_j$  for all  $j$  ( $1 \leq j \leq l$ ). Put  $y_{l+k} = x_k$  for all  $k$  ( $1 \leq k \leq m$ ), and consider the sequence  $\{y_i\}_{i=0}^{l+m}$ . Then,  $y_0 \in Y$  and  $y_{l+m} \in Y$ . The intransitivity of  $X \cup Y$  then implies that  $l = m = 0$  and  $y_0 = x_0 = x_m$ . Thus,  $x_0 \in Y$ . But then, (3) of Proposition 3.9 gives that,  $r^Y(x_0) = 1$ . But this is a contradiction, because  $x_0 \in Z$ , and  $r^Y(x_0) = \beta + 2 > 1$ , by (1). (ii) is therefore ruled out, too.

We have now ruled out every possibility for  $X \cup Z$  to fail to be intransitive. Therefore,  $X \cup Z$  is indeed intransitive, that is, (5) holds. ■

**3.28. COROLLARY.** *Let  $Z \subseteq \{\{z\} \mid z \in H_{\beta+1} \setminus H_\beta\}$ , where  $\beta$  is an arbitrary ordinal.*

*Then:*

- (1)  $Z \subseteq H$ , with  $r(y) = \beta + 2$  for all  $y \in Z$ .
- (2)  $Z \in H$ .
- (3)  $Z$  is intransitive.

**PROOF.** Apply Theorem 3.27 with  $X = \emptyset$ ,  $Y = \{\emptyset\}$ ,  $\alpha = 0$ , and  $\beta$  arbitrary, using (4) of Proposition 3.16, (2) of Proposition 3.21, and (1) of Proposition 3.9 to see that the assumptions of Theorem 3.27 are satisfied. ■

It remains to put the pieces together.

Theorem 3.35 justifies the name "α-extendable" in the following definition. We emphasize again, however, that we only give *sufficient* conditions for the existence of this kind of extensions. The word "α-extendable" should not be understood as that the conditions we give here are also *necessary* for obtaining extensions meaningful for NSA.

**3.29. DEFINITION.** A set  $X$  is called *α-extendable*, where  $\alpha$  is an ordinal, if:

- (i)  $X \in H$ .

- (ii) There is a  $y \in H \setminus X$  such that  $r(z) > \alpha$  for all  $z \in (\cup_{x \in X} x) \cup \{y\}$ , and such that  $X \cup \{y\}$  is intransitive.

3.30. REMARK. In this definition,  $x \subseteq H$ , for all  $x \in X$ , by two applications of (5) of Proposition 3.16. Therefore,  $r(z)$  is defined for all  $z \in (\cup_{x \in X} x) \cup \{y\}$ .

3.31. PROPOSITION. *Let  $\alpha$  be an ordinal. Then:*

- (1) *Every subset of an  $\alpha$ -extendable set is  $\alpha$ -extendable.*
- (2)  *$\emptyset$  is  $\alpha$ -extendable.*
- (3) *If  $X$  is  $\alpha$ -extendable, then  $\emptyset \notin X$ .*
- (4) *If  $X \neq \emptyset$  is  $\alpha$ -extendable, then  $X$  is completely grounded.*
- (5) *If  $\alpha > 0$  and if  $X \neq \emptyset$  is  $\alpha$ -extendable, then  $X \cup \{\emptyset\}$  is  $\alpha$ -grounded.*

PROOF. If  $X$  is  $\alpha$ -extendable and  $Y \subseteq X$ , then it follows from Definition 3.29 and (5) and (6) of Proposition 3.16 that  $Y \in H$ . Then, it follows from Definition 3.29 and (1) of Proposition 3.21 that  $Y$  is  $\alpha$ -extendable.

Thus (1) holds.

To prove (2), pick  $u \in H_{\alpha+1} \setminus H_\alpha$  (which is nonempty, by (5) of Proposition 3.6). Put  $y = \{u\}$ . By Corollary 3.28,  $y \in H$ ,  $r(y) = \alpha + 2$ , and  $\{y\}$  is intransitive. Since  $\emptyset \in H$ , by (6) of Proposition 3.16, Definition 3.29 gives that  $\emptyset$  is  $\alpha$ -extendable, using this  $y$ .

To prove (3): If  $X \in H$  with  $\emptyset \in X$  and  $y \in H \setminus X$ , then, by Definition 3.29 and (5) and (6) of Proposition 3.16,  $X \cup \{y\} \in H$ , and  $y \neq \emptyset$ . Then, (3) of Proposition 3.22 gives that  $X \cup \{y\}$  is not intransitive. It follows that  $X$  is cannot be  $\alpha$ -extendable. Thus, (3) holds.

(4) follows immediately from the definition, (1) of Proposition 3.21, and (1) of Proposition 3.22.

(5) follows immediately from the definition, (1) of Proposition 3.21, and (2) of Proposition 3.22. ■

3.32. REMARKS. Keeping (5) of Proposition 3.31 in mind, our objective is now to prove that there exists  $\omega$ -extendable sets with arbitrarily great transfinite cardinalities, and moreover, to prove that any every  $\omega$ -extendable set can be extended to an  $\omega$ -extendable set with arbitrary great transfinite cardinality. This is precisely what is done in Theorem 3.35, if we put  $\alpha = \omega$  there.

First, however, we obtain a slight improvement of (5) of Proposition 3.25:

3.33. PROPOSITION. *Let  $X$  be an  $\alpha$ -extendable set, for some ordinal  $\alpha > 0$ , such that it contains an  $\mathbb{N}$ -copy. Put  $X' = X \cup \{\emptyset\}$ . Let  $A$  be a set such that  $A \cap H_\alpha^{X'} \neq \emptyset$ .*

*Then,  $A \notin X'$ , and, for all  $m \geq 1$ ,  $A \in H^{X'}$  if and only if  $A^m \in H^{X'}$ .*

*If these conditions hold, then there is an ordinal  $\beta > 0$  such that (5) (a) or (5) (b) of Proposition 3.25 holds, with  $\beta$  instead of  $\alpha$ , and with  $X'$  instead of  $Y$ .*

PROOF. Since  $X$  is  $\alpha$ -extendable, it is intransitive, by definition and (1) of Proposition 3.21. By (3) of Proposition 3.31,  $\emptyset \notin X$ , and by (5) of Proposition 3.31,  $X'$  is  $\alpha$ -grounded. Clearly  $A^m \cap (H_\alpha^{X'})^m \neq \emptyset$ .

Now, by Proposition 2.9 and Lemma 3.15,  $A = \mathcal{C}(A^m) \notin X'$ .

The remainder of the conclusion now follows from (5) of Proposition 3.25. ■

3.34. LEMMA. *Assume that  $Y \setminus \{Y\} \neq \emptyset$ , and that  $H_{\alpha+1}^Y$  can be well ordered.*

*Then,  $\text{card}(H_{\alpha+1}^Y \setminus H_\alpha^Y) = \text{card } H_{\alpha+1}^Y > \text{card } H_\alpha^Y \geq \text{card } \alpha$ , whenever either the ordinal  $\alpha$  is transfinite or the set  $Y$  is infinite.*

PROOF. Assume that  $Y \setminus \{Y\} \neq \emptyset$ , and let  $\alpha$  be an ordinal such that  $H_{\alpha+1}^Y$  can be well ordered. Since  $H_\alpha^Y \subseteq H_{\alpha+1}^Y$ , by (5) of Proposition 3.6,  $H_\alpha^Y$  can also be well ordered, and so can all the other sets below, by (3) of Proposition 3.6. (See Chapter 8.)

If  $\alpha = 0$ , then the conclusion follows immediately from (1) and (2) of Proposition 3.6 (even if  $Y$  is finite), so, assume that  $\alpha > 0$ .

Then, by (14) of Proposition 3.6,  $\text{card } H_\alpha^Y \geq \text{card } \alpha$ , and by (2) and (5) of Proposition 3.6,  $\text{card } H_\alpha^Y \geq \text{card } Y$ . Thus, if either  $\alpha$  is transfinite or  $Y$  is infinite,  $H_\alpha^Y$  is infinite. Now,  $\text{card } \mathcal{P}(H_\alpha^Y) > \text{card } H_\alpha^Y$ , so  $\text{card } \mathcal{P}(H_\alpha^Y) = \text{card}(\mathcal{P}(H_\alpha^Y) \setminus \{\emptyset\}) = \text{card}((\mathcal{P}(H_\alpha^Y) \setminus \{\emptyset\}) \cup Y) = \text{card } H_{\alpha+1}^Y$ , and hence  $\text{card}(H_{\alpha+1}^Y \setminus H_\alpha^Y) = \text{card } H_{\alpha+1}^Y > \text{card } H_\alpha^Y \geq \text{card } \alpha$ , which is our desired conclusion. ■

3.35. THEOREM. *Assume that the axiom of choice holds. Then, for any ordinal  $\alpha$ :*

- (1) *For any cardinal  $b$ , if  $X$  and  $Y$  are sets such that  $X \cap Y = \emptyset$  and  $X \cup Y$  is  $\alpha$ -extendable, then, there is a set  $Z$  such that  $(X \cup Y) \cap Z = \emptyset$ ,  $X \cup Y \cup Z$  is  $\alpha$ -extendable, and  $\text{card}(X \cup Z) \geq b$ .*
- (2) *For any cardinal  $b$ , there exists an  $\alpha$ -extendable set  $X$  such that  $\text{card } X = b$ .*

PROOF. First, every set can be well ordered, since the axiom of choice is assumed to hold (see Chapter 8).

To obtain (1), pick  $y$  so that it satisfies (ii) in Definition 3.29 for  $X \cup Y$ . By (6) and (10) of Proposition 3.16,  $\{y\} \in H$  and  $\{y\} \setminus \{\{y\}\} \neq \emptyset$ . Choose a transfinite ordinal  $\beta$  such that  $\text{card } \beta \geq \max(b, \text{card } X)$ . By Lemma 3.34,  $\text{card}(H_{\beta+1}^{\{y\}} \setminus H_\beta^{\{y\}}) > \text{card } \beta$ . Since the map  $z \mapsto \{z\}$  is injective,  $\text{card}\{\{z\} \mid z \in H_{\beta+1}^{\{y\}} \setminus H_\beta^{\{y\}}\} > \text{card } \beta$ .

Thus we can take a proper subset  $Z \subset \{\{z\} \mid z \in H_{\beta+1}^{\{y\}} \setminus H_\beta^{\{y\}}\}$  such that  $\text{card}(X \cup Z) \geq \text{card } Z \geq b$ . Pick  $u \in \{\{z\} \mid z \in H_{\beta+1}^{\{y\}} \setminus H_\beta^{\{y\}}\} \setminus Z$ . Then,  $u = \{z\}$  for some  $z \in H_{\beta+1}^{\{y\}} \setminus H_\beta^{\{y\}}$ .

Now, we can apply Theorem 3.27 with  $X \cup Y$  as  $X$ ,  $\{y\}$  as  $Y$ ,  $\alpha$  as  $\alpha$ , and  $Z \cup \{u\}$  as  $Z$ : By (3) of Theorem 3.27,  $(X \cup Y) \cap Z = \emptyset$  and, since  $u \notin Z$ ,  $u \notin X \cup Y \cup Z$ . (2) of Theorem 3.27, and (5) and (6) of Proposition 3.16, give  $X \cup Y \cup Z \in H$  and  $\{u\} \subseteq H$ , and hence  $u \in H$ . Thus,  $u \in H \setminus (X \cup Y \cup Z)$ . (4) of Theorem 3.27 gives  $r(v) > \alpha$  for all  $v \in \cup_{x \in X \cup Y \cup Z} x \cup \{u\}$ , and (5) of Theorem 3.27 gives that  $X \cup Y \cup Z \cup \{u\}$  is intransitive.

All this means that  $X \cup Y \cup Z$  is  $\alpha$ -extendable.

Since  $\text{card}(X \cup Z) \geq b$ , (1) holds.

To obtain (2), we first notice that it holds for  $b = 0$ , since  $\emptyset$  is  $\alpha$ -extendable, by (2) of Proposition 3.31. We use this case to prove (2) for  $b > 0$ : just apply (1) with  $\emptyset$  as  $X$  and  $Y$ . The set  $Z = \emptyset \cup Z = \emptyset \cup \emptyset \cup Z$  thus obtained then satisfies (1) for the given  $b$ . Then, there is a subset  $X \subseteq Z$  such that  $\text{card } X = b$ . This  $X$  is  $\alpha$ -extendable, by (1) of Proposition 3.31.

Thus, (2) holds. ■

3.36. REMARKS. A point is that if we use Theorem 3.35 to extend an  $\alpha$ -extendable set, then the extended set is also  $\alpha$ -extendable, and hence it can be further extended. For every such extension, we have an element such as  $y$  in (ii) of Definition 3.29, upon which the next extension can be based.

However, if we have an infinite sequence of successive extensions, is then the union of all the successively extended sets itself  $\alpha$ -extendable? The problem is that in this case, no such element  $y$  is supplied. This is the reason why we have invoked the set  $Y$  in Theorem 3.35: We can then take such an element  $y$  from  $Y$ , and then obtain a further extension.

This is done in the proof of Theorem 7.8. In most other applications, we can put  $Y = \emptyset$ , and hence disregard  $Y$ .

## CHAPTER 4

### Proper Nonstandard Extensions

4.1. DEFINITION. The set  $H_\omega^{S'}$  is called the *superstructure over the set  $S$* , if  $S' = S \cup \{\emptyset\}$ , where  $S$  is  $\omega$ -extendable and contains an  $\mathbb{N}$ -copy.

4.2. REMARKS. Since  $S'$  in Definition 4.1 is infinite, we certainly have  $S' \setminus \{S'\} \neq \emptyset$ , so  $H_\omega^{S'}$  is defined.

In Definition 4.4 we will extend a superstructure to a larger superstructure. We will then assume that the  $\mathbb{N}$ -copy is the same in the extended superstructure as in the original one. This is also the set whose elements we use when we form  $m$ -tuples and relations on  $H_\omega^{S'}$  and its extension. (See Remarks 2.3.)

For any transfinite cardinal  $b$ , there exists a set  $S$  with cardinality  $b$  for which the superstructure over  $S$  exists, if we choose a suitable  $\mathbb{N}$ -copy, provided that the axiom of choice holds. This follows from (2) of Theorem 3.35.

The idea is that the elements in  $S$  play the role of "urelements". We have proved in Chapter 3 that they are sufficiently "urelement-like" to be able to play that role sufficiently well for our purposes.

The following proposition gives the most useful properties of the superstructure  $H_\omega^{S'}$  over  $S$ . It is just a list of properties compiled from the previous chapter. Some of them are actually implicit in Definition 4.4. In the sequel, we will freely use them without referring to them. *So, if the reader finds an argument difficult to follow, he/she may check if some of the properties in Proposition 4.3 apply.*

Since (4) of Proposition 4.7 will show that  $H_\omega^{T'}$  is the superstructure over  $T$ , with  $T$  and  $T'$  as in Definition 4.4, Proposition 4.3 is applicable with  $T$  instead of  $S$ , also.

4.3. PROPOSITION. *Let  $H_\omega^{S'}$  be the superstructure over  $S$ . Then, for  $n < \omega$ :*

- (1)  $H_0^{S'} = \emptyset$ .
  - (2)  $H_1^{S'} = S' = S \cup \{\emptyset\}$ .
  - (3)  $H_{n+1}^{S'} = \mathcal{P}(H_n^{S'}) \cup S$ .
  - (4)  $H_\omega^{S'} = \bigcup_{n=0}^{\infty} H_n^{S'}$ .
  - (5) *If  $m < n$ , then  $H_m^{S'} \subset H_n^{S'} \subset H_\omega^{S'}$  (proper subsets), and, if  $H_\omega^{S'}$  can be well ordered,  $\text{card } H_m^{S'} < \text{card } H_n^{S'} < \text{card } H_\omega^{S'}$ .*
  - (6)  $H_\omega^{S'} \cap (\bigcup_{y \in S} y) = \emptyset$ , *that is,  $S'$  is  $\omega$ -grounded.*
  - (7)  $\mathcal{P}(H_\omega^{S'}) \cap S = \emptyset$ .
- In particular,  $\emptyset \notin S$ .*
- (8)  $A \in H_{n+1}^{S'} \setminus S$  *if and only if*  $A \subseteq H_n^{S'}$ .
  - (9) *If*  $A \in H_n^{S'} \setminus S$ , *then*  $\mathcal{P}(A) \in H_{n+1}^{S'} \setminus S$ .
  - (10) *If*  $A \in H_n^{S'} \setminus S$  *and*  $B \subseteq A$ , *then*  $B \in H_n^{S'} \setminus S$ .
  - (11) *If*  $A \in H_n^{S'} \setminus S$  *and*  $B$  *is any set, then*  $A \setminus B \in H_n^{S'} \setminus S$  *and*  $A \cap B \in H_n^{S'} \setminus S$ .
  - (12) *If*  $A, B \in H_n^{S'} \setminus S$ , *then*  $A \cup B \in H_n^{S'} \setminus S$ .
  - (13)  $r^{S'}(x) \geq 1$  *for all*  $x \in H_\omega^{S'}$ .

- (14)  $r^{S'}(x) = 1$  if and only if  $x \in S'$ .
- (15) If  $x \in H_\omega^{S'} \setminus S$ , then  $(\max_{x' \in x} r^{S'}(x')) + 1 = r^{S'}(x)$ .
- (16)  $H_n^{S'} \in H_\omega^{S'}$ , and  $r^{S'}(H_n^{S'}) = n + 1$ .
- (17)  $\emptyset \in H_\omega^{S'}$  and  $\{\emptyset\} \in H_\omega^{S'}$ , with  $r^{S'}(\emptyset) = 1$ , and  $r^{S'}(\{\emptyset\}) = 2$ .
- (18)  $S$  contains no  $m$ -tuples, and no  $m$ -ary relations, for any  $m \geq 0$ .
- (19) For all  $a_1, a_2, \dots, a_m: a_1, a_2, \dots, a_m \in H_\omega^{S'}$  if and only if  $\langle a_1, a_2, \dots, a_m \rangle \in H_\omega^{S'}$ , where  $m \geq 1$ .  
 If these conditions hold, then  $(\max_{1 \leq i \leq m} r^{S'}(a_i)) + 3 = r^{S'}(\langle a_1, a_2, \dots, a_m \rangle)$ .
- (20) Let the set  $A$  be an  $m$ -ary relation, for some  $m \geq 1$ :  
 Then,  $A \subseteq (H_n^{S'})^m$ , for some  $n < \omega$ , if and only if  $A \in H_\omega^{S'}$ .  
 If these conditions hold, then  $(\sup_{a \in \mathcal{C}(A)} r^{S'}(a)) + 4 = r^{S'}(A)$ .
- (21) For all sets  $B$  such that  $B \cap H_\omega^{S'} \neq \emptyset$ , and all  $m \geq 1$ :  $B \in H_\omega^{S'}$  if and only if  $B^m \in H_\omega^{S'}$ .  
 If these conditions hold, then  $5 \leq r^{S'}(B) + 3 = r^{S'}(B^m)$ .
- (22) Assume that  $A$  and  $B$  are nonempty  $m$ -ary and  $k$ -ary relations, respectively, where  $m, k \geq 0$ .  
 Then,  $A \in H_\omega^{S'}$  and  $B \in H_\omega^{S'}$  both hold, if and only if  $A \times B \in H_\omega^{S'}$ .  
 If these conditions hold, then  $\max(r^{S'}(A), r^{S'}(B)) = r^{S'}(A \times B)$ .
- (23) Assume that  $A$  and  $B$  are nonempty  $m$ -ary and  $k$ -ary relations, respectively, where  $m, k \geq 0$ , and let  $f: A \rightarrow B$  be a function.  
 Then,  $f \in H_\omega^{S'}$  if and only if  $A \in H_\omega^{S'}$  and  $B \in H_\omega^{S'}$  both hold.  
 If these conditions hold then:  
 (a)  $G_f \in H_\omega^{S'}$ , with  $r^{S'}(A) \leq r^{S'}(G_f) \leq \max(r^{S'}(A), r^{S'}(B))$ ,  
 (b)  $r^{S'}(f) = \max(r^{S'}(A), r^{S'}(B)) + 3$ .  
 Also, if  $f$  is surjective, then the above conditions are equivalent to  $G_f \in H_\omega^{S'}$ , and if the conditions hold, then equality holds in the last step in (a).

PROOF. (1)—(4) are special cases of (1)—(4) of Proposition 3.6, respectively.

(5) follows from (5) of Proposition 3.6 and Lemma 3.34, using that  $S$  is infinite.

(6) follows from (5) of Proposition 3.31.

(7) follows from (6), (3) of Proposition 3.31, and (8) Proposition 3.14.

For (8): The direct part follows from (9) of Proposition 3.6, and (5). To get the converse part, let  $A \subseteq H_n^{S'}$ . Then, it follows from (10) of Proposition 3.6, and (7) and (4), that  $A \in H_n^{S'} \setminus S$ . Thus, (8) holds.

To obtain (9): If  $A \in H_n^{S'} \setminus S$ , then  $\mathcal{P}(A) \in H_{n+1}^{S'}$ , by (13) of Proposition 3.6. By (5),  $A \in H_\omega^{S'}$ . Since  $A \in \mathcal{P}(A)$ , we cannot have  $\mathcal{P}(A) \in S$ , by (6). Hence  $\mathcal{P}(A) \in H_{n+1}^{S'} \setminus S$ , that is, (9) holds.

(10) is vacuously true if  $n = 0$ , by (1). Assume that  $n \geq 1$ ,  $A \in H_n^{S'} \setminus S$ , and that  $B \subseteq A$ . By (8),  $A \subseteq H_{n-1}^{S'}$ . Then,  $B \subseteq H_{n-1}^{S'}$ . (8) then gives  $B \in H_n^{S'} \setminus S$ . Thus, (10) holds.

(11) is two special cases of (10).

(12) is, again, vacuously true if  $n = 0$ , by (1). Assume that  $n \geq 1$ , and that  $A, B \in H_n^{S'} \setminus S$ . Then, by (8),  $A \subseteq H_{n-1}^{S'}$  and  $B \subseteq H_{n-1}^{S'}$ . Then,  $A \cup B \subseteq H_{n-1}^{S'}$ . (8) then gives  $A \cup B \in H_n^{S'} \setminus S$ . Thus, (12) holds.

(13) and (14) are special cases of (1) and (3) of Proposition 3.9, respectively.

(15) is a special case of (5) of Proposition 3.9. (sup becomes max, because a bounded set of natural numbers has a greatest element.)

(16) follows from (7) and (8) of Proposition 3.6, (5) and (6), and (2) and (5) of Proposition 3.14.

For (17): The statement about  $\emptyset$  follows from (2), (5), and (14). The statement about  $\{\emptyset\}$  follows from this, together with (9), (5), and (15).

Thus, (17) holds.

To obtain 18:  $\emptyset \notin S$ , by (7). But  $\emptyset$  can be interpreted as the 0-tuple  $\langle \rangle$  as well as an empty relation. Also,  $\{\langle \rangle\} = \{\emptyset\} \notin S$ , by (17), (14), and (2).

Hence,  $S$  contains not the 0-tuple  $\langle \rangle$ , not the empty relation  $\emptyset$ , and not the 0-ary relation  $\{\langle \rangle\}$ .

Using this, (18) now follows from (1) of Proposition 3.25.

In the proofs of each of (19)–(23), (4) is used. Thus, using (4):

(19) follows from (2) of Proposition 3.25.

(20) follows from (4) and (3) of Proposition 3.25.

(21) follows from Proposition 3.33, and (1) and (2).

(22) follows from (6) of Proposition 3.25, and Remarks 3.26.

(23) follows from (7) of Proposition 3.25, and Remarks 3.26. ■

4.4. DEFINITION. Let  $H_\omega^{S'}$  be the superstructure over  $S$ . A **proper nonstandard extension** of  $H_\omega^{S'}$  is a map  $*$  :  $H_\omega^{S'} \rightarrow H_\omega^{T'}$ , for some set  $T' \neq \emptyset$  for which  $T = T' \setminus \{\emptyset\}$  is  $\omega$ -extendable,<sup>13</sup> such that the following twelve **axioms** are satisfied:<sup>14</sup>

- (1)  $*x = T$ , for some  $x \in H_\omega^{S'}$ .
- (2)  $*\mathbb{N} \neq \mathbb{N}$ .
- (3) For all  $s \in S$  and all  $x \in H_\omega^{S'}$ :  $*x = s$  if and only if  $x = s$ .
- (4) If  $x \in *A \setminus T$  for some  $A \in H_\omega^{S'} \setminus S$ , and if  $y \in x$ , then  $y \in *B$  for some  $B \in H_\omega^{S'} \setminus S$ .
- (5) For all  $n < \omega$  and all  $x$ : If  $x \in H_n^{S'}$ , then  $*x \in H_n^{T'}$ .
- (6) For all  $A, B \in H_\omega^{S'} \setminus S$ :  $*(A \cup B) = *A \cup *B$ , and  $*(A \setminus B) = *A \setminus *B$ .
- (7) If  $a, A \in H_\omega^{S'}$  and  $a \in A$ , then  $*a \in *A$ .
- (8) For all  $A \in H_\omega^{S'} \setminus S$ ,  $*(A^1) = (*A)^1$ .
- (9) Let  $A, B \in H_\omega^{S'}$ , and assume that, for some  $m, k \geq 0$ ,  $A$  and  $*A$  are  $m$ -ary relations, and that  $B$  and  $*B$  are  $k$ -ary relations.

Then  $*(A \times B) = *A \times *B$ .

- (10) For all  $n < \omega$ :  $*(\text{memb } H_n^{S'}) = \text{memb } *H_n^{S'}$ .
- (11) For all  $n < \omega$ , if  $1 \leq i < j \leq m$ , then  $*\{\langle a_1, a_2, \dots, a_m \rangle \in (H_n^{S'})^m \mid a_i = a_j\} = \{\langle a_1, a_2, \dots, a_m \rangle \in (*H_n^{S'})^m \mid a_i = a_j\}$ .
- (12) If  $A \in H_\omega^{S'}$ , and  $A$  and  $*A$  are  $m+1$ -ary relations ( $m \geq 0$ ), then  $*(\pi_{\frac{m+1}{m+1}}^{m+1}[A]) = \pi_{\frac{m+1}{m+1}}^{m+1}[*A]$ .

4.5. REMARKS. Since  $T$  in Definition 4.4 is  $\omega$ -extendable,  $T' \in H$ , by definition, and (4), (5), and (6) of Proposition 3.16. Thus, by (10) of Proposition 3.16,  $T' \neq \emptyset$

<sup>13</sup>This  $*$  is not a function of the type in Definitions 2.10, but an ordinary function, as in the appendix (Chapter 8).

<sup>14</sup>The reader might think that twelve axioms is rather much, and I can agree. But it is difficult to see how the number of axioms can be substantially reduced.

implies  $T' \setminus \{T'\} \neq \emptyset$ , so  $H_\omega^{T'}$  is defined.

The existence of a set  $T'$  and a mapping  $*$  as in Definition 4.4, is the object of Chapter 6, and, if we add some extra properties, of Chapter 7 (saturation). In this chapter, the object is to explore the consequences of the axioms and derive the most useful properties for applications.

For the remainder of this chapter,  $S$ ,  $S'$ ,  $T$  and  $T'$  are fixed, and we assume that they satisfy definitions 4.1 and 4.4.

4.6. REMARKS. But let us first comment on the axioms.

Seven of them are taken from Henson ([1], p 4 f, 23 f, and 37 f.), sometimes slightly modified, namely axioms 4, 5, 6, 9, 10, 11, and 12. But Henson also in practice use assumptions similar to some of the other five axioms given here, without calling them axioms.

Axiom 1 guarantees that  $T$  is not chosen unnecessarily large. It makes  $H_\omega^{T'}$  into the smallest superstructure which contains the range of  $*$ .

Axiom 2 guarantees that  $*$  is a *proper* extension. Without it, and without any similar axiom for some infinite set other than  $\mathbb{N}$ ,  $*$  might be the identity map. It is suitable to use a countably infinite set such as  $\mathbb{N}$  here. With a larger infinite set, we cannot do nonstandard arithmetic, for example.

Axiom 3 is a strong statement, which we use to establish that the restriction of  $*$  to  $S$  is an inclusion map. Together with other axioms, it also implies that  $S \subseteq T$  (thus justifying calling  $*$  an "extension"), and that  $*$  is injective. (See (1) and (2) of Proposition 4.7 and (1) of Proposition 4.9.)

Axiom 4 has perhaps no immediate intuitive appeal, but it has a natural interpretation for ultrapower extensions (see Definition 6.23 and Proposition 6.25) and it is convenient to use. It can be expressed as: "Elements of internal sets are internal." (See Definitions 4.20 and (2) of Proposition 4.24.). It may also be expressed as: "The universe of internal elements is transitive."

Axiom 5 is one half the preservation of ranks. (Compare (2) of Proposition 4.9.)

Axioms 6-12 are of algebraic nature. They say that  $*$  preserves some important algebraic relations. They are used to derive the transfer principle (Theorem 4.16.) Actually we could replace axioms 6, 9, and 12 with the transfer principle<sup>15</sup> but since we have to go through a bunch definitions of formulas and satisfaction (Definitions 4.14) before we can state the transfer principle, it is simpler to leave these three axioms as they are.

In Axiom 9, the assumptions that  $A^*$  and  $*B$  are  $m$ -ary and  $k$ -ary relations, respectively, turns out to be superfluous, by (4) of Proposition 4.10. But we need Axiom 9 to derive this, so we must leave the assumptions as they are. A similar remark can be made about Axiom 12.

Since axioms 7 and 10 both deal with the membership relation, one might suspect that one of these axioms is superfluous. But I cannot see how one of them could be derived from the other eleven axioms.

In Axiom 10, we have restricted the membership relation to  $H_n^{S'}$ , for  $n < \omega$ . We must have some kind of restriction, because  $\text{memb } H_\omega^{S'}$  does not lie in  $H_\omega^{S'}$ . A similar remark applies to the "generalized diagonals" in Axiom 11.

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<sup>15</sup>As far as I can see, we cannot replace more axioms than these three with the transfer principle.

Proposition 4.13 generalizes Axiom 12 to omission of an arbitrary coordinate, not just the last one.

#### 4.7. PROPOSITION.

- (1)  $*s = s$  for all  $s \in S'$ .
- (2)  $S \subseteq T$  and  $S' \subseteq T'$
- (3)  $T' = T \cup \{\emptyset\}$ .
- (4)  $H_\omega^{T'}$  is the superstructure over  $T$ .

PROOF. If  $s \in S$ , then  $*s = s$ , by Axiom 3. Also, by Axiom 6,  $*\emptyset = *(\emptyset \setminus \emptyset) = *\emptyset \setminus *\emptyset = \emptyset$ , so  $*\emptyset = \emptyset$ .

Thus, (1) holds.

To obtain (2), let  $s \in S' = H_1^{S'}$ . Then, by (1) and Axiom 5,  $s = *s \in H_1^{T'} = T'$ . Hence,  $s \in T'$ . It follows that  $S' \subseteq T'$ , and hence that  $S = S' \setminus \{\emptyset\} \subseteq T' \setminus \{\emptyset\} = T$ .

Thus, (2) holds.

Since  $\emptyset \in S'$ , (3) follows immediately from the Definition 4.4 and (2).

(4) now follows from (2), (3), and definitions 4.1 and 4.4. ■

#### 4.8. PROPOSITION. If $A, B \in H_\omega^{S'} \setminus S$ , and if $a \in H_\omega^{S'}$ , then:

- (1)  $*(A \cap B) = *A \cap *B$ .
- (2)  $A = \emptyset$  if and only if  $*A = \emptyset$ .
- (3)  $A \subseteq B$  if and only if  $*A \subseteq *B$ .
- (4)  $a \in A$  if and only if  $*a \in *A$ .

PROOF. (1) follows from Axiom 6: If  $A, B \in H_\omega^{S'} \setminus S$ , then  $*(A \cap B) = *(A \setminus (A \setminus B)) = *A \setminus *(A \setminus B) = *A \setminus (*A \setminus *B) = *A \cap *B$ .

To prove (2): (1) of Proposition 4.7 gives  $*\emptyset = \emptyset$ . On the other hand: If  $A \neq \emptyset$ , then  $a \in A$  for some  $a \in H_\omega^{S'}$ . Thus, by Axiom 7,  $*a \in *A$ , so  $*A \neq \emptyset$ .

Thus, (2) holds.

(3) follows from Axiom 6 and (2), since  $A \subseteq B$  is equivalent to  $A \setminus B = \emptyset$ , and similarly for  $*A \subseteq *B$ .

To prove (4), we notice that the direct part follows from Axiom 7. To get the converse part, assume that  $a \notin A$ . Then,  $a \in H_n^{S'} \setminus A$ , for some  $n$ . Then, by axioms 7 and 6,  $*a \in *H_n^{S'} \setminus *A$ . Thus,  $*a \notin *A$ . Hence, the converse part of (4) holds.

Thus, (4) holds. ■

#### 4.9. PROPOSITION.

- (1)  $*$  is an injective map.
- (2) For all  $x \in H_\omega^{S'} : r^{S'}(x) = r^{T'}(*x)$ .
- (3)  $*H_n^{S'} \subseteq H_n^{T'}$ , for all  $n$ .

PROOF. To prove (1), pick  $x, y \in H_\omega^{S'}$  and assume that  $*x = *y$ . If  $x \in S$ , then, by (1) of Proposition 4.7,  $*y = *x = x$ , and, by Axiom 3,  $x = y$ . Similarly if  $y \in S$ .

Assume, then, that  $x, y \in H_\omega^{S'} \setminus S$ . Then  $*x = *y$  means that  $*x \subseteq *y$  and  $*y \subseteq *x$ . Hence, by (3) of Proposition 4.8,  $x \subseteq y$  and  $y \subseteq x$ , that is,  $x = y$ .

We have proved that  $x = y$  in all cases. Hence, (1) holds.

Next, put  $n = r^{S'}(x)$ . Then,  $n \geq 1$ . We now prove (2) by induction on  $n$ .

It follows from (1) of Proposition 4.7 that  $r^{T'}(*x) = 1$  if  $r^{S'}(x) = 1$ . Thus, the statement is true for  $n = 1$ .

Assume now that the statement is true for some  $n \geq 1$ , and pick  $x \in H_\omega^{S'}$  with  $r^{S'}(x) = n+1$ . Then  $x \in H_{n+1}^{S'} \setminus S$ , and, by Axiom 5,  $*x \in H_{n+1}^{T'}$ , so  $r^{T'}(*x) \leq n+1$ . Now, there is a  $y \in H_\omega^{S'}$  such that  $y \in x$  and  $r^{S'}(y) = n$ . Thus, by the induction hypothesis,  $r^{T'}(*y) = n$ . Also,  $*y \in *x$ , by Axiom 7. Then, since  $*y \in H_\omega^{T'}$ ,  $*x \notin T$ . It follows that  $r^{T'}(*x) \geq n+1$ .

Hence,  $r^{T'}(*x) = n+1 = r^{S'}(x)$ . (2) now follows by induction.

(3) follows from (2):  $r^{T'}(*H_n^{S'}) = r^{S'}(H_n^{S'}) = n+1$ . If  $*H_n^{S'} \in T$ , then  $n = 0$ , and  $*H_0^{S'} = *\emptyset = \emptyset$ , by (2) of Proposition 4.8. But  $\emptyset \notin T$ . Hence,  $*H_n^{S'} \notin T$ .

It follows that  $r^{T'}(x) \leq n$  for all  $x \in *H_n^{S'}$ , which implies that  $*H_n^{S'} \subseteq H_n^{T'}$ . ■

4.10. PROPOSITION. *Let  $A \in H_\omega^{S'} \setminus S$ . Then:*

- (1)  $*(\text{diag } A) = \text{diag } *A$ , and  $*(\text{memb } A) = \text{memb } *A$ .
- (2) *If  $A \in H_\omega^{S'} \setminus S$  is finite:  $A = \{a_1, a_2, \dots, a_m\}$ , then  $*A = \{*a_1, *a_2, \dots, *a_m\}$ .*  
*Also,  $*\langle a_1, a_2, \dots, a_m \rangle = \langle *a_1, *a_2, \dots, *a_m \rangle$*
- (3)  $*(A^m) = (*A)^m$ , for all  $m \geq 0$ .
- (4) *For any  $m, k \geq 0$ : If  $A \subseteq (H_n^{S'})^m$ , for some  $n < \omega$ , then  $*A \subseteq (*H_n^{S'})^m \subseteq (H_n^{T'})^m$ .*

PROOF. First, we choose  $n$  such that  $A \subseteq H_n^{S'}$ . To prove (1), we see that  $*A \subseteq *H_n^{S'}$ , by (3) of Proposition 4.8.

The following argument proves  $*(\text{diag } A) = \text{diag } *A$ , and if we just use Axiom 10 instead of Axiom 11, and change  $\text{diag}$  to  $\text{memb}$  everywhere, this argument proves  $*(\text{memb } A) = \text{memb } *A$  instead:

By Axiom 11 and Definitions 2.8,  $*(\text{diag } H_n^{S'}) = \text{diag } *(H_n^{S'})$ . Also, by axioms 9 and 8,  $*(A^2) = *(A^1 \times A^1) = *(A^1) \times *(A^1) = (*A)^1 \times (*A)^1 = (*A)^2$ .

Then, it follows from (1) of Proposition 4.8 that  $*(\text{diag } A) = *(A^2 \cap \text{diag } H_n^{S'}) = *(A^2) \cap *( \text{diag } H_n^{S'}) = (*A)^2 \cap \text{diag } *(H_n^{S'}) = \text{diag } *A$ . Thus, (1) holds.

To prove the first part of (2), we first prove that  $*\{a\} = \{*a\}$  for all  $a \in H_\omega^{S'}$ . Clearly,  $\{a\} \in H_\omega^{S'} \setminus S$ . Since  $\{a\}$  is singleton, we have  $\text{diag } \{a\} = \{a\}^1 \times \{a\}^1$ . Hence, by (1) and axioms 9 and 8,  $\text{diag } *\{a\} = (*\{a\})^1 \times (*\{a\})^1$ . But this implies that  $*\{a\}$  has at most one element. By Axiom 7,  $*a \in *\{a\}$ , so  $*\{a\} = \{*a\}$ .

We now prove the first part of (2) by induction on  $m$ . It holds for  $m = 0$ , by (2) of Proposition 4.8. If it holds for  $m$ , and if  $A = \{a_1, a_2, \dots, a_m, a_{m+1}\} \in H_\omega^{S'} \setminus S$ , put  $B = \{a_1, a_2, \dots, a_m\}$ . Then by Axiom 6, the case just proved, and the induction hypothesis:  $*A = *(B \cup \{a_{m+1}\}) = *B \cup *\{a_{m+1}\} = \{*a_1, *a_2, \dots, *a_m\} \cup \{*a_{m+1}\} = \{*a_1, *a_2, \dots, *a_m, *a_{m+1}\}$ .

Thus, by induction, the first part of (2) holds.

The second part of (2) follows by repeated applications of the first part, since  $\langle a_1, a_2, \dots, a_m \rangle = \{\{\{1\}, \{1, a_1\}\}, \{\{2\}, \{2, a_2\}\}, \dots, \{\{m\}, \{m, a_m\}\}\}$ ,

and similarly for  $\langle *a_1, *a_2, \dots, *a_m \rangle$ , using that  $*k = k$  for all  $k \in \mathbb{N} \subseteq S$ , by (1) of Proposition 4.9. Thus, (2) holds.

We prove (3) by induction on  $m$ .

$A^0 = (*A)^0 = \{\langle \rangle\} = \{\emptyset\}$ , and  $*\{\emptyset\} = \{\emptyset\}$ , by (2) of Proposition 4.8, and (2). Hence,  $*(A^0) = (*A)^0$ .

Now, assume that  $*(A^m) = (*A)^m$ , for some  $m \geq 0$ . Then, by axioms 9 and 8:  $*(A^{m+1}) = *(A^m \times A^1) = *(A^m) \times *(A^1) = (*A)^m \times (*A)^1 = (*A)^{m+1}$ . Thus,  $*(A^{m+1}) = (*A)^{m+1}$ .

(3) now follows by induction.

(4) now follows from (3) of Proposition 4.8, (3), and (3) of Proposition 4.9. ■

4.11. PROPOSITION.  $*S = T$  and  $*S' = T'$ .

PROOF. We first prove that  $*S' = T'$ . That  $*S' \subseteq T'$  is a special case of (3) of Proposition 4.9. Conversely,  $T = *X$  for some  $X \in H_\omega^{S'}$ , by Axiom 1.  $r^{S'}(X) = 2$ , by (2) of Proposition 4.9, so  $X \notin S$ . Put  $X' = X \cup \{\emptyset\}$ . Then, by Axiom 6, (2) of Proposition 4.8, and (2) of Proposition 4.10, we obtain  $*X' = *X \cup \{\emptyset\} = T \cup \{\emptyset\} = T'$ . Again, by (3) of Proposition 4.9,  $r^{S'}(X') = 2$ .

Then,  $X' \subseteq H_1^{S'} = S'$ , so (3) of Proposition 4.8 gives  $T' = *X' \subseteq *S'$ . Thus,  $*S' = T'$ .

Next,  $S = S' \setminus \{\emptyset\}$  and  $T = T' \setminus \{\emptyset\}$ , so by Axiom 6, (2) of Proposition 4.8, (2) of Proposition 4.10, and the above case,  $*S = T$ . ■

4.12. PROPOSITION. Let  $\sigma : \{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, k\}$ , for some  $m, k \geq 0$ , and assume that  $A \subseteq (H_n^{S'})^m$ , for some  $n < \omega$ .

Then,  $*\{\langle a_1, a_2, \dots, a_k \rangle \in (H_n^{S'})^k \mid \langle a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(m)} \rangle \in A\} = \{\langle a_1, a_2, \dots, a_k \rangle \in (*H_n^{S'})^k \mid \langle a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(m)} \rangle \in *A\}$ .

PROOF. First,  $*A \subseteq (*H_n^{S'})^m$ , by (4) of Proposition 4.10.

If  $m = 0$ , then, either  $A = \emptyset$  or  $A = \{\langle \rangle\}$ , and then  $*A = \emptyset$  and  $*A = \{\emptyset\}$ , respectively, by (2) of Proposition 4.8 and 2 of Proposition 4.10. What is claimed in these cases is then  $*\emptyset = \emptyset$  or  $*((H_n^{S'})^k) = (*H_n^{S'})^k$ , which hold, by (2) of Proposition 4.8 and (3) of Proposition 4.10.

So, assume that  $m \geq 1$ .

Consider the set

$$B = ((H_n^{S'})^k \times A) \cap (\bigcap_{i=1}^m \{\langle a_1, a_2, \dots, a_k, a_{k+1}, \dots, a_{k+m} \rangle \in (H_n^{S'})^{k+m} \mid a_{\sigma(i)} = a_{k+i}\}).$$

By axioms 8, 9, and 11, (1) of Proposition 4.8, and (3) of Proposition 4.10,

$$*B = ((*H_n^{S'})^k \times *A) \cap (\bigcap_{i=1}^m \{\langle a_1, a_2, \dots, a_k, a_{k+1}, \dots, a_{k+m} \rangle \in (*H_n^{S'})^{k+m} \mid a_{\sigma(i)} = a_{k+i}\}).$$

Also, it is clear that

$$\begin{aligned} \{\langle a_1, a_2, \dots, a_k \rangle \in (H_n^{S'})^k \mid \langle a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(m)} \rangle \in A\} \\ = \pi_{k+1}^{k+1}[\pi_{k+2}^{k+2} \dots [\pi_{k+m}^{k+m}[B]] \dots], \end{aligned}$$

and

$$\begin{aligned} \{\langle a_1, a_2, \dots, a_k \rangle \in (*H_n^{S'})^k \mid \langle a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(m)} \rangle \in *A\} \\ = \pi_{k+1}^{k+1}[\pi_{k+2}^{k+2} \dots [\pi_{k+m}^{k+m}[*B]] \dots]. \end{aligned}$$

The conclusion now follows by repeated applications of Axiom 12. ■

4.13. PROPOSITION. Let  $A \in H_\omega^{S'}$  be an  $m + 1$ -ary relation for some  $m \geq 0$ . Then, for any  $i: 1 \leq i \leq m + 1$ ,  $*(\pi_i^{m+1}[A]) = \pi_i^{m+1}[*A]$ .

PROOF.  $A \subseteq (H_n^{S'})^{m+1}$ , for some  $n$ . We define  $\sigma : \{1, 2, \dots, m, m + 1\} \rightarrow \{1, 2, \dots, m, m + 1\}$  by

$$\sigma(j) = \begin{cases} j, & \text{if } 1 \leq j < i, \\ m + 1, & \text{if } j = i, \\ j - 1, & \text{if } i < j \leq m + 1. \end{cases}$$

Then,

$$\pi_i^{m+1}[A] = \pi_{\frac{m+1}{m+1}}^{m+1}[\{\langle a_1, a_2, \dots, a_m, a_{m+1} \rangle \in (H_n^{S'})^{m+1} \mid \langle a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(m)}, a_{\sigma(m+1)} \rangle \in A\}],$$

and, by (4) of Proposition 4.10,

$$\pi_i^{m+1}[*A] = \pi_{\frac{m+1}{m+1}}^{m+1}[\{\langle a_1, a_2, \dots, a_m, a_{m+1} \rangle \in (*H_n^{S'})^{m+1} \mid \langle a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(m)}, a_{\sigma(m+1)} \rangle \in *A\}].$$

The conclusion now follows from Proposition 4.12 and Axiom 12. ■

We have now gathered enough material to be able to prove the transfer principle. But first, we need to make some definitions.

4.14. DEFINITIONS. As variables over  $S$  we use the symbols  $x_1, x_2, \dots$ . We use  $X_1, X_2, \dots$  as variables over  $T$ .

An atomic formula over  $S$  is an expression of the type  $\langle x_{i_1}, x_{i_2}, \dots, x_{i_m} \rangle \in A$ , where  $m \geq 0$ ,  $x_{i_1}, x_{i_2}, \dots, x_{i_m}$  are (not necessarily distinct) variables over  $S$ , and  $A \in H_\omega^{S'}$  is an  $m$ -ary relation.

We define formula over  $S$  recursively by the following:

- (i) An atomic formula over  $S$  is a formula over  $S$ .
- (ii) If  $\phi$  and  $\psi$  are formulas over  $S$ ,  $x_i$  is a variable over  $S$ , and  $B \in H_\omega^{S'} \setminus S$ , then  $\neg\phi$ ,  $(\phi \vee \psi)$ ,  $(\phi \wedge \psi)$ ,  $(\phi \rightarrow \psi)$ ,  $(\phi \leftrightarrow \psi)$ ,  $(\exists x_i \in B)\phi$ , and  $(\forall x_i \in B)\phi$  are formulas over  $S$ .
- (iii) A formula over  $S$  can be proved to be one using only (i) and (ii).

Atomic formulas and formulas over  $T$  are defined similarly, with  $T$  substituted for  $S$  everywhere above, and using variables over  $T$  instead of over  $S$ .

When confusion cannot arise, we sometimes will not write out the outermost parentheses around a formula.

We regard formulas involving  $\wedge$ ,  $\rightarrow$ ,  $\leftrightarrow$ , and  $\forall$  as abbreviations for formulas involving only  $\neg$ ,  $\vee$ , and  $\exists$ , by the following "expansion" rules:

$$\begin{array}{ll} (\phi \wedge \psi) & \text{is an abbreviation of } \neg(\neg\phi \vee \neg\psi), \\ (\phi \rightarrow \psi) & \text{is an abbreviation of } (\neg\phi \vee \psi), \\ (\phi \leftrightarrow \psi) & \text{is an abbreviation of } \neg(\neg(\neg\phi \vee \psi) \vee \neg(\neg\psi \vee \phi)), \\ (\forall x_i \in B)\phi & \text{is an abbreviation of } \neg(\exists x_i \in B)\neg\phi, \end{array}$$

for formulas over  $S$ , and similarly for formulas over  $T$ , with  $X_i$  instead of  $x_i$ .

We also sometimes write

$$\begin{array}{lll} \phi_1 \vee \phi_2 \vee \phi_3 & \text{instead of} & (\phi_1 \vee \phi_2) \vee \phi_3, \quad \text{and} \\ \phi_1 \vee \phi_2 \vee \phi_3 \vee \phi_4 & \text{instead of} & ((\phi_1 \vee \phi_2) \vee \phi_3) \vee \phi_4, \quad \text{etc.} \end{array}$$

and likewise,

$$\begin{array}{lll} \phi_1 \wedge \phi_2 \wedge \phi_3 & \text{instead of} & (\phi_1 \wedge \phi_2) \wedge \phi_3, \quad \text{and} \\ \phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4 & \text{instead of} & ((\phi_1 \wedge \phi_2) \wedge \phi_3) \wedge \phi_4, \quad \text{etc.} \end{array}$$

where  $\phi_1, \phi_2, \dots$  are formulas either over  $S$  or over  $T$ .

An occurrence of a variable  $x_i$  in a formula  $\phi$  over  $S$  is called *bounded* in  $\phi$  if it occurs within a subformula of  $\phi$  of the type  $(\exists x_i \in B)\psi$  or  $(\forall x_i \in B)\psi$ , where  $\psi$  is a formula, over  $S$ , itself. An occurrence of  $x_i$  in  $\phi$  is called *free* in  $\phi$  if it is not bounded in  $\phi$ . Similarly for a variable over  $T$  in a formula over  $T$ .

Now, let  $a_1, a_2, \dots, a_k \in H_\omega^{S'}$ , for some  $k \geq 0$ , and let  $\phi$  be a formula over  $S$ , such that all free variables in  $\phi$  (that is, all variables which have free occurrences in  $\phi$ ) are among  $x_1, x_2, \dots, x_k$ . (We do not require that all these variables have free occurrences in  $\phi$ , only that no *other* variables have.) We then say that  $\langle a_1, a_2, \dots, a_k \rangle \in H_\omega^{S'}$  *satisfies*  $\phi$  if, recursively, either

- (i)  $\phi$  is atomic:  $\langle x_{i_1}, x_{i_2}, \dots, x_{i_m} \rangle \in A$ , and  $\langle a_{i_1}, a_{i_2}, \dots, a_{i_m} \rangle \in A$ .
- (ii)  $\phi$  is  $\neg\psi$ , and  $\langle a_1, a_2, \dots, a_k \rangle$  does not satisfy  $\psi$ .
- (iii)  $\phi$  is  $\psi \vee \chi$ , and  $\langle a_1, a_2, \dots, a_k \rangle$  satisfies at least one of  $\psi$  and  $\chi$ .
- (iv)  $\phi$  is  $(\exists x_i \in B)\psi$ , and there exists an  $a_{k+1} \in B$  such that  $\langle a_1, a_2, \dots, a_k, a_{k+1} \rangle$  satisfies  $\psi'$ , where  $\psi'$  is obtained from  $\psi$  thus:

If  $x_i$  is  $x_{k+1}$ , then  $\psi'$  is  $\psi$ . Otherwise, let  $l$  be the smallest number such that the variable  $x_l$  occurs neither in  $\phi$  nor among  $x_1, x_2, \dots, x_k$ . Then, replace every occurrence of  $x_{k+1}$  in  $\psi$  by  $x_l$ , and then replace every occurrence of  $x_i$  in  $\psi$  by  $x_{k+1}$ . The resulting formula is then  $\psi'$ .

- (v)  $\phi$  is  $\psi \wedge \chi$ ,  $\psi \rightarrow \chi$ ,  $\psi \leftrightarrow \chi$ , or  $(\forall x_i \in B)\psi$ , and  $\langle a_1, a_2, \dots, a_k \rangle$  satisfies  $\phi'$ , where  $\phi'$  is obtained from  $\phi$  by using the appropriate "expansion rule" above.

We use the following notation: We write a formula over  $S$  as  $\phi(x_1, x_2, \dots, x_k)$  if we want to indicate that all free variables in  $\phi$  are among  $x_1, x_2, \dots, x_k$ , for some  $k \geq 0$ . For  $a_1, a_2, \dots, a_k \in H_\omega^{S'}$ , we then write  $\phi(a_1, a_2, \dots, a_k)$  if  $\langle a_1, a_2, \dots, a_k \rangle$  satisfies  $\phi(x_1, x_2, \dots, x_k)$ .

A *sentence* over  $S$  is formula  $\phi$  over  $S$  with no free variables. Such a  $\phi$  is *true* if the 0-tuple  $\langle \rangle = \emptyset$  satisfies it. Otherwise,  $\phi$  is *false*.

We make the corresponding definitions for formulas over  $T$ .

Now, let  $\phi$  be a formula over  $S$ . For every variable  $x_i$  over  $S$ , replace every occurrence of  $x_i$  in  $\phi$  with  $X_i$ . Also, for every  $m$ -ary relation  $A \in H_\omega^{S'}$  ( $m \geq 0$ ), replace every occurrence of  $A$  in  $\phi$  with  $*A$ . (Then,  $*A \in H_\omega^{T'}$  is an  $m$ -ary relation, by (4) of Proposition 4.10.) Likewise, for any set  $B \in H_\omega^{S'} \setminus S$  in  $\phi$ , replace every occurrence of  $B$  in  $\phi$  with  $*B$  (which lies in  $H_\omega^{T'} \setminus T$ , by (4) of Proposition 4.8 and Proposition 4.11).

The result is a formula over  $T$  which we denote  $*\phi$ , and which we call the *\*-transform* of  $\phi$ .

We write the  $*$ -transform of  $\phi(x_1, x_2, \dots, x_k)$ , as  $*\phi(X_1, X_2, \dots, X_k)$ .

4.15. REMARKS. There are, of course, many similarities to ordinary formulas in predicate logic. We have the same connectives  $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$ , and the same quantifiers  $\exists$  and  $\forall$ , with the usual interpretations: negation, disjunction, conjunction, implication, equivalence, and the existential and universal quantifiers. There are differences, though. The major difference is that the quantifiers here are *bounded*, i.e. the quantifier expressions have the forms  $\exists x_i \in B$  and  $\forall x_i \in B$ , where  $B$  is a set. For the reason to this restriction, see Remarks 4.17.

The change of variable in (iv) above is admittedly not very beautiful. It is one attempt to solve a problem which probably does not have any really nice solution.

For example, consider the formula  $(\exists x_3 \in S) \langle x_1, x_3 \rangle \in S^2$  over  $S$ . Its only free variable is  $x_1$ . Therefore, the question whether a 1-tuple  $\langle a_1 \rangle$  satisfies this formula or not, should be meaningful. And the answer should be yes if and only if there exists an  $a_2 \in S$  such that  $\langle a_1, a_2 \rangle \in S^2$ . But  $a_2$  should be substituted for the variable  $x_2$  in the formula  $\langle x_1, x_2 \rangle \in S^2$ , not for  $x_3$  in  $\langle x_1, x_3 \rangle \in S^2$ . We choose to solve this problem by simply change the bounded variable to  $x_2$ .

An alternative approach would be to assign variables to a formula before we ask whether or not an  $m$ -tuple satisfies it. For example, if  $\psi$  denotes the formula  $\langle x_1, x_2 \rangle \in \mathbb{N} \times S$ , then the questions whether or not  $\langle a_1, a_2 \rangle$  satisfies  $\psi(x_1, x_2)$  and  $\psi(x_2, x_1)$ , respectively, may have different answers. This makes the definition of satisfaction more complicated, and we have not chosen this approach.

Notice also that the number  $m$  for which is meaningful to ask whether or not an  $m$ -tuple satisfies a given formula is *not* uniquely determined by the formula. Any  $m$  such that the formula has no free occurrences of any  $x_k$  with  $k > m$  will do.

4.16. THEOREM. (Transfer Principle). *Let  $\phi(x_1, x_2, \dots, x_k)$  be a formula over  $S$ , ( $k \geq 0$ ), whose free variables are among  $x_1, x_2, \dots, x_k$ , and let  $C \in H_\omega^{S'}$  be a  $k$ -ary relation. Then*

(1)

$$*\{\langle a_1, a_2, \dots, a_k \rangle \in C \mid \phi(a_1, a_2, \dots, a_k)\} = \{\langle a_1, a_2, \dots, a_k \rangle \in *C \mid *\phi(a_1, a_2, \dots, a_k)\}.$$

(2) *For all  $b_1, b_2, \dots, b_k \in H_\omega^{S'}$ :  $\phi(b_1, b_2, \dots, b_k)$  if and only if  $*\phi(*b_1, *b_2, \dots, *b_k)$ .*

(3) *If  $\phi$  is a sentence in  $S$ , then  $\phi$  is true if and only if  $*\phi$  is.*

PROOF. We prove (1) by induction on the complexity of  $\phi(x_1, x_2, \dots, x_k)$ .

If  $\phi(x_1, x_2, \dots, x_k)$  is atomic:  $\langle x_{i_1}, x_{i_2}, \dots, x_{i_m} \rangle \in A$ , then  $*\phi(X_1, X_2, \dots, X_k)$  is  $\langle X_{i_1}, X_{i_2}, \dots, X_{i_m} \rangle \in *A$ .

Choose  $n < \omega$  such that  $C \subseteq (H_n^{S'})^m$ . Then, by (4) of Proposition 4.10,  $*C \subseteq (*H_n^{S'})^m$ .

Then,

$$*\{\langle a_1, a_2, \dots, a_k \rangle \in C \mid \phi(a_1, a_2, \dots, a_k)\} = *(C \cap \{\langle a_1, a_2, \dots, a_k \rangle \in (H_n^{S'})^k \mid \langle a_{i_1}, a_{i_2}, \dots, a_{i_m} \rangle \in A\}),$$

and

$$\begin{aligned} \{\langle a_1, a_2, \dots, a_k \rangle \in {}^*C \mid {}^*\phi(a_1, a_2, \dots, a_k)\} = \\ {}^*C \cap \{\langle a_1, a_2, \dots, a_k \rangle \in ({}^*H_n^{S'})^k \mid \langle a_{i_1}, a_{i_2}, \dots, a_{i_m} \rangle \in {}^*A\}. \end{aligned}$$

That these two are equal follows from (1) of Proposition 4.8 and Proposition 4.12. Thus, the statement is true if  $\phi$  is atomic.

Next, let  $\psi(x_1, x_2, \dots, x_k)$ ,  $\chi(x_1, x_2, \dots, x_k)$ , and  $\mu(x_1, x_2, \dots, x_k, x_{k+1})$  be formulas over  $S$ , and assume that

$$\begin{aligned} {}^*\{\langle a_1, a_2, \dots, a_k \rangle \in C \mid \psi(a_1, a_2, \dots, a_k)\} = \\ \{\langle a_1, a_2, \dots, a_k \rangle \in {}^*C \mid {}^*\psi(a_1, a_2, \dots, a_k)\}, \end{aligned}$$

and

$$\begin{aligned} {}^*\{\langle a_1, a_2, \dots, a_k \rangle \in C \mid \chi(a_1, a_2, \dots, a_k)\} = \\ \{\langle a_1, a_2, \dots, a_k \rangle \in {}^*C \mid {}^*\chi(a_1, a_2, \dots, a_k)\}. \end{aligned}$$

Then, if  $\phi(x_1, x_2, \dots, x_k)$  is  $\neg\psi(x_1, x_2, \dots, x_k)$ ,  ${}^*\phi(X_1, X_2, \dots, X_k)$  is  $\neg {}^*\psi(X_1, X_2, \dots, X_k)$ .

Thus,

$$\begin{aligned} & {}^*\{\langle a_1, a_2, \dots, a_k \rangle \in C \mid \phi(a_1, a_2, \dots, a_k)\} = \\ & {}^*\{\langle a_1, a_2, \dots, a_k \rangle \in C \mid \neg\psi(a_1, a_2, \dots, a_k)\} = \\ & {}^*(C \setminus \{\langle a_1, a_2, \dots, a_k \rangle \in C \mid \psi(a_1, a_2, \dots, a_k)\}) = \\ & {}^*C \setminus {}^*\{\langle a_1, a_2, \dots, a_k \rangle \in C \mid \psi(a_1, a_2, \dots, a_k)\} = \\ & {}^*C \setminus \{\langle a_1, a_2, \dots, a_k \rangle \in {}^*C \mid {}^*\psi(a_1, a_2, \dots, a_k)\} = \\ & \{\langle a_1, a_2, \dots, a_k \rangle \in {}^*C \mid \neg {}^*\psi(a_1, a_2, \dots, a_k)\} = \\ & \{\langle a_1, a_2, \dots, a_k \rangle \in {}^*C \mid {}^*\phi(a_1, a_2, \dots, a_k)\}, \end{aligned}$$

using the induction hypothesis, the definition, and Axiom 6.

Thus the statement is true if  $\phi(x_1, x_2, \dots, x_k)$  is  $\neg\phi(x_1, x_2, \dots, x_k)$ .

If  $\phi(x_1, x_2, \dots, x_k)$  is  $\psi(x_1, x_2, \dots, x_k) \vee \chi(x_1, x_2, \dots, x_k)$ , then  ${}^*\phi(X_1, X_2, \dots, X_k)$  is  ${}^*\psi(X_1, X_2, \dots, X_k) \vee {}^*\chi(X_1, X_2, \dots, X_k)$ .

Thus,

$$\begin{aligned} & {}^*\{\langle a_1, a_2, \dots, a_k \rangle \in C \mid \phi(a_1, a_2, \dots, a_k)\} = \\ & {}^*\{\langle a_1, a_2, \dots, a_k \rangle \in C \mid \psi(a_1, a_2, \dots, a_k) \vee \chi(a_1, a_2, \dots, a_k)\} = \\ & {}^*(\{\langle a_1, a_2, \dots, a_k \rangle \in C \mid \psi(a_1, a_2, \dots, a_k)\} \\ & \cup \{\langle a_1, a_2, \dots, a_k \rangle \in C \mid \chi(a_1, a_2, \dots, a_k)\}) = \\ & {}^*\{\langle a_1, a_2, \dots, a_k \rangle \in C \mid \psi(a_1, a_2, \dots, a_k)\} \\ & \cup {}^*\{\langle a_1, a_2, \dots, a_k \rangle \in C \mid \chi(a_1, a_2, \dots, a_k)\} = \\ & \{\langle a_1, a_2, \dots, a_k \rangle \in {}^*C \mid {}^*\psi(a_1, a_2, \dots, a_k)\} \\ & \cup \{\langle a_1, a_2, \dots, a_k \rangle \in {}^*C \mid {}^*\chi(a_1, a_2, \dots, a_k)\} = \\ & \{\langle a_1, a_2, \dots, a_k \rangle \in {}^*C \mid {}^*\psi(a_1, a_2, \dots, a_k) \vee {}^*\chi(a_1, a_2, \dots, a_k)\} \\ & \{\langle a_1, a_2, \dots, a_k \rangle \in {}^*C \mid {}^*\phi(a_1, a_2, \dots, a_k)\}, \end{aligned}$$

using the induction hypothesis, the definition, and Axiom 6.

Hence, the statement is true in this case too.

Next, assume that  $\phi(x_1, x_2, \dots, x_k)$  is  $(\exists x_{k+1} \in B)\mu(x_1, x_2, \dots, x_k, x_{k+1})$ . Then,  ${}^*\phi(X_1, X_2, \dots, X_k)$  is  $(\exists X_{k+1} \in {}^*B) {}^*\mu(X_1, X_2, \dots, X_k, X_{k+1})$ .

Assume (as induction hypothesis) that

$${}^*\{\langle a_1, a_2, \dots, a_k, a_{k+1} \rangle \in C \times B^1 \mid \mu(a_1, a_2, \dots, a_k, a_{k+1})\} = \{\langle a_1, a_2, \dots, a_k, a_{k+1} \rangle \in {}^*(C \times B^1) \mid {}^*\mu(a_1, a_2, \dots, a_k, a_{k+1})\}.$$

Now,

$${}^*\{\langle a_1, a_2, \dots, a_k \rangle \in C \mid \exists a_{k+1} \in B : \mu(a_1, a_2, \dots, a_k, a_{k+1})\} = {}^*(\pi_{\frac{k+1}{k+1}}^{\frac{k+1}{k+1}}[\{\langle a_1, a_2, \dots, a_k, a_{k+1} \rangle \in C \times B^1 \mid \mu(a_1, a_2, \dots, a_k, a_{k+1})\}]),$$

and,

$$\{\langle a_1, a_2, \dots, a_k \rangle \in {}^*C \mid \exists a_{k+1} \in {}^*B : \mu(a_1, a_2, \dots, a_k, a_{k+1})\} = \pi_{\frac{k+1}{k+1}}^{\frac{k+1}{k+1}}[\{\langle a_1, a_2, \dots, a_k, a_{k+1} \rangle \in {}^*C \times ({}^*B)^1 \mid {}^*\mu(a_1, a_2, \dots, a_k, a_{k+1})\}].$$

That these two are equal follows from axioms 8, 9, and 12, the definition, and the induction hypothesis. Thus, the statement is true also in this case. The case when  $\phi(x_1, x_2, \dots, x_k)$  is  $(\exists x_i \in B)\mu$  and  $i \neq k+1$ , can be reduced to the preceding case by a change of variables, as in (iv) in the definition.

Finally, if  $\phi(x_1, x_2, \dots, x_n)$  is  $\psi \wedge \chi$ ,  $\psi \rightarrow \chi$ ,  $\psi \leftrightarrow \chi$ , or  $(\forall x_i \in B)\mu$ , then we can use the appropriate expansion rule to reduce it to the cases above.

Thus, (1) holds.

To obtain (2), it is clear that  $\phi(b_1, b_2, \dots, b_k)$  holds if and only if

$$\{\langle a_1, a_2, \dots, a_k \rangle \in \{\langle b_1, b_2, \dots, b_k \rangle\} \mid \phi(a_1, a_2, \dots, a_k)\} = \{\langle b_1, b_2, \dots, b_k \rangle\}.$$

Similarly,  ${}^*\phi({}^*b_1, {}^*b_2, \dots, {}^*b_k)$  holds if and only if

$$\{\langle a_1, a_2, \dots, a_k \rangle \in \{\langle {}^*b_1, {}^*b_2, \dots, {}^*b_k \rangle\} \mid {}^*\phi(a_1, a_2, \dots, a_k)\} = \{\langle {}^*b_1, {}^*b_2, \dots, {}^*b_k \rangle\}.$$

(2) now follows from (1), and (2) of Proposition 4.10.

(3) is a special case of (2) ( $k = 0$ ). ■

4.17. REMARKS. In (1) of the transfer principle, it is necessary to restrict the formula to a relation  $C$ . The reason is that the set of all  $\langle a_1, a_2, \dots, a_k \rangle \in (H_\omega^{S'})^m$  satisfying the formula might not lie in  $H_\omega^{S'}$ . This is the case for the formula  $\neg \langle x_1 \rangle \in \emptyset^1$  over  $S$ , for example.

In the induction step in the proof of (1) of the transfer principle, for a formula of the form  $(\exists x_{k+1} \in B)\psi(x_1, x_2, \dots, x_k, x_{k+1})$ , we use Axiom 12. Now, if we did not have bounded quantifiers, but just  $\exists x_{k+1}$  instead of  $\exists x_{k+1} \in B$ , Axiom 12 would not be applicable. This is why we only have bounded quantifiers. It is difficult to see how we could change the theory so that we could prove a transfer principle where we have unbounded quantifiers.

The most common is that one uses the sets  $H_n^{S'}$  ( $n < \omega$ ) as bounds for quantifiers (in formulas over  $S$ ).

4.18. REMARKS. In the next proposition, we prove that functions behave as they should under the map  $*$ . These results are important if we want to generalize the atomic formulas in Definitions 4.14 to include function symbols and constants, and extend the transfer principle to formulas of this type too. We will not get into this here, and we will not use any such formulas in this text, so we leave to the interested reader to work out the details. It should be pointed out, however, that one in practice needs such formulas if one uses NSA in some field which contains a lot of functions, because otherwise, many formulas would be too long to comprehend. In some of the formulas later in this chapter and in Chapter 7, this "comprehension limit" is approached, but, hopefully, the reader will not think that this limit is exceeded.

Nevertheless, we give the proposition:

4.19. PROPOSITION. *Let  $A, B, C \in H_\omega^{S'}$  be  $m$ -ary,  $k$ -ary, and  $l$ -ary relations, respectively, for some  $m, k, l \geq 0$ , and let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be functions. Then:*

- (1)  $*f$  is a function:  $*f : *A \rightarrow *B$ , with graph  $G_{*f} = *G_f$ .
- (2)  $*(f(a_1, a_2, \dots, a_m)) = (*f)(*a_1, *a_2, \dots, *a_m)$ , for all  $\langle a_1, a_2, \dots, a_m \rangle \in A$ .
- (3)  $*(f[D]) = (*f)[*D]$  for all  $D \subseteq A$ . In particular,  $*R_f = R_{*f}$ .
- (4)  $f$  is injective (surjective, bijective) if and only if  $*f$  is.
- (5)  $*(g \circ f) = *g \circ *f$ .
- (6)  $*(\text{id}_A) = \text{id}_{*A}$
- (7)  $f$  has an inverse if and only if  $*f$  has, and if these inverses exist,  $*(f^{-1}) = (*f)^{-1}$ .
- (8) If  $D \subseteq A$  and  $f[D] \subseteq E \subseteq B$ , and if  $h$  is the restriction of  $f$  to  $D$  and  $E$ , then  $*h$  is the restriction of  $*f$  to  $*D$  and  $*E$ .

PROOF. Let us once for all choose  $n < \omega$  such that  $A \subseteq (H_n^{S'})^m$ ,  $B \subseteq (H_n^{S'})^k$ , and  $C \subseteq (H_n^{S'})^l$ .

Then,  $*A \subseteq (*H_n^{S'})^m$ ,  $*B \subseteq (*H_n^{S'})^k$ , and  $*C \subseteq (*H_n^{S'})^l$ , by (4) of Proposition 4.10.

Now, to obtain (1),  $G_f \subseteq A \times B$ . Hence, by (3) of Proposition 4.8 and Axiom 9:  $*G_f \subseteq *A \times *B$ . Hence  $*G_f \subseteq (*H_n^{S'})^{m+k}$ .

Now, since  $f : A \rightarrow B$  is a function, the following sentence over  $S$  is true, if  $k \geq 1$ :

$$\begin{aligned}
& (\forall x_1 \in H_n^{S'}) (\forall x_2 \in H_n^{S'}) \cdots (\forall x_m \in H_n^{S'}) (\langle x_1, x_2, \dots, x_m \rangle \in A \rightarrow \\
& \quad (\exists x_{m+1} \in H_n^{S'}) (\exists x_{m+2} \in H_n^{S'}) \cdots (\exists x_{m+k} \in H_n^{S'}) \\
& \quad (\forall x_{m+k+1} \in H_n^{S'}) (\forall x_{m+k+2} \in H_n^{S'}) \cdots (\forall x_{m+2k} \in H_n^{S'}) \\
& \quad (\langle x_1, x_2, \dots, x_m, x_{m+k+1}, x_{m+k+2}, \dots, x_{m+2k} \rangle \in G_f \\
& \Leftrightarrow (\langle x_{m+1}, x_{m+k+1} \rangle \in \text{diag } H_n^{S'} \wedge \langle x_{m+2}, x_{m+k+2} \rangle \in \text{diag } H_n^{S'} \wedge \cdots \\
& \quad \wedge \langle x_{m+k}, x_{m+2k} \rangle \in \text{diag } H_n^{S'})).
\end{aligned}$$

Then, by the (3) of the transfer principle, using (1) of Proposition 4.10, the following sentence over  $T$  is also true:

$$\begin{aligned}
& (\forall X_1 \in {}^*H_n^{S'}) (\forall X_2 \in {}^*H_n^{S'}) \cdots (\forall X_m \in {}^*H_n^{S'}) (\langle X_1, X_2, \dots, X_m \rangle \in A \rightarrow \\
& \quad (\exists X_{m+1} \in {}^*H_n^{S'}) (\exists X_{m+2} \in {}^*H_n^{S'}) \cdots (\exists X_{m+k} \in {}^*H_n^{S'}) \\
& \quad (\forall X_{m+k+1} \in {}^*H_n^{S'}) (\forall X_{m+k+2} \in {}^*H_n^{S'}) \cdots (\forall X_{m+2k} \in {}^*H_n^{S'}) \\
& \quad (\langle X_1, X_2, \dots, X_m, X_{m+k+1}, X_{m+k+2}, \dots, X_{m+2k} \rangle \in {}^*G_f \\
& \Leftrightarrow (\langle X_{m+1}, X_{m+k+1} \rangle \in \text{diag } {}^*H_n^{S'} \wedge \langle X_{m+2}, X_{m+k+2} \rangle \in \text{diag } {}^*H_n^{S'} \wedge \cdots \\
& \quad \wedge \langle X_{m+k}, X_{m+2k} \rangle \in \text{diag } {}^*H_n^{S'})).
\end{aligned}$$

Since  ${}^*f = {}^*\langle A, G_f, B \rangle = \langle {}^*A, {}^*G_f, {}^*B \rangle$ , by (2) of Proposition 4.10, all this means that  ${}^*f$  is a function,  ${}^*f : {}^*A \rightarrow {}^*B$ , with graph  $G_{{}^*f} = {}^*G_f$ . This is true also if  $k = 0$ , because then  $G_f = A$ ,  ${}^*G_f = {}^*A$ , and either  $B = \{\emptyset\} = {}^*B$  or  $B = \emptyset = {}^*B$ , in which case also  $A = \emptyset = {}^*A$  ((2) of Proposition 4.8 and (2) of Proposition 4.10 were used here.).

Thus, (1) holds.

It follows from (1), Axiom 7, and (2) of Proposition 4.10 that

$$\langle a_1, a_2, \dots, a_m, a_{m+1}, a_{m+2}, \dots, a_{m+k} \rangle \in G_f$$

implies

$$\langle {}^*a_1, {}^*a_2, \dots, {}^*a_m, {}^*a_{m+1}, {}^*a_{m+2}, {}^*a_{m+k} \rangle \in G_{{}^*f}.$$

This means that  $f(a_1, a_2, \dots, a_m) = \langle a_{m+1}, a_{m+2}, \dots, a_{m+k} \rangle$  implies that  $({}^*f)({}^*a_1, {}^*a_2, \dots, {}^*a_m) = \langle {}^*a_{m+1}, {}^*a_{m+2}, \dots, {}^*a_{m+k} \rangle$ . Thus, (2) holds.

To obtain (3), we first note that if  $C \subseteq A$ , then  ${}^*C \subseteq {}^*A$ , by (3) of Proposition 4.8.

Now,

$$\begin{aligned}
f[D] &= \{ \langle x_1, x_2, \dots, x_k \rangle \in B \mid \\
& \quad (\exists x_{k+1} \in H_n^{S'}) (\exists x_{k+2} \in H_n^{S'}) \cdots (\exists x_{k+m} \in H_n^{S'}) \\
& \quad (\langle x_{k+1}, x_{k+2}, \dots, x_{k+m} \rangle \in D \wedge \langle x_{k+1}, x_{k+2}, \dots, x_{k+m}, x_1, x_2, \dots, x_k \rangle \in G_f) \}.
\end{aligned}$$

Thus, by (1) of the transfer principle, and (1):

$$\begin{aligned}
{}^*(f[D]) &= \{ \langle X_1, X_2, \dots, X_k \rangle \in {}^*B \mid \\
& \quad (\exists X_{k+1} \in {}^*H_n^{S'}) (\exists X_{k+2} \in {}^*H_n^{S'}) \cdots (\exists X_{k+m} \in {}^*H_n^{S'}) \\
& \quad (\langle X_{k+1}, X_{k+2}, \dots, X_{k+m} \rangle \in {}^*D \wedge \langle X_{k+1}, X_{k+2}, \dots, X_{k+m}, X_1, X_2, \dots, X_k \rangle \in G_{{}^*f}) \} \\
&= ({}^*f)[{}^*D].
\end{aligned}$$

Hence, (3) holds.

To prove (4):  $f$  is injective if and only if the following sentence over  $S$  is true, if  $m \geq 1$ :

$$\begin{aligned}
& (\forall x_1 \in H_n^{S'}) (\forall x_2 \in H_n^{S'}) \cdots (\forall x_{2m+k} \in H_n^{S'}) \\
& \quad (\langle x_1, x_2, \dots, x_m, x_{2m+1}, x_{2m+1}, \dots, x_{2m+k} \rangle \in G_f \wedge \\
& \quad \langle x_{m+1}, x_{m+2}, \dots, x_{2m}, x_{2m+1}, \dots, x_{2m+k} \rangle \in G_f) \rightarrow \\
& \quad (\langle x_1, x_{m+1} \rangle \in \text{diag } H_n^{S'} \wedge \langle x_2, x_{m+2} \rangle \in \text{diag } H_n^{S'} \wedge \cdots \wedge \langle x_m, x_{2m} \rangle \in \text{diag } H_n^{S'})).
\end{aligned}$$

Using that  $*(\text{diag } H_n^{S'}) = \text{diag } *H_n^{S'}$ , by (1) of Proposition 4.10, and using (1), we see that  $*f$  is injective if and only if the  $*$ -transform of this sentence is true.

Thus, it follows from (3) of the transfer principle that  $f$  is injective if and only if  $*f$  is.

This holds also if  $m = 0$ , because then, by (1), both  $D_f = A$  and  $D_{*f} = *A$  are 0-ary relations, and then both  $f$  and  $*f$  are injective.

Since, by (1), surjectivity of  $f$  and  $*f$  means that  $R_f = B$  and  $R_{*f} = *B$ , respectively, it follows from (3) that  $f$  is surjective if and only if  $*f$  is.

That  $f$  is bijective if and only if  $*f$  is, now follows from the previous cases.

(4) is proved.

For (5), it follows from (1) that  $*(g \circ f)$  and  $*g \circ *f$  are functions with domain  $*A$  and codomain  $*C$ , and that  $*G_{g \circ f} = G_{*(g \circ f)}$ . We must prove that  $G_{*(g \circ f)} = G_{*g \circ *f}$ . To do this, let  $\phi(x_1, x_2, \dots, x_{m+l})$  be the following formula over  $S$ :

$$\begin{aligned} & (\exists x_{m+l+1} \in H_n^{S'}) (\exists x_{m+l+2} \in H_n^{S'}) \cdots (\exists x_{m+l+k} \in H_n^{S'}) \\ & \quad \langle x_1, x_2, \dots, x_m, x_{m+l+1}, x_{m+l+2}, \dots, x_{m+l+k} \rangle \in G_f \wedge \\ & \quad \langle x_{m+l+1}, x_{m+l+2}, \dots, x_{m+l+k}, x_{m+1}, x_{m+2}, \dots, x_{m+l} \rangle \in G_g. \end{aligned}$$

It is clear that

$$G_{g \circ f} = \{ \langle a_1, a_2, \dots, a_m, c_1, c_2, \dots, c_l \rangle \in A \times C \mid \phi(a_1, a_2, \dots, a_m, c_1, c_2, \dots, c_l) \},$$

and, by (1),

$$G_{*g \circ *f} = \{ \langle a_1, a_2, \dots, a_m, c_1, c_2, \dots, c_l \rangle \in *A \times *C \mid * \phi(a_1, a_2, \dots, a_m, c_1, c_2, \dots, c_l) \}.$$

Now, by the above, it follows from (1) of the transfer principle and Axiom 9, and (1), that  $G_{*(g \circ f)} = *G_{g \circ f} = G_{*g \circ *f}$ .

Hence, (5) holds.

To prove (6): It is clear from (1) that  $*\text{id}_A$  is a function with domain and codomain  $*A$ . We must prove  $*G_{\text{id}_A} = G_{\text{id}_{*A}}$ .

If  $m = 0$ , then  $A = \emptyset$  or  $A = \{ \langle \rangle \} = \{ \emptyset \}$ . By Remarks 2.12,  $G_{\text{id}_A} = \emptyset$  or  $G_{\text{id}_A} = \{ \emptyset \}$ , respectively. Thus, all we have to prove in these cases is that  $*\emptyset$  and  $*\{ \emptyset \} = \{ \emptyset \}$ . But these are true, by (2) of Proposition 4.8 and (2) of Proposition 4.10.

So, assume that  $m \geq 1$ :

Then,

$$G_{\text{id}_A} = (A \times A) \bigcap \bigcap_{i=1}^m \{ \langle a_1, a_2, \dots, a_{2m} \rangle \in (H_n^{S'})^{2m} \mid a_i = a_{m+i} \},$$

and likewise

$$G_{\text{id}_{*A}} = (*A \times *A) \bigcap \bigcap_{i=1}^m \{ \langle a_1, a_2, \dots, a_{2m} \rangle \in (*H_n^{S'})^{2m} \mid a_i = a_{m+i} \}.$$

Now, it follows from axioms 9 and 11, and (1) of Proposition 4.8, that  $*G_{\text{id}_A} = G_{\text{id}_{*A}}$ .

Thus, (6) holds.

To prove (7): It follows from (4) that  $f$  has an inverse if and only if  $*f$  has. If so, it follows from (1) that  $*(f^{-1})$  is a function with the same domain and codomain as  $(*f)^{-1}$ , that is  $*B$  and  $*A$ , respectively.

We must prove that  $G_{*(f^{-1})} = G_{(*f)^{-1}}$ .

It follows from (1) and Proposition 4.12 that

$$\begin{aligned} G_{*(f^{-1})} &= * \{ \langle b_1, b_2, \dots, b_k, a_1, a_2, \dots, a_m \rangle \in (H_n^{S'})^{k+m} \mid \\ &\quad \langle a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_k \rangle \in G_f \} = \\ &= \{ \langle b_1, b_2, \dots, b_k, a_1, a_2, \dots, a_m \rangle \in (*H_n^{S'})^{k+m} \mid \\ &\quad \langle a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_k \rangle \in G_{*f} \} = G_{(*f)^{-1}}. \end{aligned}$$

(7) now follows.

To obtain (8): It follows from (1) that  $*h$  is a function with domain  $*D$  and codomain  $*E$ . That  $*D \subseteq *A$  and  $*E \subseteq *B$  follows from (3) of Proposition 4.8. Since  $G_h \subseteq G_f$ , (3) of Proposition 4.8 and (1) also give  $G_{*h} \subseteq G_{*f}$ . But this implies that  $G_{*h}$  is the restriction of  $G_{*f}$  to  $*D$  and  $*E$ , that is, (8) holds. ■

We now introduce four central concepts in NSA.

4.20. DEFINITIONS. Let  $b \in H_\omega^{T'}$ .

$b$  is called *standard* if  $b = *a$ , for some  $a \in H_\omega^{S'}$ .  $b$  is called *nonstandard* if it is not standard.

$b$  is called *internal* if  $b \in *A$  for some  $A \in H_\omega^{S'} \setminus S$ .  $b$  is called *external* if it is not internal.

4.21. PROPOSITION. Let  $A \in H_\omega^S \setminus S$ . Then:

- (1)  $\{ *a \mid a \in A \} \subseteq *A$ , with equality if  $A$  is finite.  
The inclusion is proper if  $\mathbb{N} \lesssim A$  (which holds e.g. if  $A$  is infinite and can be well ordered, see Chapter 8).
- (2) All elements in  $*A \setminus \{ *a \mid a \in A \}$  (if there are any) are nonstandard.
- (3)  $\{ *a \mid a \in \mathbb{N} \} = \mathbb{N} \subset * \mathbb{N}$  (proper inclusion), and all elements in  $* \mathbb{N} \setminus \mathbb{N}$  are nonstandard.

PROOF. The first part of (1), and (2), follow easily from (4) of Proposition 4.8 and (2) of Proposition 4.10. We wait with the second part of (1), and prove (3) first: Since  $\mathbb{N} \subseteq S$ , the first part of (3) follows from (1) of Proposition 4.7, the first part of (1), and Axiom 2. The second part then follows from (2).

To prove the second part of (1): That equality holds in (1) if  $A$  is finite is a part of (2) of Proposition 4.10.

It remains to prove that  $*A \setminus \{ *a \mid a \in A \}$  is nonempty if  $\mathbb{N} \lesssim A$ .

That  $\mathbb{N} \lesssim A$  means that there is an injective function  $f : \mathbb{N}^1 \rightarrow A^1$ . Now, it follows from (1), (2) and (4) of Proposition 4.19, Axiom 8, and (1) of Proposition 4.7, that  $*f : (*\mathbb{N})^1 \rightarrow (*A)^1$  is an injective function, and that  $(*f)(n) = *(f(n))$  for all  $n \in \mathbb{N}$ .

By (3), we can pick  $p \in * \mathbb{N} \setminus \mathbb{N}$ . Assume, to get a contradiction, that  $(*f)(p) = \langle *a \rangle$ , for some  $a \in A$ . Since  $*f$  is injective,  $(*f)(n) \neq \langle *a \rangle$  for all  $n \in \mathbb{N}$ .

Thus, for any fixed  $n \in \mathbb{N}$ ,  $\langle *a, n \rangle$  does not satisfy the following formula over  $T$ :  $\langle X_2, X_1 \rangle \in G_{*f}$ .

Thus, by (2) of the transfer principle, backwards, (1) of Proposition 4.7, and (1) of Proposition 4.19,  $\langle a, n \rangle$  does not satisfy this formula over  $S$ :  $\langle x_2, x_1 \rangle \in G_f$ . This holds for all  $n \in \mathbb{N}$ , so  $\langle a \rangle$  satisfies the following formula over  $S$ :  $\neg(\exists x_2 \in \mathbb{N}) \langle x_2, x_1 \rangle \in G_f$ . Thus, by (2) of the transfer principle, forwards, (2) of Proposition 4.10, and (1) of Proposition 4.19, that  $\langle *a \rangle$  satisfies the formula  $\neg(\exists X_2 \in {}^*\mathbb{N}) \langle X_2, X_1 \rangle \in G_{*f}$ . But this contradicts  $(*f)(p) = \langle *a \rangle$ .

Therefore, the assumption that  $(*f)(p) = \langle *a \rangle$  for some  $a \in A$  is false, which means that  $(*f)(p) \in ({}^*A)^1 \setminus \{\langle *a \rangle \mid a \in A\}$ . Hence,  $*A \setminus \{\langle *a \mid a \in A\} \neq \emptyset$ , which completes the proof. ■

The usage of the transfer principle in the above proof, going back and forward, is typical for many arguments in NSA. For example, in nonstandard real analysis, one gives alternative definitions of limits and derivatives, without using  $\epsilon$  and  $\delta$ . To prove that these definitions are equivalent to the standard ones, one uses arguments of this kind. (See e.g. Cutland, [1], p. 59 f.)

4.22. DEFINITIONS. The elements in  ${}^*\mathbb{N}$  are called *hypernatural numbers*. We say *nonstandard natural number* instead of nonstandard hypernatural number. Sometimes, we also say *standard natural number* instead of (ordinary) natural number.

We will return to hypernatural numbers towards the end of this chapter. Mainly because we need them to define *hyperfinite sets* (Definition 4.38).

4.23. REMARKS. The internal elements in  $H_\omega^{T'}$  are the only elements whose values the variables in the  $*$ -transform of a formula over  $S$  can adopt. Thus, one can say that the external elements is "unreachable" by  $*$ -transforms, or that transformed formulas only "talk about" internal elements.

It is therefore important to be able to decide whether or not a given element is internal. Therefore, we prove some important results about internal elements. The key result is the internal definition principle (Theorem 4.27).

By (3) of Proposition 4.24 below, all external elements are "sets" and not "urelements". Thus, there is no point of talking about external *elements*, but only to talk about external *sets*.

4.24. PROPOSITION.

- (1) All standard elements  $b \in H_\omega^{T'}$  are internal.
- (2) If  $A \in H_\omega^{T'} \setminus T$  is internal, then every element  $b \in A$  is internal.
- (3) All elements in  $T'$  are internal.

PROOF. To prove (1), if  $b \in H_\omega^{T'}$  is standard, then  $b = *a$  for some  $a \in H_\omega^{S'}$ . But  $a \in H_n^{S'}$  for some  $n$ . Hence  $b = *a \in {}^*H_n^{S'}$ , by Axiom 7.

Thus,  $b$  is internal.

(2) follows from Axiom 4.

Since  $S' \notin S$ , (3) follows from Proposition 4.11 and the definitions. ■

4.25. THEOREM. Let  $n < \omega$ .

- (1) If  $n \geq 1$  and  $x \in {}^*H_n^{S'} \setminus T$ , then  $x \subseteq {}^*H_{n-1}^{S'}$ .
- (2)  ${}^*H_n^{S'}$  is the set of all internal elements in  $H_n^{T'}$ .
- (3) For all  $m \geq 0$ : If  $A \subseteq (H_n^{T'})^m$  is internal, then  $A \subseteq ({}^*H_n^{S'})^m$ .

- (4) If  $A \in H_\omega^{S'} \setminus S$ , then  ${}^*\mathcal{P}(A)$  is the set of all internal subsets of  ${}^*A$ .  
Thus,  ${}^*\mathcal{P}(A) \subseteq \mathcal{P}({}^*A)$ .

PROOF. If  $n \geq 1$ , we note that for  $k \geq n$ , the following sentence over  $S$  is true:

$$(*) \quad (\forall x_1 \in H_n^{S'}) (\forall x_2 \in H_k^{S'}) (\langle x_2, x_1 \rangle \in \text{memb } H_k^{S'} \rightarrow \langle x_2 \rangle \in (H_{n-1}^{S'})^1).$$

Then, by (3) of the transfer principle, (1) of Proposition 4.10, and Axiom 8, the following sentence over  $T$  is true:

$$(**) \quad (\forall X_1 \in {}^*H_n^{S'}) (\forall X_2 \in {}^*H_k^{S'}) (\langle X_2, X_1 \rangle \in \text{memb } {}^*H_k^{S'} \rightarrow \langle X_2 \rangle \in ({}^*H_{n-1}^{S'})^1).$$

Now, let  $x \in {}^*H_n^{S'} \setminus T$ , and pick  $y \in x$ . By Axiom 4,  $y \in {}^*A$ , for some  $A \in H_\omega^{S'} \setminus S$ . But  $A \subseteq H_m^{S'}$ , for some  $m$ . Put  $k = \max(m, n)$ . Then,  $A \subseteq H_k^{S'}$ . Hence, by (3) of Proposition 4.8,  ${}^*A \subseteq {}^*H_k^{S'}$ , and  $y \in {}^*H_k^{S'}$ . Also,  ${}^*H_n^{S'} \subseteq {}^*H_k^{S'}$ , by (3) of Proposition 4.8. Applying (\*\*) with this  $k$ , and with  $x$  as  $X_1$  and  $y$  as  $X_2$ , we get  $y \in {}^*H_{n-1}^{S'}$ . This holds for all  $y \in x$ , so  $x \subseteq {}^*H_{n-1}^{S'}$ . Thus, (1) holds.

We prove (2) by induction on  $n$ .

For  $n = 0$ , the statement reduces to the triviality  $\emptyset = \emptyset$ , by (2) of Proposition 4.8.

Assume now that the statement is true for some  $n \geq 0$ .

First, let  $x \in {}^*H_{n+1}^{S'}$ . By definition,  $x$  is internal, and by (3) of Proposition 4.9,  $x \in H_{n+1}^{T'}$ .

Conversely, assume that  $x \in H_{n+1}^{T'}$  is internal. If  $x \in T'$ , then  $x \in {}^*H_{n+1}^{S'}$ , by Proposition 4.11 and (3) of Proposition 4.8. Otherwise,  $x \neq \emptyset$ , and  $x \subseteq H_n^{T'}$ . Then, for every  $y \in x$ ,  $y \in H_n^{T'}$  and  $y$  is internal, by (2) of Proposition 4.24. Thus, by the induction hypothesis,  $y \in {}^*H_n^{S'}$ , for all  $y \in x$ .

Next, since  $x$  is internal,  $x \in {}^*C$ , for some  $C \in H_\omega^{S'} \setminus S$ . Then,  $C \subseteq H_m^{S'}$ , for some  $m$  which could be chosen so that  $m \geq n + 1$ . Then, it follows from (3) of Proposition 4.8 that  $x \in {}^*H_m^{S'}$ , and hence that  $y \in {}^*H_{m-1}^{S'} \subseteq {}^*H_n^{S'}$  for all  $y \in x$ , by (1), and (3) of Proposition 4.8.

Now, the following sentence over  $S$  is true:

$$(\forall x_1 \in H_m^{S'}) \\ ((\forall x_2 \in H_{m-1}^{S'}) (\langle x_2, x_1 \rangle \in \text{memb } H_m^{S'} \rightarrow \langle x_2 \rangle \in (H_n^{S'})^1) \rightarrow \langle x_1 \rangle \in (H_{n+1}^{S'})^1).$$

Then, by (3) of the transfer principle, its  ${}^*$ -transform is also true, which is, by (1) of Proposition 4.10, and Axiom 8:

$$(\forall X_1 \in {}^*H_m^{S'}) \\ ((\forall X_2 \in {}^*H_{m-1}^{S'}) (\langle X_2, X_1 \rangle \in \text{memb } {}^*H_m^{S'} \rightarrow \langle X_2 \rangle \in ({}^*H_n^{S'})^1) \rightarrow \langle X_1 \rangle \in ({}^*H_{n+1}^{S'})^1).$$

Hence,  $x \in {}^*H_{n+1}^{S'}$  follows in this case too.

So, for all internal  $x \in H_{n+1}^{T'}$ ,  $x \in {}^*H_{n+1}^{S'}$ .

Thus,  ${}^*H_{n+1}^{S'}$  is the set of all internal elements in  $H_{n+1}^{T'}$ .

(2) now follows by induction.

(3) is clear if  $m = 0$ . If  $m > 0$  and  $A \subseteq (H_n^{T'})^m$  is internal, then it follows from definitions 2.2, 2.7 and 2.6, and several applications of (2) of Proposition 4.24, that all elements in  $\mathcal{C}(A)$  are internal. Thus, by (2),  $A \subseteq (*H_n^{S'})^m$ .

Hence, (3) holds.

To prove (4), let  $A \in H_\omega^{S'} \setminus S$  and put  $n = r^{S'}(A)$ . Then,  $n \geq 1$  and  $r^{S'}(\mathcal{P}(A)) = n + 1$ . By (2) of Proposition 4.9,  $r^{T'}(*A) = n$  and  $r^{T'}(*\mathcal{P}(A)) = n + 1$ . Now,  $\mathcal{P}(A) \notin S$  and  $\mathcal{P}(A) \cap S = \emptyset$ . Hence, by (4), (1) and (2) of Proposition 4.8, and Proposition 4.11,  $*\mathcal{P}(A) \notin T$  and  $*\mathcal{P}(A) \cap T = \emptyset$ , and also,  $*A \notin T$ . Then, by definition and (2) of Proposition 4.24, all elements in  $*A$ , in  $*\mathcal{P}(A)$ , and in elements in  $*\mathcal{P}(A)$  (if there are any), are internal elements in  $H_\omega^{T'}$ , and their  $T'$ -ranks are at most  $n - 1$ ,  $n$ , and  $n - 1$ , respectively. Thus, by (2), all elements in  $*A$  lie in  $*H_{n-1}^{S'}$ , all elements in  $*\mathcal{P}(A)$  lie in  $*H_n^{S'}$ , and all elements in elements in  $*\mathcal{P}(A)$  lie in  $*H_{n-1}^{S'}$ . Also, by (2), all internal subsets of  $*A$  lie in  $*H_n^{S'} \setminus T$ , since they lie in  $H_n^{T'} \setminus T$ . Also,  $*H_{n-1}^{S'} \subseteq *H_n^{S'}$ , by (3) of Proposition 4.8.

Now, the following sentence over  $S$  is true:

$$(\forall x_1 \in H_n^{S'}) (\langle x_1 \rangle \in (\mathcal{P}(A))^1 \leftrightarrow (\neg \langle x_1 \rangle \in S^1 \wedge (\forall x_2 \in H_{n-1}^{S'}) (\langle x_2, x_1 \rangle \in \text{memb } H_n^{S'} \rightarrow \langle x_2 \rangle \in A^1))).$$

Thus, by (3) of the transfer principle, its  $*$ -transform is true, that is (using Proposition 4.11, (1) of Proposition 4.10 and Axiom 8):

$$(\forall X_1 \in *H_n^{S'}) (\langle X_1 \rangle \in (*\mathcal{P}(A))^1 \leftrightarrow (\neg \langle X_1 \rangle \in T^1 \wedge (\forall X_2 \in *H_{n-1}^{S'}) (\langle X_2, X_1 \rangle \in \text{memb } *H_n^{S'} \rightarrow \langle X_2 \rangle \in (*A)^1))).$$

By the above considerations, this means that  $*\mathcal{P}(A)$  is the set of all internal subsets of  $*A$ . Thus, (4) holds. ■

The following theorem is crucial for the proof of the internal definition principle, and it is also useful for obtaining many other results.

4.26. THEOREM. *Let  $C \in H_\omega^{S'}$  be an  $m + 1$ -ary relation, for some  $m \geq 0$ .*

*Then*

$$*\{\langle a_1, a_2, \dots, a_m, a_{m+1} \rangle \in C \mid \langle a_1, a_2, \dots, a_m \rangle \in a_{m+1}\} = \{\langle a_1, a_2, \dots, a_m, a_{m+1} \rangle \in *C \mid \langle a_1, a_2, \dots, a_m \rangle \in a_{m+1}\}.$$

PROOF. There is an  $n$  such that  $C \subseteq (H_n^{S'})^{m+1}$ . This  $n$  can be chosen so that  $n \geq 5$ . Then, by (4) of Proposition 4.10,  $*C \subseteq (*H_n^{S'})^{m+1} \subseteq (H_n^{T'})^{m+1}$ .

For each  $i$  ( $1 \leq i \leq m$ ), let  $\psi_i$  be the following formula over  $S$ , which intuitively asserts that<sup>16</sup>  $x_{m+3} = \{\{i\}, \{i, x_i\}\}$ , given that  $x_i \in H_n^{S'}$  and  $x_{m+3} \in H_{n-2}^{S'} \setminus S$ :

$$\begin{aligned} & (\forall x_{m+4} \in H_{n-3}^{S'}) (\langle x_{m+4}, x_{m+3} \rangle \in \text{memb } H_{n-2}^{S'} \leftrightarrow (\neg \langle x_{m+4} \rangle \in S^1 \wedge \\ & \quad ((\forall x_{m+5} \in H_{n-4}^{S'}) (\langle x_{m+5}, x_{m+4} \rangle \in \text{memb } H_{n-3}^{S'} \leftrightarrow x_{m+5} \in \{i\}) \\ & \quad \vee (\forall x_{m+5} \in H_{n-4}^{S'}) (\langle x_{m+5}, x_{m+4} \rangle \in \text{memb } H_{n-3}^{S'} \\ & \quad \leftrightarrow (x_{m+5} \in \{i\} \vee \langle x_{m+5}, x_i \rangle \in \text{diag } H_n^{S'}))))). \end{aligned}$$

<sup>16</sup>We say *intuitively asserts*, because we cannot say that  $\langle a_1, a_2 \rangle$  satisfies the formula if  $a_2 = \{\{i\}, \{i, a_1\}\}$ , because the free variables are not  $x_1$  and  $x_2$ . If we change variables so that these become the free variables, then  $\langle a_1, a_2 \rangle$  will satisfy the formula, if the  $S'$ -ranks are the right ones.

Then, by Axiom 8, (1) and (2) of Proposition 4.10, (1) of Proposition 4.7, and Proposition 4.11,  ${}^*\psi_i$  is:

$$\begin{aligned} & (\forall X_{m+4} \in {}^*H_{n-3}^{S'}) (\langle X_{m+4}, X_{m+3} \rangle \in \text{memb } {}^*H_{n-2}^{S'} \leftrightarrow (\neg \langle X_{m+4} \rangle \in T^1 \wedge \\ & ((\forall X_{m+5} \in {}^*H_{n-4}^{S'}) (\langle X_{m+5}, X_{m+4} \rangle \in \text{memb } {}^*H_{n-3}^{S'} \leftrightarrow X_{m+5} \in \{i\}) \\ & \vee (\forall X_{m+5} \in {}^*H_{n-4}^{S'}) (\langle X_{m+5}, X_{m+4} \rangle \in \text{memb } {}^*H_{n-3}^{S'} \\ & \leftrightarrow (X_{m+5} \in \{i\} \vee \langle X_{m+5}, X_i \rangle \in \text{diag } {}^*H_n^{S'}))))). \end{aligned}$$

Using (2) of Proposition 4.24 and (2) of Theorem 4.25, we see that  ${}^*\psi$  intuitively asserts that  $X_{m+3} = \{\{i\}, \{i, X_i\}\}$ , given that  $X_i \in {}^*H_n^{S'}$  and  $X_{m+3} \in {}^*H_{n-2}^{S'} \setminus T$ .

Next, let  $\phi(x_1, x_2, \dots, x_m, x_{m+1})$  be the following formula over  $S$ :

$$\begin{aligned} & (\exists x_{m+2} \in H_{n-1}^{S'} \setminus S) (\langle x_{m+2}, x_{m+1} \rangle \in \text{memb } H_n^{S'} \wedge \\ & (\forall x_{m+3} \in H_{n-2}^{S'}) (\langle x_{m+3}, x_{m+2} \rangle \in \text{memb } H_{n-1}^{S'} \leftrightarrow \\ & (\neg \langle x_{m+3} \rangle \in S^1 \wedge (x_{m+3} \in \emptyset \vee \psi_1 \vee \psi_2 \vee \dots \vee \psi_m))))). \end{aligned}$$

By axioms 6 and 8, Proposition 4.11, (1) of Proposition 4.10, and (2) of Proposition 4.8,  ${}^*\phi$  is

$$\begin{aligned} & (\exists X_{m+2} \in {}^*H_{n-1}^{S'} \setminus T) (\langle X_{m+2}, X_{m+1} \rangle \in \text{memb } {}^*H_n^{S'} \wedge \\ & (\forall X_{m+3} \in {}^*H_{n-2}^{S'}) (\langle X_{m+3}, X_{m+2} \rangle \in \text{memb } {}^*H_{n-1}^{S'} \leftrightarrow \\ & (\neg \langle X_{m+3} \rangle \in T^1 \wedge (X_{m+3} \in \emptyset \vee {}^*\psi_1 \vee {}^*\psi_2 \vee \dots \vee {}^*\psi_m))))). \end{aligned}$$

Now, for  $\langle a_1, a_2, \dots, a_m, a_{m+1} \rangle \in C \subseteq (H_n^{S'})^{m+1}$ ,  $\langle a_1, a_2, \dots, a_m \rangle \in a_{m+1}$  implies  $\langle a_1, a_2, \dots, a_m \rangle \in H_{n-1}^{S'}$  (this holds also if  $m = 0$ , since  $n \geq 5$  and  $\langle \rangle = \emptyset \in H_1^{S'}$ ), and  $a_i \in H_{n-4}^{S'}$ , for all  $i$  ( $1 \leq i \leq m$ ).

It follows that, for  $\langle a_1, a_2, \dots, a_m, a_{m+1} \rangle \in C$ ,  $\phi(a_1, a_2, \dots, a_m, a_{m+1})$  holds if and only if  $\langle a_1, a_2, \dots, a_m \rangle \in a_{m+1}$ . (We put in the clause  $x_{m+3} \in \emptyset$  in  $\phi$  in order to get this to work also in the case  $m = 0$ .)

Repeating this argument with  $T$  instead of  $S$ , and using (2) of Proposition 4.24 and (2) of Theorem 4.25 several times each, we obtain that for  $\langle a_1, a_2, \dots, a_m, a_{m+1} \rangle \in {}^*C$ ,  ${}^*\phi(a_1, a_2, \dots, a_m, a_{m+1})$  holds if and only if  $\langle a_1, a_2, \dots, a_m \rangle \in a_{m+1}$ .

The desired conclusion now follows from (1) of the transfer principle. ■

Next, we state and prove the internal definition principle. We have to divide it into two parts, because we strictly uphold the difference between a set  $A$  and the set  $A^1$  of 1-tuples from  $A$ .

4.27. THEOREM. (Internal Definition Principle). *Let  $\phi(x_1, x_2, \dots, x_{k+m})$  be a formula over  $S$ , and let  $b_1, b_2, \dots, b_m \in H_\omega^{T'}$  be internal ( $k, m \geq 0$ ). Then:*

(1) *If  $C \in H_\omega^{S'}$  is a  $k$ -ary relation, then the set*

$$\{\langle a_1, a_2, \dots, a_k \rangle \in {}^*C \mid {}^*\phi(a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_m)\}$$

*is internal.*

(2) *If  $k = 1$  and  $A \in H_\omega^{S'} \setminus S$ , then the set*

$$\{a \in {}^*A \mid {}^*\phi(a, b_1, b_2, \dots, b_m)\}$$

*is internal.*

PROOF. We give a proof of (1). The proof of (2) can then be obtained from the proof of (1) by going through it, line by line, and make the following changes:

Put  $k = 1$ .

Write  $a$  instead of  $a_1$  and instead of  $\langle a_1, a_2, \dots, a_k \rangle$ .

Write  $A$  instead of  $C$ , except possibly in formulas (see the next paragraph).

In the formulas, write  $A^1$  instead of  $C$ , and  $(^*A)^1$  instead  $^*C$ , except when  $C$  occurs in  $C^1$ , and  $^*C$  in  $(^*C)^1$ , in which cases the previous paragraph should apply.

Write  $H_n^{S'}$  instead of  $(H_n^{S'})^k$ , and remove the second sentence of the proof.

Write  $n$  instead of  $n + 3$ , and  $n + 1$  instead of  $n + 4$ .

Remove the sentences defining  $D$  and characterizing  $^*D$ , respectively. Elsewhere, write  $\text{memb } H_{n+1}^{S'}$  instead of  $D$  and  $\text{memb } ^*H_{n+1}^{S'}$  instead of  $^*D$ , and use Axiom 10 to obtain  $^*(\text{memb } H_{n+1}^{S'}) = \text{memb } ^*H_{n+1}^{S'}$ .

This transforms the proof of (1) into a proof of (2), although this proof of (2) can be simplified.

The proof of (1) goes as follows:

Choose  $n < \omega$  such that  $C \subseteq (H_n^{S'})^k$ . By (4) of Proposition 4.10,  $^*C \subseteq (^*H_n^S)^k \subseteq (H_n^{T'})^k$ .

Now,  $C \subseteq H_{n+3}^{S'}$  (this is also true for  $k = 0$ ), and, by (3) of Proposition 4.8 and (3) of Proposition 4.9,  $^*C \subseteq ^*H_{n+3}^{S'} \subseteq H_{n+3}^{T'}$

Furthermore, for every subset  $M \subseteq C$ ,  $M \in H_{n+4}^{S'}$ , and for every internal subset  $U \subseteq ^*C$ , we obtain likewise  $U \in ^*H_{n+4}^{S'}$ , using the above and (2) of Theorem 4.25.

Next, put

$$D = \{ \langle a_1, a_2, \dots, a_k, a_{k+1} \rangle \in (H_{n+4}^{S'})^{k+1} \mid \langle a_1, a_2, \dots, a_k \rangle \in a_{k+1} \}.$$

By Theorem 4.26 and (3) of Proposition 4.10:

$$^*D = \{ \langle a_1, a_2, \dots, a_k, a_{k+1} \rangle \in (^*H_{n+4}^{S'})^{k+1} \mid \langle a_1, a_2, \dots, a_k \rangle \in a_{k+1} \}.$$

Now, for each  $i$  ( $1 \leq i \leq m$ ), there exists a set  $B_i \in H_\omega^{S'} \setminus S$  such that  $b_i \in ^*B_i$ , since  $b_i$  is internal. Since  $b_i \in ^*B_i$ , (2) of Proposition 4.8 gives  $B_i \neq \emptyset$ .

Choose  $b'_i \in B_i$  for each  $i$  ( $1 \leq i \leq m$ ), and put

$$M = \{ \langle a_1, a_2, \dots, a_k \rangle \in C \mid \phi(a_1, a_2, \dots, a_k, b'_1, b'_2, \dots, b'_m) \}.$$

Since  $M \subseteq C$ ,  $M \in H_{n+4}^{S'}$ .

Since the  $b'_i \in B_i$  ( $1 \leq i \leq m$ ) were arbitrary, it follows from the definition of  $D$  that the following sentence over  $S$  is true, since it simply asserts the existence of a set such as  $M$  for each choice of  $b'_i \in B_i$  ( $1 \leq i \leq m$ ):

$$\begin{aligned} & (\forall x_{k+1} \in B_1)(\forall x_{k+2} \in B_2) \cdots (\forall x_{k+m} \in B_m)(\exists x_{k+m+1} \in H_{n+4}^{S'} \setminus S) \\ & ((\forall x_{k+m+2} \in H_{n+3}^{S'}) (\langle x_{k+m+2}, x_{k+m+1} \rangle \in \text{memb } H_{n+4}^{S'} \rightarrow \langle x_{k+m+2} \rangle \in C^1) \wedge \\ & (\forall x_1 \in H_n^{S'}) (\forall x_2 \in H_n^{S'}) \cdots (\forall x_k \in H_n^{S'}) (\langle x_1, x_2, \dots, x_k, x_{k+m+1} \rangle \in D \leftrightarrow \\ & (\langle x_1, x_2, \dots, x_k \rangle \in C \wedge \phi(x_1, x_2, \dots, x_k, x_{k+1}, \dots, x_{k+m}))). \end{aligned}$$

Hence, by (3) of the transfer principle, its  $*$ -transform is also true. The  $*$ -transform is, by axioms 6, 8, and 10, and Proposition 4.11:

$$\begin{aligned} & (\forall X_{k+1} \in {}^*B_1)(\forall X_{k+2} \in {}^*B_2) \cdots (\forall X_{k+m} \in {}^*B_m)(\exists X_{k+m+1} \in {}^*H_{n+4}^{S'} \setminus T) \\ & ((\forall X_{k+m+2} \in {}^*H_{n+3}^{S'}) (\langle X_{k+m+2}, X_{k+m+1} \rangle \in \text{memb } {}^*H_{n+4}^{S'} \rightarrow \langle X_{k+m+2} \rangle \in ({}^*C)^1) \wedge \\ & (\forall X_1 \in {}^*H_n^{S'}) (\forall X_2 \in {}^*H_n^{S'}) \cdots (\forall X_k \in {}^*H_n^{S'}) (\langle X_1, X_2, \dots, X_k, X_{k+m+1} \rangle \in {}^*D \leftrightarrow \\ & (\langle X_1, X_2, \dots, X_k \rangle \in {}^*C \wedge {}^*\phi(X_1, X_2, \dots, X_k, X_{k+1}, \dots, X_{k+m}))). \end{aligned}$$

It follows that there exists a set  $U \in {}^*H_{n+4}^{S'} \setminus T$ , which hence is internal, such that the following holds:

$$\begin{aligned} & (\forall X_{k+m+2} \in {}^*H_{n+3}^{S'}) (\langle X_{k+m+2}, U \rangle \in \text{memb } {}^*H_{n+4}^{S'} \rightarrow \langle X_{k+m+2} \rangle \in ({}^*C)^1) \wedge \\ & (\forall X_1 \in {}^*H_n^{S'}) (\forall X_2 \in {}^*H_n^{S'}) \cdots (\forall X_k \in {}^*H_n^{S'}) (\langle X_1, X_2, \dots, X_k, U \rangle \in {}^*D \leftrightarrow \\ & (\langle X_1, X_2, \dots, X_k \rangle \in {}^*C \wedge {}^*\phi(X_1, X_2, \dots, X_k, b_1, b_2, \dots, b_m))). \end{aligned}$$

From this, the above considerations, and several applications of (2) of Proposition 4.24 and (2) of Theorem 4.25, it follows that

$$U = \{ \langle a_1, a_2, \dots, a_k \rangle \in {}^*C \mid {}^*\phi(a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_m) \},$$

which is therefore internal.

Thus (1) holds, and we obtain a proof of (2) by doing as described above. ■

#### 4.28. COROLLARIES.

- (1) *If  $A, B \in H_\omega^{T'} \setminus T$  are internal, then  $A \cup B$ ,  $A \setminus B$ , and  $A \cap B$  are internal.*
- (2) *Let  $a_1, a_2, \dots, a_m \in H_\omega^{T'}$  ( $m \geq 0$ ). Then, the following conditions are equivalent:*
  - (a) *All  $a_i$  ( $1 \leq i \leq m$ ) are internal.*
  - (b)  *$\{a_1, a_2, \dots, a_m\}$  is internal.*
  - (c)  *$\langle a_1, a_2, \dots, a_m \rangle$  is internal.*

- (3) *All kinds of omitting, repeating, and permuting variables preserve internality.*

*More precisely, let  $A \in H_\omega^{T'}$  be an internal  $m$ -ary relation, for some  $m \geq 0$ , and let  $i_1, i_2, \dots, i_k$  ( $k \geq 0$ ) be, not necessarily distinct, integers among  $1, 2, \dots, m$ . Let  $j_1, j_2, \dots, j_l$  be those integers, now distinct, among  $1, 2, \dots, m$  which do not occur among  $i_1, i_2, \dots, i_k$ .*

*Finally, let  $B$  be the set of all  $\langle a_{i_1}, a_{i_2}, \dots, a_{i_k} \rangle$  for which there exist  $a_{j_1}, a_{j_2}, \dots, a_{j_l}$  such that  $\langle a_1, a_2, \dots, a_m \rangle \in A$ .*

*Then,  $B$  is an internal  $k$ -ary relation in  $H_\omega^{T'}$ .*

- (4) *If  $A \in H_\omega^{T'}$  is an internal  $m$ -ary relation ( $m \geq 0$ ), then  $\pi_i^m[A]$  ( $1 \leq i \leq m$ ) is an internal set in  $H_\omega^{T'} \setminus T$ .*
- (5) *If  $A \in H_\omega^{T'}$  is an  $m$ -ary relation and if  $B \in H_\omega^{T'}$  is a  $k$ -ary relation ( $m, k \geq 0$ ), then  $A$  and  $B$  are internal if and only if  $A \times B$  is internal.*
- (6) *For each  $m \geq 1$  and each  $A \in H_\omega^{T'} \setminus T$ :  $A$  is internal if and only if  $A^m$  is internal.*

- (7) Let  $A, B, C \in H_\omega^{T'}$   $m$ -ary,  $k$ -ary, and  $l$ -ary relations, respectively ( $m, k, l \geq 0$ ). Also, let  $f : A \rightarrow B$  and  $g : B \rightarrow C$  be functions. Then:
- (a)  $f$  is internal if and only if both the codomain  $B$  and the graph  $G_f$  are internal.
- If  $f$  is internal, then  $A = D_f$  and  $R_f$  are internal too.
- (b) If  $f$  is surjective and  $G_f$  is internal, then  $f$  is internal.
- (c) If  $f$  and  $g$  are internal, then  $g \circ f$  is internal.
- (d)  $A$  is internal if and only if  $\text{id}_A$  is internal.
- (e) If  $f$  has an inverse  $f^{-1}$ , then  $f$  is internal if and only if  $f^{-1}$  is internal.

PROOF. To prove (1): Let  $A, B \in H_\omega^{T'} \setminus T$  be internal. Choose  $n$  such that  $A, B \in H_n^{T'}$ . Then,  $n \geq 1$ . By (2) of Theorem 4.25,  $A, B \in {}^*H_n^{S'}$ .

Next, we apply (2) of the internal definition principle and (1) of Proposition 4.10, to obtain that the following sets are internal:

$$\begin{aligned} & \{a \in {}^*H_{n-1}^{S'} \mid \langle a, A \rangle \in \text{memb } {}^*H_n^{S'} \vee \langle a, B \rangle \in \text{memb } {}^*H_n^{S'}\}. \\ & \{a \in {}^*H_{n-1}^{S'} \mid \langle a, A \rangle \in \text{memb } {}^*H_n^{S'} \wedge \neg \langle a, B \rangle \in \text{memb } {}^*H_n^{S'}\}. \\ & \{a \in {}^*H_{n-1}^{S'} \mid \langle a, A \rangle \in \text{memb } {}^*H_n^{S'} \wedge \langle a, B \rangle \in \text{memb } {}^*H_n^{S'}\}. \end{aligned}$$

But, by (2) of Proposition 4.24, and (1) and (2) of Theorem 4.25, these sets are  $A \cup B$ ,  $A \setminus B$ , and  $A \cap B$ , respectively. These sets are therefore internal.

Thus, (1) holds.

To prove (2), we first note that the implication  $(b) \Rightarrow (a)$  follows immediately from (2) of Proposition 4.24.

Next, we prove that if  $a \in H_\omega^{T'}$  is internal, then so is  $\{a\}$ .

If  $a \in H_\omega^{T'}$  is internal, then  $a \in {}^*H_n^{S'}$  for some  $n \geq 1$ , by (2) of Theorem 4.25.

Then, we apply (2) of the internal definition principle and (1) of Proposition 4.10 to obtain that the set  $\{b \in {}^*H_n^{S'} \mid \langle a, b \rangle \in \text{diag } {}^*H_n^{S'}\}$  is internal. But this set is  $\{a\}$ . Thus,  $\{a\}$  is internal.

Now, we prove the implication  $(a) \Rightarrow (b)$  by induction on  $m$ . If  $m = 0$ , then this is trivially true, since  $\emptyset$  is internal, by (3) of Proposition 4.24. Assume that the implication holds for  $m \geq 0$ , and assume that  $a_1, a_2, \dots, a_m, a_{m+1} \in H_\omega^{S'}$  are internal. By the induction hypothesis,  $\{a_1, a_2, \dots, a_m\}$  is internal, and by the case just proved,  $\{a_{m+1}\}$  is internal. Thus, by (1),  $\{a_1, a_2, \dots, a_m, a_{m+1}\} = \{a_1, a_2, \dots, a_m\} \cup \{a_{m+1}\}$  is internal.

By induction  $(a) \Rightarrow (b)$ .

Hence,  $(a) \Leftrightarrow (b)$ .

Next, since all  $i \in \mathbb{N}$  are internal, by (3) of Proposition 4.24, the equivalence  $(a) \Leftrightarrow (c)$  now follows by 3 successive applications, in each direction, of the equivalence  $(a) \Leftrightarrow (b)$ , using Definitions 2.2.

It follows that (a), (b), and (c) are equivalent, that is, (2) holds.

To obtain (3), assume that  $A$  is an internal  $m$ -ary relation, and that  $i_1, i_2$  etc. are as in the statement. Then,  $A \subseteq ({}^*H_n^{S'})^m$  and  $A \in {}^*H_{n+4}^{S'}$  for some  $n \geq 0$ , by (3) and (2) of Theorem 4.25.

Put

$$D = \{\langle a_1, a_2, \dots, a_k, a_{k+1} \rangle \in (H_{n+4}^{S'})^{k+1} \mid \langle a_1, a_2, \dots, a_k \rangle \in a_{k+1}\}.$$

Now, we apply (1) of the internal definition principle, and (1) and (3) of Proposition 4.10, to obtain that the set

$$\begin{aligned} & \{ \langle c_1, c_2, \dots, c_k \rangle \in (*H_n^{S'})^k \mid (\exists X_1 \in *H_n^{S'}) (\exists X_2 \in *H_n^{S'}) \dots (\exists X_m \in *H_n^{S'}) \\ & \langle c_1, X_{i_1} \rangle \in \text{diag } *H_n^{S'} \wedge \langle c_2, X_{i_2} \rangle \in \text{diag } *H_n^{S'} \wedge \dots \wedge \langle c_k, X_{i_k} \rangle \in \text{diag } *H_n^{S'} \\ & \wedge \langle X_1, X_2, \dots, X_m, A \rangle \in *D \} \end{aligned}$$

is internal. Using (3) of Proposition 4.8 and Theorem 4.26, we see that this set is the desired set, which is therefore an internal set in  $H_\omega^{T'}$ .

Thus, (3) holds.

(4) follows in the same way as (3), with  $k = 1$ , using  $\text{memb } H_{n+1}^{S'}$  instead of  $D$ ,  $c \in *H_n^{S'}$  instead of  $\langle c_1, c_2, \dots, c_k \rangle \in (*H_n^{S'})^k$ , and  $\langle c_1, X_i \rangle \in \text{diag } *H_n^{S'}$  as the only term (except the last one) in the conjunction above. We also refer to Axiom 10 instead of Theorem 4.26, and (2) of the internal definition principle instead of (1).

The converse part of (5) is just two special cases of (3). To obtain the direct part, we use (3) and (2) of Theorem 4.25 to find an  $n$  such that  $A \subseteq (*H_n^{S'})^m$  and  $B \subseteq (*H_n^{S'})^k$ , and  $A, B \in *H_{n+4}^{S'} \setminus T$ .

We define  $D_1$  and  $D_2$  as

$$D_1 = \{ \langle a_1, a_2, \dots, a_k, a_{m+1} \rangle \in (H_{n+4}^{S'})^{m+1} \mid \langle a_1, a_2, \dots, a_m \rangle \in a_{m+1} \},$$

$$D_2 = \{ \langle a_1, a_2, \dots, a_k, a_{k+1} \rangle \in (H_{n+4}^{S'})^{k+1} \mid \langle a_1, a_2, \dots, a_k \rangle \in a_{k+1} \}.$$

Now, we apply (1) of the internal definition principle and (3) of Proposition 4.10 to obtain that the set

$$\begin{aligned} & \{ \langle a_1, a_2, \dots, a_{m+k} \rangle \in (*H_n^{S'})^{m+k} \mid \\ & \langle a_1, a_2, \dots, a_m, A \rangle \in *D_1 \wedge \langle a_{m+1}, a_{m+2}, \dots, a_{m+k}, B \rangle \in *D_2 \} \end{aligned}$$

is internal.

By (3) of Proposition 4.8, (3) of Proposition 4.10, and Theorem 4.26, this set is  $A \times B$ , which is therefore internal.

Thus, (5) holds.

To obtain (6), we see that the converse part is a special case of (4).

It suffices to prove the direct part for  $m = 1$ , because then, the general case will follow easily by induction using (5).

So, let  $A \in H_\omega^{T'} \setminus T$  be internal. (2) of Theorem 4.25, (3) of Proposition 4.8, and (2) of Proposition 4.24 then give an  $n$  such that  $A \subseteq *H_n^{S'} \subseteq *H_{n+1}^{S'}$  and  $A \in *H_{n+1}^{S'}$ .

Now, we apply (1) and (3) of Proposition 4.10, and (1) of the internal definition principle, to obtain that the set

$$\{ \langle a \rangle \in (*H_n^{S'})^1 \mid \langle a, A \rangle \in \text{memb } *H_{n+1}^{S'} \}$$

is internal. But by the above, this set is  $A^1$ .

Thus, (6) holds.

To obtain (7): It is clear from (2) that  $f = \langle A, G_f, B \rangle$  is internal if and only if  $A$ ,  $G_f$ , and  $B$  all are internal.

But if  $G_f$  is internal, then, by taking appropriate projections in (3),  $A = D_f$  and  $R_f$  are internal.

(a) follows from these considerations, and so does (b), since  $B = R_f$  if  $f$  is surjective.

To obtain (c): If  $f$  and  $g$  are internal, then  $A$ ,  $B$ ,  $C$ ,  $G_f$ , and  $G_g$  are all internal, by (a). Therefore, it suffices to show that  $G_{g \circ f}$  is internal, by (a).

By (5),  $G_f \times C$  and  $A \times G_g$  are internal. Hence, by (1),  $(G_f \times C) \cap (A \times G_g)$  is internal.

Now,  $G_{g \circ f}$  is the set of all  $\langle a_1, a_2, \dots, a_m, c_1, c_2, \dots, c_l \rangle$  for which there exist  $b_1, b_2, \dots, b_k$  such that  $\langle a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_k, c_1, c_2, \dots, c_l \rangle \in (G_f \times C) \cap (A \times G_g)$ . By (3),  $G_{g \circ f}$  is internal. Hence, (c) holds.

To obtain (d): The converse part follows directly from (a).

To get the direct part, assume that  $A$  is internal. Since  $A$  is the codomain of  $\text{id}_A$ , it suffices to prove that  $G_{\text{id}_A}$  is internal, by (a).

Now,  $G_{\text{id}_A}$  is the set of all  $\langle a_1, a_2, \dots, a_m, a_1, a_2, \dots, a_m \rangle$  such that  $\langle a_1, a_2, \dots, a_m \rangle \in A$ . Thus, by (3),  $G_{\text{id}_A}$  is internal, and (d) holds.

To obtain (e): If  $f$  has an inverse  $f^{-1}$ , then it follows from (3) that  $G_f$  is internal if and only if  $G_{f^{-1}}$  is, since all elements in  $G_{f^{-1}}$  can be obtained from elements in  $G_f$  by permuting variables, and vice versa.

Since  $f$  and  $f^{-1}$  are surjective, (e) follows from (a) and (b).

Thus, (7) holds. ■

We conclude this chapter by giving a few results on hypernatural numbers. In particular, we define *hyperfinite set* (Definition 4.38), which we will need in the definition of *enlargement* in Chapter 7 (Definition 7.17).

We also get our first two examples of *external sets*:  $\mathbb{N}$  and  ${}^*\mathbb{N} \setminus \mathbb{N}$  (see Proposition 4.36).

4.29. REMARKS. Consider the ordinary strict total ordering relation  $<$  on  $\mathbb{N}$ . As usual, we write  $a < b$  instead of  $\langle a, b \rangle \in <$  for all  $a$  and  $b$ , and likewise for  ${}^* <$ .

That  $<$  is a strict total order on  $\mathbb{N}$  means that the following sentence over  $S$  is true:

$$(\forall x_1 \in \mathbb{N})(\forall x_2 \in \mathbb{N})(\forall x_3 \in \mathbb{N})(\neg x_1 < x_1 \wedge ((x_1 < x_2 \wedge x_2 < x_3) \rightarrow x_1 < x_3) \wedge (x_1 < x_2 \vee x_2 < x_1 \vee \langle x_1, x_2 \rangle \in \text{diag } \mathbb{N})).$$

By (1) and (3) of Proposition 4.10 and (3) of Proposition 4.8,  ${}^* <$  is a binary relation on  ${}^*\mathbb{N}$ .

If we apply (3) of the transfer principle, and (1) of Proposition 4.10, to the sentence above, we then see that  ${}^* <$  is a strict total order on  ${}^*\mathbb{N}$ . Also, if  $j, k \in \mathbb{N}$ , then  $j < k$  if and only if  $j {}^* < k$ , by (1) of Proposition 4.7, (2) of Proposition 4.10 and (4) of Proposition 4.8.

By (3) of Proposition 4.21, this means that  $<$  is the restriction of  ${}^* <$  to  $\mathbb{N}$ .

Also, we, as usual, define the (nonstrict) total order relation  $\leq$  on  $\mathbb{N}$  by putting  $k \leq l$  if and only if  $k < l$  or  $k = l$ , for all  $k, l \in \mathbb{N}$ .

By the same reason as for  ${}^* <$ ,  ${}^* \leq$  is a binary relation on  ${}^*\mathbb{N}$ .

By applying (3) of the transfer principle, and (1) of Proposition 4.10 to the true sentence (over  $S$ )

$$(\forall x_1 \in \mathbb{N})(\forall x_2 \in \mathbb{N})(x_1 \leq x_2 \leftrightarrow (x_1 < x_2 \vee \langle x_1, x_2 \rangle \in \text{diag } \mathbb{N})),$$

we see that  $^*\leq$  has the same relation to  $^*$  as  $\leq$  has to  $\mathbb{N}$ . Therefore,  $^*\leq$  is a nonstrict total order on  $^*\mathbb{N}$ , and  $\leq$  is the restriction of  $^*\leq$  to  $\mathbb{N}$ .

As usual, we may also write  $a > b$  instead of  $b < a$  and  $a \geq b$  instead of  $a > b$  or  $a = b$ , for  $a, b \in \mathbb{N}$ . These relations, and  $^*>$  and  $^*\geq$ , have similar properties.

4.30. DEFINITION. A subset of  $A \subseteq ^*\mathbb{N}$  is called *bounded* if there is an  $m \in ^*\mathbb{N}$  such that  $k \leq m$  for all  $k \in A$ .

The following proposition explains why we may call the nonstandard natural numbers "infinite". It says that  $\mathbb{N}$  is an *initial segment* of  $^*\mathbb{N}$ , with respect to  $^*<$ .

4.31. PROPOSITION. *If  $k \in \mathbb{N}$  and  $m \in ^*\mathbb{N} \setminus \mathbb{N}$ , then  $k \leq m$ .*

PROOF. Let  $\phi(x_1)$  and  $\psi(x_1, x_2)$  be the following formulas over  $S$ , respectively:

$$\neg(\exists x_2 \in \mathbb{N}) x_2 < x_1, \quad \text{and}$$

$$\neg(\exists x_3 \in \mathbb{N})(x_1 < x_3 \wedge x_3 < x_2).$$

Then  $\phi(0)$  and  $\psi(j, j+1)$  hold, for all  $j \in \mathbb{N}$ . By (2) of the transfer principle and (1) of Proposition 4.7,  $^*\phi(0)$  and  $^*\psi(j, j+1)$  hold, for all  $j \in \mathbb{N}$ . This means that no hypernatural number is smaller than 0, nor lies between two consecutive standard natural numbers (remembering that  $<$  is the restriction of the total ordering relation  $^*<$  on  $^*\mathbb{N}$  to  $\mathbb{N}$ ).

Then, if  $k \in \mathbb{N}$  and  $m \in ^*\mathbb{N} \setminus \mathbb{N}$ , we must then have  $m \leq k$  or  $k \leq m$ , since  $^*<$  is a total ordering relation on  $^*\mathbb{N}$ . If  $m \leq k$ , then there must be a smallest  $l \in \mathbb{N}$  such that  $m \leq l$ . If  $l = 0$ , this contradicts that  $^*\phi(0)$  holds. Hence,  $l = j + 1$  for some  $j \in \mathbb{N}$  such that  $j \leq m$ . But this contradicts that  $^*\psi(j, j+1)$  holds.

This contradiction gives that  $k \leq m$  must hold. ■

4.32. PROPOSITION. *For every  $m \in ^*\mathbb{N}$ , the set  $\{j \in ^*\mathbb{N} \mid 1 \leq j \wedge j \leq m\}$  is internal.*

PROOF. By definition and (3) of Proposition 4.21, 1 and  $m$  are internal. The conclusion then follows by (2) of the internal definition principle. ■

4.33. REMARKS. If  $m \in \mathbb{N}$ , then the set in Proposition 4.32 is  $\mathbb{N}_m = ^*\mathbb{N}_m$  (see Definition 2.4), by (1) of Proposition 4.21, Proposition 4.7, and Proposition 4.31, since  $\leq$  is the restriction of  $^*\leq$  to  $\mathbb{N}$ .

Thus, the following definition will agree with the usual definition, for  $m \in \mathbb{N}$ :

4.34. DEFINITION. For each  $m \in ^*\mathbb{N}$ ,  $\mathbb{N}_m$  denotes the (internal) set in Proposition 4.32.

## 4.35. PROPOSITION.

- (1) *Every nonempty internal set of hypernatural numbers has a smallest element, with respect to  $^*\leq$ .*
- (2) *Every nonempty bounded internal set of hypernatural numbers has a greatest element, with respect to  $^*\leq$ .*

PROOF. (1) follows by applying (3) of the transfer principle to the true sentence (over  $S$ )

$$(\forall x_1 \in \mathcal{P}(\mathbb{N}) \setminus \{\emptyset\})(\exists x_2 \in \mathbb{N})(\forall x_3 \in \mathbb{N}) \\ ((\langle x_3, x_1 \rangle \in \text{memb } H_2^{S'} \wedge x_3 \leq x_2) \leftrightarrow \langle x_3, x_2 \rangle \in \text{diag } \mathbb{N}),$$

using (2) and (4) of Theorem 4.25, Axiom 6, (2) and (3) of Proposition 4.8, and (1) and (2) of Proposition 4.10.

(2) follows by applying the same argument as in (1) to the true sentence (over  $S$ )

$$(\forall x_1 \in \mathcal{P}(\mathbb{N}) \setminus \{\emptyset\})(\exists x_2 \in \mathbb{N})(\forall x_3 \in \mathbb{N})(\langle x_3, x_1 \rangle \in \text{memb } H_2^{S'} \rightarrow x_3 \leq x_2) \rightarrow \\ (\exists x_2 \in \mathbb{N})(\forall x_3 \in \mathbb{N})(\langle x_3, x_1 \rangle \in \text{memb } H_2^{S'} \wedge x_2 \leq x_3 \leftrightarrow \langle x_3, x_2 \rangle \in \text{diag } \mathbb{N}).$$

■

## 4.36. PROPOSITION (Spillover Principle for Hypernatural Numbers).

- (1) *Every internal set of hypernatural numbers which contains arbitrarily large standard natural numbers, contains some nonstandard natural number.*  
*Thus,  $\mathbb{N}$  is external.*
- (2) *There exists no smallest nonstandard natural number, and every internal set of hypernatural numbers which contains arbitrarily small nonstandard natural numbers, contains some standard natural number.*  
*Thus,  $^*\mathbb{N} \setminus \mathbb{N}$  is external.*

PROOF. (1) follows directly from (2) of Proposition 4.35.

If  $m$  was the smallest nonstandard natural number, then  $\mathbb{N} = (\mathbb{N}_m \cup \{0\}) \setminus \{m\}$ , by Proposition 4.31. But then,  $\mathbb{N}$  would be internal, by Proposition 4.33 and (1) and (2) of Corollaries 4.28. This contradicts (1), so no such  $m$  exists.

The remainder of (2) now follows directly from this, and (1) of Proposition 4.35.

■

4.37. REMARKS. There is also a spillover principle for hyperreal numbers. (See e.g. [1], p. 33)

(2) of Proposition 4.36 may also be called the "spillunder" principle.

4.38. DEFINITION. A set  $A \in H_\omega^{T'} \setminus T$  is said to be *hyperfinite* if there exists an  $m \in ^*\mathbb{N}$  and an internal bijection  $f : \mathbb{N}_m^1 \rightarrow A^1$ .

4.39. PROPOSITION. *Every hyperfinite set  $A \in H_\omega^{T'} \setminus T$  is internal.*

PROOF. This follows immediately from Definition 4.38, and (7) (a) and (6) of Corollaries 4.28. ■

4.40. PROPOSITION. *If  $A$  is a finite set of internal elements in  $H_\omega^{T'}$ , then  $A$  is hyperfinite.*

PROOF. Assume that  $A = \{a_1, a_2, \dots, a_m\}$  is finite, where each  $a_i$  is an internal element in  $H_\omega^{T'}$  ( $1 \leq i \leq m$ ). Since  $\mathbb{N} \subseteq S$ , (1) of Proposition 4.7, (1) of Proposition 4.24, and (2) of Corollaries 4.28 give that each  $\langle i, a_i \rangle$  is internal. Again, by (2) of Corollaries 4.28,  $G = \{\langle 1, a_1 \rangle, \langle 2, a_2 \rangle, \dots, \langle m, a_m \rangle\}$  is internal. But  $G$  is the graph of a bijection  $f : \mathbb{N}_m^1 \rightarrow A^1$ . By 7 (b) of Corollaries 4.28,  $f$  is internal. Hence,  $A$  is hyperfinite. ■

The final two results in this chapter are used in the proof of Theorem 7.19. They make it possible to use hyperfinite sets without invoking bijections from sets of hypernatural numbers.

4.41. THEOREM. *Let  $M \in H_\omega^{S'} \setminus S$ , and let  $A$  be the set of all elements in  $M \setminus S$  which are finite sets.*

*Then  $*A$  is the set of all elements in  $*M \setminus T$  which are hyperfinite sets.*

PROOF. For each  $n \geq 1$  ( $n < \omega$ ), let  $F_n = \{\langle a, b, c \rangle \in (H_{n+4}^{S'})^3 \mid \langle a, b \rangle \in c\}$ . By Theorem 4.26 and (3) of Proposition 4.10,  $*F_n = \{\langle a, b, c \rangle \in (*H_{n+4}^{S'})^3 \mid \langle a, b \rangle \in c\}$ .

Next, let  $\psi_n(x_1, x_2)$  be the following formula over  $S$ :

$$\begin{aligned} & (\exists x_4 \in H_{n+4}^{S'} \setminus S) ((\forall x_5 \in H_{n+3}^{S'}) (\langle x_5, x_4 \rangle \in \text{memb } H_{n+4}^{S'} \rightarrow \langle x_5 \rangle \in ((H_n^{S'})^2)^1) \wedge \\ & (\forall x_6 \in H_n^{S'}) (\forall x_7 \in H_n^{S'}) (\langle x_6, x_7, x_4 \rangle \in F_n \rightarrow (\langle x_6, x_2 \rangle \in \text{memb } H_{n+1}^{S'} \wedge \\ & \langle x_7, x_1 \rangle \in \text{memb } H_{n+1}^{S'}) \wedge (\forall x_8 \in H_n^{S'}) (\langle x_8, x_2 \rangle \in \text{memb } H_{n+1}^{S'} \rightarrow \\ & (\exists x_9 \in H_n^{S'}) (\forall x_{10} \in H_n^{S'}) (\langle x_8, x_{10}, x_4 \rangle \in F_n \leftrightarrow \langle x_9, x_{10} \rangle \in \text{diag } H_n^{S'})) \wedge \\ & (\forall x_{11} \in H_n^{S'}) (\langle x_{11}, x_1 \rangle \in \text{memb } H_{n+1}^{S'} \rightarrow \\ & (\exists x_{12} \in H_n^{S'}) (\forall x_{13} \in H_n^{S'}) (\langle x_{13}, x_{11}, x_4 \rangle \in F_n \leftrightarrow \langle x_{12}, x_{13} \rangle \in \text{diag } H_n^{S'}))). \end{aligned}$$

For  $B, C \subseteq H_n^{S'}$ ,  $\psi_n(B, C)$  holds if and only if there exists a bijection  $f : C^1 \rightarrow B^1$ . (Intuitively,  $x_4$  is the graph of this bijection, and it is sufficient to establish that such a graph exists.)

Then, it follows from, Axiom 6, Proposition 4.11, (2) of Proposition 4.24, (2) and (3) of Theorem 4.25, (1) and (3) of Proposition 4.10, and (7) (a) and (7) (b) of Corollaries 4.28, that if  $B, C \subseteq *H_n^{S'}$  are internal, then  $*\psi_n(B, C)$  holds if and only if there exists an *internal* bijection  $f : C^1 \rightarrow B^1$ .

Next, let  $\phi_n(x_1)$  be the following formula over  $S$ :

$$\begin{aligned} & (\exists x_2 \in \mathcal{P}(\mathbb{N})) ((\exists x_3 \in \mathbb{N}) (\forall x_5 \in \mathbb{N}) (\forall x_6 \in \{1\}) (\langle x_5, x_2 \rangle \in \text{memb } H_{n+1}^{S'} \leftrightarrow \\ & (x_6 \leq x_5 \wedge x_5 \leq x_3)) \wedge \psi_n(x_1, x_2)). \end{aligned}$$

Then, for  $B \subseteq H_n^{S'}$ ,  $\phi_n(B)$  holds if and only if  $B$  is finite. Also, it follows from the above, Definition 4.38, (2) and (4) of Theorem 4.25, (1) and (2) of Proposition 4.10, (1) of Proposition 4.7, (3) of Proposition 4.8, and (7) (a) and (7) (b) of Corollaries 4.28, that for an internal set  $B \subseteq *H_n^{S'} \subseteq H_n^{T'}$ ,  $*\phi(B)$  holds if and only if  $B$  is hyperfinite.

Now, choose  $n \geq 1$  such that  $M \subseteq H_{n+1}^{S'}$ , and let  $A$  be the set of all elements in  $M \setminus S$  which are finite sets. Then,  $*A \subseteq *M \subseteq *H_{n+1}^{S'}$ , by (3) of Proposition 4.8.

Now, the following sentence over  $S$  is true:

$$(\forall x_1 \in H_{n+1}^{S'}) (x_1 \in A \leftrightarrow (x_1 \in M \setminus S \wedge \phi_n(x_1))).$$

Its  $*$ -transform is then also true, by (3) of the transfer principle. This  $*$ -transform is, by Axiom 6 and Proposition 4.11:

$$(\forall X_1 \in {}^*H_{n+1}^{S'}) (X_1 \in {}^*A \leftrightarrow (X_1 \in {}^*M \setminus T \wedge {}^*\phi_n(X_1))).$$

Now, if  $B \in {}^*A$ , then  $B \in {}^*H_{n+1}^{S'} \setminus T$ , by the above, Axiom 6, and Proposition 4.11. It follows that  $B$  is an internal subset of  ${}^*H_n^{S'}$ , by (1) of Theorem 4.25. This holds for all  $B \in {}^*A$ .

If, instead,  $B \in {}^*M \setminus T \subseteq {}^*H_{n+1}^{S'}$ , then, by (2) of Theorem 4.25,  $B$  is an internal subset of  ${}^*H_n^{S'}$ .

It follows from the above considerations that  ${}^*A$  is the set of all elements in  ${}^*M \setminus T$  which are hyperfinite sets.

The desired conclusion now follows from (3) of the transfer principle. ■

4.42. COROLLARY. *Let  $A \in H_\omega^{S'} \setminus S$ .*

*Then,  $A$  is a subset of  $H_\omega^{S'} \setminus S$  whose elements are finite sets if and only if  ${}^*A$  is a subset of  $H_\omega^{T'} \setminus T$  whose elements are hyperfinite sets.*

PROOF. (4), (1), and (2) of Proposition 4.8, and Proposition 4.11, give  ${}^*A \notin T$ , and  $A \cap S = \emptyset$  if and only if  ${}^*A \cap T = \emptyset$ .

Let  $B$  be the set of all elements in  $A \setminus S$  which are finite sets. Then, by Theorem 4.41 and Axiom 6,  ${}^*B$  is the set of all elements in  ${}^*A \setminus T$  which are hyperfinite sets. Now,  $B = A$  if and only if  ${}^*B = {}^*A$ , by (1) of Proposition 4.9.

The desired conclusion now follows. ■

## CHAPTER 5

### Filters and Ultrafilters

In this short chapter, we give the most important properties of *filters* and *ultrafilters*. These will be used in Chapter 6, where we use an ultrafilter to construct a proper nonstandard extension of a superstructure, called an *ultrapower extension*.

5.1. DEFINITIONS. Let  $X$  be a nonempty set. A *filter* on  $X$  is a subset  $\mathcal{U} \subseteq \mathcal{P}(X)$  such that:

- (1)  $X \in \mathcal{U}$ .
- (2)  $\emptyset \notin \mathcal{U}$ .
- (3) If  $A \in \mathcal{U}$  and if  $A \subseteq B \subseteq X$ , then  $B \in \mathcal{U}$ .
- (4) If  $A, B \in \mathcal{U}$ , then  $A \cap B \in \mathcal{U}$ .

A filter  $\mathcal{U}$  on  $X$  is called an *ultrafilter* if there is no filter  $\mathcal{V}$  on  $X$  such that  $\mathcal{U} \subset \mathcal{V}$  (proper subset).

5.2. REMARKS. If  $X$  is a nonempty set, and if  $Y \subseteq X$  is nonempty, then it follows easily from Definitions 5.1 that  $\mathcal{U} = \{Z \subseteq X \mid Y \subseteq Z\}$  is a filter on  $X$ .

Also, this  $\mathcal{U}$  is an ultrafilter if and only if  $Y$  is a singleton set. For a proper singleton subset of  $Y$  gives rise to a filter  $\mathcal{V}$  of the same type such that  $\mathcal{U} \subset \mathcal{V}$ , and if  $Y$  is a singleton set  $\{x\}$ , then  $\{x\} \cap Z = \emptyset$  for all  $Z \in \mathcal{P}(X) \setminus \mathcal{U}$ , so (4) and (2) of the Definitions 5.1 implies that  $\mathcal{U}$  is an ultrafilter.

5.3. DEFINITIONS. A filter of the type in Remarks 5.2 is called a *principal filter* on  $X$ .

A filter on  $X$  which is not principal is called a *nonprincipal filter* on  $X$ .

5.4. PROPOSITION. *Let  $\mathcal{U}$  be a filter on  $X \neq \emptyset$ . Then:*

- (1) *If there is a finite set  $A$  such that  $A \in \mathcal{U}$ , then  $\mathcal{U}$  is a principal filter.*
- (2) *If  $X$  is finite, then  $\mathcal{U}$  is a principal filter.*

PROOF. Assume that there is a finite set  $A \in \mathcal{U}$ . Choose such a finite set  $A$  with a minimal number of elements. By (3) of Definitions 5.1, if  $A \subseteq B \subseteq X$ , then  $B \in \mathcal{U}$ . Also, for any  $B \in \mathcal{U}$ ,  $A \cap B \in \mathcal{U}$ , by (4) of Definitions 5.1. Then, by our minimality assumption,  $A \cap B = A$ . Thus  $A \subseteq B$ . Hence, for  $B \subseteq X$ ,  $B \in \mathcal{U}$  if and only if  $A \subseteq B$ . This means that  $\mathcal{U}$  is a principal filter. Hence, (1) holds.

If  $X$  is finite, then so are all elements in  $\mathcal{U}$ . Hence, (2) follows from (1). ■

5.5. LEMMA. *If  $X \neq \emptyset$ ,  $\mathcal{U}$  is a filter on  $X$ , and  $A \subseteq X$ , then:*

*Either  $B \cap A \notin \mathcal{U}$  for all  $B \in \mathcal{U}$ , or  $B \setminus A \notin \mathcal{U}$  for all  $B \in \mathcal{U}$ .*

PROOF. If this condition does not hold then there must exist sets  $B, C \in \mathcal{U}$  such that both  $B \cap A \in \mathcal{U}$  and  $C \setminus A \in \mathcal{U}$  hold. Then, by (4) of Definitions 5.1,  $\emptyset = (B \cap A) \cap (C \setminus A) \in \mathcal{U}$ , which contradicts (2) of Definitions 5.1.

Thus, the condition holds. ■

5.6. PROPOSITION. *A filter  $\mathcal{U}$  on  $X \neq \emptyset$  is an ultrafilter if and only if the following holds:*

*For every  $A \subseteq X$ , exactly one of  $A \in \mathcal{U}$  and  $X \setminus A \in \mathcal{U}$  holds.*

PROOF. First, assume that  $\mathcal{U}$  is an ultrafilter on  $X$ , and pick  $A \subseteq X$ . By Lemma 5.5, either  $B \cap A \notin \mathcal{U}$  for all  $B \in \mathcal{U}$ , or  $B \setminus A \notin \mathcal{U}$  for all  $B \in \mathcal{U}$ .

So, assume first that  $B \setminus A \notin \mathcal{U}$  for all  $B \in \mathcal{U}$ .

Let  $\mathcal{V}$  be the set of all  $B \subseteq X$  for which there exists a  $C \in \mathcal{U}$  such that  $C \cap A \subseteq B$ .

We show that  $\mathcal{V}$  is a filter on  $X$ . To do this, we verify (1)–(4) of the definition.

(1) clearly holds, just take  $C = X$ .

If (2) fails, then  $C \cap A = \emptyset$  for some  $C \in \mathcal{U}$ . Then  $C \setminus A = C \in \mathcal{U}$ . But this contradicts our assumption.

Thus, (2) holds.

(3) follows immediately from the definition of  $\mathcal{V}$ .

To obtain (4), pick  $B_1, B_2 \in \mathcal{V}$ . Then, there are  $C_1, C_2 \in \mathcal{U}$  such that  $C_1 \cap A \subseteq B_1$ , and  $C_2 \cap A \subseteq B_2$ . But, by (4) for  $\mathcal{U}$ ,  $C_1 \cap C_2 \in \mathcal{U}$ . Since  $(C_1 \cap C_2) \cap A \subseteq B_1 \cap B_2$ ,  $B_1 \cap B_2 \in \mathcal{V}$ . Hence, (4) holds.

Thus,  $\mathcal{V}$  is a filter on  $X$ .

Also, taking  $C = X$ , we see that  $A \in \mathcal{V}$ .

Next, for all  $C \in \mathcal{U}$ ,  $C \cap A \subseteq C$ . Hence  $C \in \mathcal{V}$ . It follows that  $\mathcal{U} \subseteq \mathcal{V}$ . Since  $\mathcal{U}$  is an ultrafilter,  $\mathcal{V} = \mathcal{U}$ , and thus  $A \in \mathcal{U}$ .

If, instead,  $B \cap A \notin \mathcal{U}$ , for all  $B \in \mathcal{U}$ , then applying the above to  $A' = X \setminus A$ , we obtain  $X \setminus A \in \mathcal{U}$ . That not both of  $A \in \mathcal{U}$  and  $X \setminus A \in \mathcal{U}$  hold follows easily from (4) and (2) of Definitions 5.1.

Thus, since  $A \subseteq X$  was arbitrary, the condition holds.

Conversely, assume that the condition holds. Let  $\mathcal{V}$  be a filter on  $X$  such that  $\mathcal{U} \subseteq \mathcal{V}$ , and pick  $A \in \mathcal{V}$ . By our assumption, either  $A \in \mathcal{U}$  or  $X \setminus A \in \mathcal{U}$ . If  $X \setminus A \in \mathcal{U}$ , then  $X \setminus A \in \mathcal{V}$ . Since  $A \in \mathcal{V}$ , (4) of the definition, for  $\mathcal{V}$ , gives  $\emptyset \in \mathcal{V}$ , which contradicts (2) of the definition. Thus,  $A \in \mathcal{U}$ . Since this holds for all  $A \in \mathcal{V}$ ,  $\mathcal{V} = \mathcal{U}$ .

It follows that  $\mathcal{U}$  is an ultrafilter. ■

5.7. COROLLARY. *If  $\mathcal{U}$  is an ultrafilter on  $X \neq \emptyset$ , and if  $A_1 \cup A_2 \cup \cdots \cup A_n \in \mathcal{U}$ , ( $n \geq 1$ ), then  $A_i \in \mathcal{U}$  for some  $i$  ( $1 \leq i \leq n$ ).*

PROOF. We prove this by induction on  $n$ . The case  $n = 1$  is trivial.

Assume that it holds for  $n$ , and assume that  $A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1} \in \mathcal{U}$ . Put  $A = A_1 \cup A_2 \cup \cdots \cup A_n$ . By Proposition 5.6, either  $A \in \mathcal{U}$  or  $X \setminus A \in \mathcal{U}$ . If  $A \in \mathcal{U}$ , then the induction hypothesis gives that  $A_i \in \mathcal{U}$  for some  $i$  such that  $1 \leq i \leq n$ .

If, instead,  $X \setminus A \in \mathcal{U}$ , then, by (4) of the definition,  $A_{n+1} \setminus A = (A \cup A_{n+1}) \cap (X \setminus A) \in \mathcal{U}$ . Then, by (3) of the definition,  $A_{n+1} \in \mathcal{U}$ .

Thus, in any case  $A_i \in \mathcal{U}$  for some  $i$  ( $1 \leq i \leq n + 1$ ).

The desired conclusion now follows by induction. ■

5.8. THEOREM. *If the axiom of choice holds, then: For any filter  $\mathcal{U}$  on  $X \neq \emptyset$ , there exists an ultrafilter  $\mathcal{V}$  on  $X$  such that  $\mathcal{U} \subseteq \mathcal{V}$ .*

PROOF. Let  $S$  be the family of all filters  $\mathcal{W}$  on  $X$  such that  $\mathcal{U} \subseteq \mathcal{W}$ . Let  $S$  be partially ordered by set inclusion: " $\subseteq$ ". Since  $\mathcal{U} \in S$ ,  $S \neq \emptyset$ .

Now, let  $T \neq \emptyset$  be a chain in  $S$ . Let  $\mathcal{T} = \cup_{\mathcal{W} \in T} \mathcal{W}$ . We show that  $\mathcal{T}$  is a filter on  $X$ . To do this, we must verify (1)–(4) of Definitions 5.1.

Since  $X \in \mathcal{W}$  for all  $\mathcal{W} \in T$ ,  $X \in \mathcal{T}$ . Thus, (1) holds.

Also,  $\emptyset \notin \mathcal{W}$ , for all  $\mathcal{W} \in T$ . Thus,  $\emptyset \notin \mathcal{T}$ . Hence, (2) holds.

If  $A \in \mathcal{T}$ , then  $A \in \mathcal{W}$  for some  $\mathcal{W} \in T$ . Then, if  $A \subseteq B \subseteq X$ , (3), for  $\mathcal{W}$ , gives that  $B \in \mathcal{W} \subseteq \mathcal{T}$ . Thus, (3) holds.

If  $A, B \in \mathcal{T}$ , then  $A \in \mathcal{W}$  and  $B \in \mathcal{W}'$  for some  $\mathcal{W}, \mathcal{W}' \in T$ . Since  $T$  is a chain, we have either  $\mathcal{W} \subseteq \mathcal{W}'$  or  $\mathcal{W}' \subseteq \mathcal{W}$ , say  $\mathcal{W}' \subseteq \mathcal{W}$ . Then  $B \in \mathcal{W}$  too, and (4), for  $\mathcal{W}$ , gives  $A \cap B \in \mathcal{W} \subseteq \mathcal{T}$ . Thus, (4) holds.

It follows that  $\mathcal{T}$  is a filter on  $X$ .

Also, since  $\mathcal{U} \subseteq \mathcal{W}$  for all  $\mathcal{W}$  in  $T$ ,  $\mathcal{U} \subseteq \mathcal{T}$ . Since  $\mathcal{W} \subseteq \mathcal{T}$ , for all  $\mathcal{W} \in T$ , this means that  $\mathcal{T}$  is an upper bound to the chain  $T$ . Since the chain  $T \subseteq S$  was arbitrarily chosen, we have proved that every nonempty chain  $T$  in  $S$  has an upper bound. This is also true if  $T = \emptyset$ , when we could take  $S$  as an upper bound. Hence, by Zorn's lemma, which is equivalent to the axiom of choice (see Chapter 8),  $S$  has a maximal element  $\mathcal{V}$ , and it is easy to see that this means that  $\mathcal{V}$  is an ultrafilter on  $X$  such that  $\mathcal{U} \subseteq \mathcal{V}$ . ■

5.9. DEFINITION. A filter  $\mathcal{U}$  on  $X \neq \emptyset$  is called *countably incomplete* if there exists a countably infinite sequence  $\{A_n\}_{n=1}^{\infty}$  of sets in  $\mathcal{U}$  such that  $\cap_{n=1}^{\infty} A_n = \emptyset$ .

5.10. REMARKS. Clearly, if  $\mathcal{U}, \mathcal{V}$  are filters on  $X \neq \emptyset$  such that  $\mathcal{U} \subseteq \mathcal{V}$ , and if  $\mathcal{U}$  is countably incomplete, then so is  $\mathcal{V}$ .

A countably incomplete filter must be nonprincipal, because every family of sets in a principal filter has an intersection containing a nonempty set such as  $Y$  in Remarks 5.2. By Proposition 5.4, then, no countably incomplete filter contains a finite set, and there is no countably incomplete filter on a finite set.

5.11. PROPOSITION. *If  $X$  is countably infinite, then every nonprincipal ultrafilter on  $X$  is countably incomplete.*

PROOF. Let  $\mathcal{U}$  be a nonprincipal ultrafilter on a countably infinite set  $X$ . For every  $x \in X$ , (1) of Proposition 5.4 gives  $\{x\} \notin \mathcal{U}$ . Thus, by Proposition 5.6,  $X \setminus \{x\} \in \mathcal{U}$ , for all  $x \in X$ . Since  $X$  is countably infinite, it can be enumerated,  $X = \{x_1, x_2, x_3, \dots\}$ . Now  $\cap_{n=1}^{\infty} (X \setminus \{x_n\}) = \emptyset$ .

Hence,  $\mathcal{U}$  is countably incomplete. ■

5.12. REMARKS. We will now see that one can construct a countably incomplete ultrafilter on any infinite set, provided that the axiom of choice holds.

So, assume that the axiom of choice holds, and let  $X$  be an infinite set. Let  $\hat{X}$  be the set of all nonempty finite subsets of  $X$ . We will construct a countably incomplete ultrafilter on  $\hat{X}$ . Since  $X$  and  $\hat{X}$  have the same cardinality (see Chapter 8), there is a bijection between  $X$  and  $\hat{X}$ . Applying this bijection, we can then obtain a countably incomplete ultrafilter on  $X$  too.

So, let us construct a countably incomplete ultrafilter on  $\hat{X}$ . This explicit ultrafilter will be used in the proof of Theorem 7.13, to obtain a  $\kappa$ -saturated extension of a given superstructure.

For every  $\hat{x} \in \hat{X}$ , put  $A_{\hat{x}} = \{\hat{u} \in \hat{X} \mid \hat{x} \subseteq \hat{u}\}$ . Next, let  $\mathcal{V}$  be the set of all  $\hat{A} \subseteq \hat{X}$  for which there is an  $\hat{x} \in \hat{X}$  such that  $A_{\hat{x}} \subseteq \hat{A}$ .

We prove that  $\mathcal{V}$  is a filter on  $\hat{X}$ . Verifying (1)–(4) of the definition, we see that (1), (2), and (3) are obvious. To obtain (4), pick  $\hat{A}, \hat{B} \in \mathcal{V}$ . Then there are  $\hat{x}, \hat{y} \in \hat{X}$

such that  $A_{\hat{x}} \subseteq \hat{A}$  and  $A_{\hat{y}} \subseteq \hat{B}$ . Then  $A_{\hat{x}} \cap A_{\hat{y}} \subseteq \hat{A} \cap \hat{B}$ . But it is easy to see that  $A_{\hat{x}} \cap A_{\hat{y}} = A_{\hat{x} \cup \hat{y}}$ . Hence,  $\hat{A} \cap \hat{B} \in \mathcal{V}$ . Since  $\hat{A}, \hat{B} \in \mathcal{V}$  were arbitrarily chosen, (4) holds.

We have thus proved that  $\mathcal{V}$  is a filter on  $\hat{X}$ . Also, we see that  $A_{\hat{x}} \in \mathcal{V}$  for every  $\hat{x} \in \hat{X}$ .

Next, we show that  $\mathcal{V}$  is countably incomplete. Since  $X$  is infinite, there is a countably infinite sequence  $\{x_n\}_{n=1}^{\infty}$  of distinct elements from  $X$ . For each  $n$  ( $n \geq 1$ ), put  $\hat{x}_n = \{x_n\}$ , and consider the countably infinite sequence  $\{A_{\hat{x}_n}\}_{n=1}^{\infty}$  of sets from  $\mathcal{V}$ . If  $\hat{x} \in \bigcap_{n=1}^{\infty} A_{\hat{x}_n}$ , then  $\hat{x}$  is a finite set such that  $x_n \in \hat{x}$  for all  $n$  ( $n \geq 1$ ), which is a contradiction. It follows that  $\bigcap_{n=1}^{\infty} A_{\hat{x}_n} = \emptyset$ .

Hence,  $\mathcal{V}$  is countably incomplete.

Now, by Theorem 5.8, there is an ultrafilter  $\mathcal{U}$  on  $\hat{X}$  such that  $\mathcal{V} \subseteq \mathcal{U}$ , and this  $\mathcal{U}$  is countably incomplete, by Remarks 5.10.

As remarked above, there is now a countably incomplete ultrafilter on  $X$  too.

We have thus proved:

**5.13. THEOREM.** *If the axiom of choice holds, and if  $X$  is an infinite set, then there is a countably incomplete ultrafilter on  $X$ .*

## CHAPTER 6

### Ultrapower Extensions

We will now see how one can construct a proper nonstandard extension of any given superstructure. This type of extension that we use here is called an *ultrapower extension*. This is the most popular type of explicit proper nonstandard extension. But there are also other types of extensions. For example, one can obtain extensions using the *upward Löwenheim-Skolem-Tarski theorem* in model theory ([6], Prop 2.38, p.100).

**In this chapter, the axiom of choice is assumed to hold.**

6.1. REMARK. The idea is to use an ultrafilter to construct a structure which in a sense is "isomorphic" to a superstructure, and then "transform" this into a true superstructure, which is a proper nonstandard extension of the original one.

6.2. DEFINITIONS. Let  $S$  be an  $\omega$ -extendable set which contains an  $\mathbb{N}$ -copy, and put  $S' = S \cup \{\emptyset\}$ . (Thus,  $H_\omega^{S'}$  is the superstructure over  $S$ .)

Let  $I$  be an infinite set, which we call the *index set*, and let  $\mathcal{U}$  be a countably incomplete ultrafilter on  $I$ . Such an ultrafilter exists, by Theorem 5.13.

We keep  $S$ ,  $I$ , and  $\mathcal{U}$  fixed in the rest of this chapter.

Let  $V$  be the set of all functions  $x : I \rightarrow H_\omega^{S'}$ , for which there is an  $n < \omega$  (which may depend upon  $x$ ) such that  $\{i \mid x(i) \in H_n^{S'}\} \in \mathcal{U}$ .

For  $k$  elements  $x_1, x_2, \dots, x_k \in V$  ( $k \geq 0$ ), we say that  $x_1(i), x_2(i), \dots, x_k(i)$  satisfy some condition  $\phi(a_1, a_2, \dots, a_k)$  *almost everywhere*, abbreviated a. e., if the set  $M$  of all  $i \in I$  such that  $\phi(x_1(i), x_2(i), \dots, x_k(i))$  holds lies in  $\mathcal{U}$ .<sup>17</sup> (Thus, for every  $x \in V$ , there is an  $n < \omega$  such that  $x(i) \in H_n^{S'}$  a.e.)

Using that  $\mathcal{U}$  is a filter, we see that the condition  $x(i) = y(i)$  a.e. ( $x, y \in V$ ) defines an equivalence relation on  $V$  (see Chapter 8). We define  $W$  as the family of equivalence classes corresponding to this equivalence relation. This  $W$  is called an *ultrapower* of  $H_\omega^{S'}$ .<sup>18</sup>  $[x]$  denotes the equivalence class in  $W$  which contains  $x \in V$ .

For every  $a \in H_\omega^{S'}$ ,  ${}^\circ a$  denotes the equivalence class  $[x] \in W$ , where  $x$  is the constant function such that  $x(i) = a$  for all  $i \in I$ .

For  $[x], [y] \in W$ , we write  $[x] \in [y]$  if  $x(i) \in y(i)$  a.e., and we write  $[x] \subseteq [y]$  if  $x(i) \subseteq y(i)$  a.e. and  $x(i), y(i) \notin S$  a.e. It follows from (4) and (3) of Definitions 5.1 that these are both well defined, i.e. independent of which representatives we choose from the equivalence classes.

If the function  $f : (H_\omega^{S'})^m \rightarrow H_\omega^{S'}$ , for some  $m \geq 0$ , is such that to every  $n < \omega$  there is a  $k < \omega$  such that  $f[(H_n^{S'})^m] \subseteq H_k^{S'}$ , then we say that  $f$  is *rank consistent*.

For such a function, we define the function  ${}^\circ f : W^m \rightarrow W$ , for all  $\langle [x_1], [x_2], \dots, [x_m] \rangle \in W^m$ , by  $({}^\circ f)([x_1], [x_2], \dots, [x_m]) = [y]$ , where  $y(i) =$

<sup>17</sup>This suggestive terminology is borrowed from measure theory. It has the drawback, though, that one might think that if a property holds a.e., then it holds for an overwhelming majority of the elements in  $I$ . This need not be true at all, which we can see if we partition  $I$  into, say, 1,000,000 subsets with the same cardinality and then apply Corollary 5.7.

<sup>18</sup>Actually, according to the usual definition of ultrapower, the condition on  $n$  above should not be there. This condition is necessary here, however.

$(x_1(i), x_2(i), \dots, x_m(i))$ , for all  $i \in I$ . It follows by induction on (4) and (3) of Definitions 5.1, that  ${}^\circ f$  is well defined.

6.3. REMARKS. The functions  $f : (H_\omega^{S'})^m \rightarrow H_\omega^{S'}$  are not of the type in Definitions 2.20 and Chapter 4, but of the type defined in the appendix (Chapter 8), because its values are not  $k$ -tuples, for any  $k \geq 0$  (except accidentally). Also, the domain of such a function is *all of*  $(H_\omega^{S'})^m$ , for some  $m \geq 0$

Functions we view in this way are e.g.  $\cup, \setminus, \cap$  (with domain  $(H_\omega^{S'})^2$ ), and  $\pi_i^m$  for  $1 \leq i \leq m$ , (with domain  $(H_\omega^{S'})^m$ , for  $m \geq 0$ ). See (3) of Proposition 6.5.

In a natural way, we may identify such a function with domain  $(H_\omega^{S'})^0 = \{\langle \rangle\} = \{\emptyset\}$ , with a constant: its constant value. Doing this, our two usages of the symbol  ${}^\circ$  are consistent with each other.<sup>19</sup>

The reason that we introduce this general type of functions, is that we can use it to unify some results which we need when we verify that our final extension satisfies the twelve axioms of Definition 4.4, and thus substantially shorten this verification.

6.4. PROPOSITION. *Let  $[x], [y] \in W$ , and assume that  $x(i), y(i) \notin S$  a.e. Then:*

- (1)  $[x] = [y]$  if and only if  $[x] {}^\circ \subseteq [y]$  and  $[y] {}^\circ \subseteq [x]$ .
- (2)  $[x] {}^\circ \subseteq [y]$  if and only if, for all  $[z] \in W$ :  $[z] {}^\circ \in [x] \Rightarrow [z] {}^\circ \in [y]$ .
- (3)  $[x] = [y]$  if and only if, for all  $[z] \in W$ :  $[z] {}^\circ \in [y] \Leftrightarrow [z] {}^\circ \in [x]$ .
- (4) The map  ${}^\circ : H_\omega^{S'} \rightarrow W$  is injective.

PROOF. If  $[x] = [y]$ , then  $x(i) = y(i)$  a.e. Since  $x(i), y(i) \notin S$  a.e.,  $x(i) \subseteq y(i)$  a.e. and  $y(i) \subseteq x(i)$  a.e. But then,  $[x] {}^\circ \subseteq [y]$  and  $[x] {}^\circ \subseteq [y]$ .

Conversely, if  $[x] {}^\circ \subseteq [y]$  and  $[x] {}^\circ \subseteq [y]$ , then  $x(i) \subseteq y(i)$  a.e. and  $y(i) \subseteq x(i)$  a.e. By (4) of Definitions 5.1,  $x(i) \subseteq y(i)$  and  $y(i) \subseteq x(i)$  a.e., simultaneously. This means, by (3) of Definitions 5.1, that  $x(i) = y(i)$  a.e., that is  $[x] = [y]$ .

Thus, (1) holds.

To prove (2), assume that  $[x] {}^\circ \subseteq [y]$ , and take  $[z] \in W$  such that  $[z] {}^\circ \in [x]$ . Then  $z(i) \in x(i)$  a.e. Since also  $x(i) \subseteq y(i)$  a.e., (4) and (3) of Definitions 5.1 give  $z(i) \in y(i)$  a.e., i.e.  $[z] {}^\circ \in [y]$ . Thus  $[z] {}^\circ \in [x] \Rightarrow [z] {}^\circ \in [y]$  for all  $[z] \in W$ .

Conversely, assume that  $[x] {}^\circ \subseteq [y]$  does not hold. This means that it is not the case that  $x(i) \subseteq y(i)$  a.e. Since  $\mathcal{U}$  is an ultrafilter, this implies that  $x(i) \not\subseteq y(i)$  a.e. by Proposition 5.6. Then, we can choose  $[z] \in W$  such that  $z(i) \in x(i) \setminus y(i)$  a.e. Then  $[z] {}^\circ \in [x]$ , but not  $[z] {}^\circ \in [y]$ . The implication  $[z] {}^\circ \in [x] \Rightarrow [z] {}^\circ \in [y]$  is thus false for this  $[z] \in W$ , which gives the converse part of (2).

Thus, (2) holds.

(3) follows immediately from (1) and (2).

(4) follows easily from the definition, using (1) and (2) of Definitions 5.1. ■

6.5. PROPOSITION. *Let  $f : (H_\omega^{S'})^m \rightarrow H_\omega^{S'}$  be rank consistent, for some  $m \geq 0$ , Then:*

- (1)  $({}^\circ f)({}^\circ a_1, {}^\circ a_2, \dots, {}^\circ a_m) = {}^\circ(f(a_1, a_2, \dots, a_m))$ , for all  $a_1, a_2, \dots, a_m \in H_\omega^{S'}$ .
- (2) If, for some  $k \geq 0$ ,  $g_1, g_2, \dots, g_k : (H_\omega^{S'})^k \rightarrow H_\omega^{S'}$  are rank consistent, then, so is the function  $q : (H_\omega^{S'})^k \rightarrow H_\omega^{S'}$  defined by

$$q(x_1, x_2, \dots, x_k) = f(g_1(x_1, x_2, \dots, x_k), g_2(x_1, x_2, \dots, x_k), \dots, g_m(x_1, x_2, \dots, x_k)),$$

<sup>19</sup>The map  ${}^\circ$ , for both its usages, is also an ordinary function as in Chapter 8. The same is true for the functions  $h_1, h$ , and  $*$  introduced in Remarks 6.12 and definition 6.13 and 6.16.

for all  $x_1, x_2, \dots, x_k \in H_\omega^{S'}$ .

Also,

$$(\circ q)([x_1], [x_2], \dots, [x_k]) = (\circ f)((\circ g_1)([x_1], [x_2], \dots, [x_k]), (\circ g_2)([x_1], [x_2], \dots, [x_k]), \dots, (\circ g_m)([x_1], [x_2], \dots, [x_k])),$$

for all  $[x_1], [x_2], \dots, [x_k] \in W$ .

- (3) For any  $m \geq 0$  and any  $k: (1 \leq k \leq m)$ , the projection  $\pi_k^m: (H_\omega^{S'})^m \rightarrow H_\omega^{S'}$ , defined as in Definitions 2.6 for all  $a_1, a_2, \dots, a_m \in H_\omega^{S'}$ , is rank consistent. Also,  $(\circ \pi_k^m)([x_1], [x_2], \dots, [x_m]) = [x_k]$ , for all  $[x_1], [x_2], \dots, [x_m] \in W$ .

PROOF. These properties are immediate from the definitions. ■

6.6. REMARK. In this chapter,  $\pi_k^m$  will all the time be as in (3) of Proposition 6.5, ( $1 \leq k \leq m$ ), that is, with domain  $(H_\omega^{S'})^m$  and codomain  $H_\omega^{S'}$ .

6.7. PROPOSITION. Let  $a \in H_\omega^{S'}$ , and let  $A, B \in H_\omega^{S'} \setminus S$ .

Then:

- (1)  $a \in A$  if and only if  $\circ a \circ \in \circ A$ .
- (2)  $A \subseteq B$  if and only if  $\circ A \circ \subseteq \circ B$ .

PROOF. These follow immediately from the definitions and (1) and (2) of Definitions 5.1. ■

6.8. PROPOSITION. For every  $[x] \in W$ , there is a unique  $n < \omega$  such that  $r^{S'}(x(i)) = n$  a.e. This  $n$  is independent of the choice of the representative  $x$ .

PROOF. By definition, there is an  $n' < \omega$  such that  $r^{S'}(x(i)) \leq n'$  a.e. Thus, the union of the sets  $\{i \in I \mid r^{S'}(x(i)) = n\}$ , ( $1 \leq n \leq n'$ ), belongs to  $\mathcal{U}$ . Hence one of these sets belongs to  $\mathcal{U}$ , by Corollary 5.7. Thus,  $r^{S'}(x(i)) = n$  a.e. for some  $n$ . That this  $n$  is unique and independent of the choice of representative follows from (2), (4), and (3) of Definitions 5.1. ■

6.9. DEFINITION. For  $[x] \in W$ , we define  $\circ r^{S'}([x])$  as the unique number  $n$  in Proposition 6.8.

6.10. PROPOSITION. If  $[x] \in W$ , then:

- (1)  $\circ r^{S'}([x]) \geq 1$ .
- (2) If  $\circ r^{S'}([x]) = n \geq 2$ , then  $\circ r^{S'}([y]) \leq n - 1$  for all  $[y] \in W$  such that  $[y] \circ \in [x]$ , and there is such a  $[y]$  with  $\circ r^{S'}([y]) = n - 1$ .
- (3)  $\circ r^{S'}([x]) = 1$  if and only if  $[x] \circ \in \circ S$  or  $[x] = \circ \emptyset$ .  
If these conditions hold, then there are no  $[y] \in W$  such that  $[y] \circ \in [x]$ .
- (4) For all  $a \in H_\omega^{S'}$ ,  $\circ r^{S'}(\circ a) = r^{S'}(a)$ .

PROOF. (1) follows directly from the definition, and (2) of Definitions 5.1.

To prove (2): If  $[x], [y] \in W$ ,  $\circ r^{S'}([x]) = n \geq 2$  and  $[y] \circ \in [x]$ , then  $r^{S'}(x(i)) = n$  and  $y(i) \in x(i)$  a.e. Hence,  $r^{S'}(y(i)) \leq n - 1$  a.e. Hence, by Proposition 6.8 and the definition,  $\circ r^{S'}([y]) \leq n - 1$ . This holds for all  $[y] \in W$  such that  $[y] \circ \in [x]$ .

Next, assume that  $[x] \in W$ , with  $\circ r^{S'}([x]) = n \geq 2$ . Then  $r^{S'}(x(i)) = n$  a.e. Then, we can choose  $[y] \in W$  such that  $y(i) \in x(i)$  and  $r^{S'}(y(i)) = n - 1$  a.e. Thus  $[y] \circ \in [x]$  and  $\circ r^{S'}([y]) = n - 1$ . Hence, (2) holds.

$\circ r^{S'}([x]) = 1$  holds if and only if to  $r^{S'}(x(i)) = 1$  a.e. that is,  $x(i) = S'$  a.e. By Corollary 5.7, this holds if and only if either of the following two conditions holds:  $x(i) \in S$  a.e., that is  $[x] \circ \in \circ S$ , or  $x(i) = \emptyset$  a.e., that is  $[x] = \circ \emptyset$ .

In both cases  $x(i)$  has no elements in  $H_\omega^{S'}$ , a.e. Hence, for no  $[y] \in W$ ,  $[y] \circ \in [x]$ . Thus, (3) holds.

(4) is immediate from the definitions. ■

6.11. REMARKS. As the preceding results show,  $W$  can be considered as a structure "isomorphic" to a superstructure. We may express it thus, that we have a "quasi-membership relation", a "quasi-subset relation" and a "quasi-rank function", all behaving as they should. By applying the injection  $\circ$ , we also see that we within  $W$  has an "isomorphic copy" of  $H_\omega^{S'}$ .

Our next objective is to "transform" this "quasi-superstructure"  $W$  into a true superstructure, and in the process obtain a proper nonstandard extension from  $H_\omega^{S'}$  into this new superstructure.

Actually, this transformation turns out to be only an injection. The elements in  $W$  will only correspond to the *internal* elements in the new superstructure, by Proposition 6.25.

Some readers may wonder if it is really necessary to make this transformation. Couldn't we just consider  $W$  as a model of a true superstructure, using  $\circ \in$  as if this was the true membership relation, etc.?

But if we do so, then external sets such as  $\mathbb{N}$  (see Proposition 4.36) will not be elements (or "quasi-elements") in the superstructure, and this not desirable.

So, we want to transform  $W$  into a true superstructure.

6.12. REMARKS. We now proceed as follows:

Let  $Y \in H$  be a set such that  $S \cap Y = \emptyset$  and  $S \cup Y$  is  $\omega$ -extendable. For example, we may take  $Y = \emptyset$ , since  $\emptyset \in H$ , by (6) of Proposition 3.16. (In the proof of Theorem 7.8, though, we need to consider nonempty sets  $Y$  also.)

By (1) of Theorem 3.35, we can choose a set  $Z$  such that  $(S \cup Y) \cap Z = \emptyset$ ,  $S \cup Y \cup Z$  is  $\omega$ -extendable, and  $\text{card } S \cup Z > \text{card } W \geq \text{card } S$ . (The last inequality holds because the map  $\circ : H_\omega^{S'} \rightarrow W$  is injective, by (4) of Proposition 6.4.) Then,  $\text{card } Z = \text{card } S \cup Z$  (see Chapter 8).

Therefore, there is an injection  $h_1 : \{[x] \in W \mid [x] \circ \in \circ S\} \setminus \{\circ a \mid a \in S\} \rightarrow Z$ . Put  $T = S \cup R_{h_1}$  (where  $R_{h_1}$  is the range of  $h_1$ ), and put  $T' = T \cup \{\emptyset\}$ .

Then,  $T \cap Y = \emptyset$ , and it follows from (1) of Proposition 3.31 that  $T \cup Y$  and  $T$  both are  $\omega$ -extendable, with  $Y$  as above.

*T, T', and h<sub>1</sub> are kept fixed throughout this chapter.*

6.13. DEFINITION. We now define a function  $h$  on  $W$ , by recursion on  $\circ r^{S'}([x])$ , for  $[x] \in W$ :

If  $\circ r^{S'}([x]) = 1$ , then we put  $h([x]) = a$  if  $[x] = \circ a$  for some  $a \in S'$ , and we put  $h([x]) = h_1([x])$  otherwise.

If  $h$  is defined on all  $[y] \in W$  with  $n > \circ r^{S'}([y]) \geq 1$ , and if  $[x] \in W$  with  $\circ r^{S'}[x] = n$ , then we put  $h([x]) = \{h([y]) \mid \circ r^{S'}([y]) < n \wedge [y] \circ \in [x]\}$ .

6.14. REMARKS. It follows from (4) of Proposition 6.4, (3) of Proposition 6.10, and Remarks 6.12 that  $h([x])$  is well defined for  $[x] \in W$  with  $\circ r^{S'}([x]) = 1$ .

Since  $\circ r^{S'}([x]) \geq 1$  for all  $[x] \in W$ , by (1) of Proposition 6.10,  $h$  is therefore defined on all of  $W$ .

Since  $\mathbb{N} \subseteq S \subseteq T$ ,  $H_\omega^{T'}$  is the superstructure over  $T$ .

### 6.15. PROPOSITION.

- (1)  $h$  maps  $W$  into  $H_\omega^{T'}$ , and  $r^{T'}(h([x])) = \circ r^{S'}([x])$ , for all  $[x] \in W$ .
- (2) For all  $[x], [y] \in W$ ,  $[x] \circ \in [y]$  if and only if  $h([x]) \in h([y])$ .
- (3) For all  $[x], [y] \in W$  such that neither  $[x] \circ \in \circ S$  nor  $[y] \circ \in \circ S$  holds,  $[x] \circ \subseteq [y]$  if and only if  $h([x]) \subseteq h([y])$ .
- (4)  $h$  is injective.
- (5)  $h(\circ S) = T$  and  $h(\circ S') = T'$ .
- (6) If  $y' \in R_h \setminus T$ , then  $y' \subseteq R_h$  (where  $R_h$  is the range of  $h$ ).

PROOF. We prove by induction on  $n = \circ r^{S'}([x])$ , that  $h([x]) \in H_\omega^{T'}$  with  $r^{T'}(h([x])) = n$ , for all  $[x] \in W$  with  $\circ r^{S'}([x]) = n$ , for all  $n \geq 1$ . (We must have  $\circ r^{S'}([x]) \geq 1$ , by (1) of Proposition 6.10.)

If  $n = 1$  then  $h([x]) \in T' \subseteq H_\omega^{T'}$ , by definition, and so  $r^{T'}(h([x])) = 1$ .

Assume now that  $\circ r^{S'}([x]) > 0$ , and that (1) holds for all  $k$  such that  $1 \leq k < n$ . Then, if  $y' \in h([x])$ ,  $y' = h([y])$  for some  $[y] \in W$  with  $\circ r^{S'}([y]) \leq n - 1$  and  $[y] \circ \in [x]$ , by Definition 6.13. By the induction hypothesis,  $r^{T'}(y') \leq n - 1$ . This is true for all  $y' \in h([x])$ . Moreover, by (2) of Proposition 6.10, there are such  $y'$  and  $[y]$  with  $\circ r^{S'}([y]) = n - 1$ , and  $r^{T'}(y') = n - 1$ .

Thus,  $h([x]) \subseteq H_\omega^{T'}$ , and  $\max_{y' \in h([x])} r^{T'}(y') = n - 1$ . Also, since  $T'$  is  $\omega$ -grounded,  $h([x]) \notin T'$ . Hence,  $r^{T'}(h([x])) = n$ .

(1) now follows by induction.

To prove (2), pick  $[x], [y] \in W$ , and assume that  $[x] \circ \in [y]$ . Then,  $\circ r^{S'}([y]) > 1$ , by (1) and (3) of Proposition 6.10. Thus, by (2) of Proposition 6.10 and Definition 6.13,  $h([x]) \in h([y])$ .

Conversely, assume that  $h([x]) \in h([y])$ . Then, by (1),  $h([y]) \in H_\omega^{T'} \setminus T'$  and  $\circ r^{S'}([y]) > 1$ . Then, by Definition 6.13, we must have  $[x] \circ \in [y]$ .

Thus, (2) holds.

To prove (3), take  $[x], [y] \in W$  such that neither  $[x] \circ \in \circ S$  nor  $[y] \circ \in \circ S$  holds.

Assume first that  $[x] \circ \subseteq [y]$ .

Then, by the definitions,  $x(i) \subseteq y(i)$  and  $x(i), y(i) \notin S$  (simultaneously) a.e. For any  $i$  where this holds, either  $y(i) = \emptyset$ , in which case  $x(i) = \emptyset$ , or  $n > 1$ , where  $n = r^{S'}(y(i))$ . In the latter case,  $x(i) \subseteq y(i) \subseteq H_{n-1}^{S'}$ . Thus,  $x(i) \in \mathcal{P}(H_{n-1}^{S'})$ , which implies either  $x(i) = \emptyset$  or  $r^{S'}(x(i)) > 1$ .

By Corollary 5.7, either  $x(i) = \emptyset$  a.e., that is,  $[x] = \circ \emptyset$ , or  $r^{S'}(x(i)) > 1$  a.e., which implies  $\circ r^{S'}([x]) > 1$ . In the first case,  $h([x]) = \emptyset$  and then, certainly,  $h([x]) \subseteq h([y])$ . In the latter case, pick  $z' \in h([x])$ . Then, by definition,  $z' = h([z])$  for some  $[z] \in W$  such that  $[z] \circ \in [x]$ . By (2) of Proposition 6.4,  $[z] \circ \in [y]$ . Thus, by (2),  $z' = h([z]) \in h([y])$ . Since this holds for all  $z' \in h([x])$ ,  $h([x]) \subseteq h([y])$ .

Thus, in both cases,  $h([x]) \subseteq h([y])$ .

Conversely, assume that  $h([x]) \subseteq h([y])$ . Now,  $x(i), y(i) \notin S$  a.e., by the definitions. Take  $[z] \in W$  such that  $[z] \circ \in [x]$ . Then, by (2),  $h([z]) \in h([x])$ , and hence  $h([z]) \in h([y])$ , and, again by (2),  $[z] \circ \in [y]$ . This holds for all  $[z] \in W$  with  $[z] \circ \in [x]$ . Thus, by (2) of Proposition 6.4,  $[x] \circ \subseteq [y]$ . Thus, (3) holds.

To prove (4), take  $[x], [y] \in W$  such that  $h([x]) = h([y])$ . By (1),  ${}^\circ r^{S'}([x]) = {}^\circ r^{S'}([y]) = n$ , say. If  $n = 1$ , then  $[x] = [y]$  follows immediately from the definition of  $h$  in that case.

If  $n > 1$ , then neither  $[x] {}^\circ \in {}^\circ S$  nor  $[y] {}^\circ \in {}^\circ S$  holds, by (3) of Proposition 6.10. Thus, by (3),  $[x] {}^\circ \subseteq [y]$  and  $[y] {}^\circ \subseteq [x]$ , and, by (1) of Proposition 6.4,  $[x] = [y]$  holds in this case too.

Thus, (4) holds.

To obtain (5): we first notice that  ${}^\circ r^{S'}({}^\circ S) = {}^\circ r^{S'}({}^\circ S') = 2$ , by (4) of Proposition 6.10.

Pick  $z' \in h({}^\circ S)$ . By Definition 6.13, then,  $z' = h([z])$  for some  $[z] \in W$  such that  $[z] {}^\circ \in {}^\circ S$ . By definition, this means that  $z(i) \in S$  a.e. Then, (2) and (4) of Definitions 5.1 give that  $z(i) = \emptyset$  a.e. cannot hold, whence  $[z] \neq {}^\circ \emptyset$ . Also,  ${}^\circ r^{S'}([x]) = 1$ , by (3) of Proposition 6.10. Hence, (4) of Proposition 6.4, Definition 6.13, and Remarks 6.12 give  $z' = h([z]) \in T$ . This holds for all  $z' \in h({}^\circ S)$ , so  $h({}^\circ S) \subseteq T$ .

Conversely, if  $z' \in T$ , then  $z' \neq \emptyset$ . Then, by Remarks 6.12, (1) of Proposition 6.7, (3) of Proposition 6.10, and Definition 6.13,  $z' = h([z])$  for some  $[z] \in W$  such that  $[z] {}^\circ \in {}^\circ S$ . Hence, by (2),  $z' = h([z]) \in h({}^\circ S)$ . This holds for all  $z' \in T$ , so  $T \subseteq h({}^\circ S)$ .

Hence,  $h({}^\circ S) = T$ .

To obtain  $h({}^\circ S') = T'$ , it suffices, by the previous case, to prove that  $h({}^\circ S') = h({}^\circ S) \cup \{\emptyset\}$ .

To do this, pick  $z' \in h({}^\circ S')$ . By definition, then,  $z' = h([z])$  for some  $[z] \in W$  such that  $[z] {}^\circ \in S'$ . By definition, this means that  $z(i) \in S$  or  $z = \emptyset$ , a.e. By Corollary 5.7, either  $z(i) \in S$  a.e. in which case  $[z] {}^\circ \in {}^\circ S$ , or  $z(i) = \emptyset$  a.e., in which case  $[z] = {}^\circ \emptyset$ . In the first case (2) gives  $z' = h([z]) \in h({}^\circ S)$ . In the second case, Definition 6.13 give  $h([z]) = \emptyset$ .

Thus, in both cases,  $z' \in h({}^\circ S) \cup \{\emptyset\}$ . This holds for all  $z' \in h({}^\circ S')$ , so  $h({}^\circ S') \subseteq h({}^\circ S) \cup \{\emptyset\}$ .

Conversely, since  $S' \notin S$ ,  $S \notin S$  and  $\emptyset \notin S$ , (1) of Proposition 6.7 gives that neither  ${}^\circ S' {}^\circ \in {}^\circ S$  nor  ${}^\circ S {}^\circ \in {}^\circ S$  holds.

Then, by (2) of Proposition 6.7, and (3):  $h({}^\circ S) \subseteq h({}^\circ S')$ . Also, since, by (3) of Proposition 6.10 and Definition 6.13,  $h({}^\circ \emptyset) = \emptyset$ , (1) of Proposition 6.7 and (2) give  $\emptyset \in h({}^\circ S')$ .

Thus,  $h({}^\circ S) \cup \{\emptyset\} \subseteq h({}^\circ S')$ .

Hence,  $h({}^\circ S') = h({}^\circ S) \cup \{\emptyset\}$ , that is, (5) holds.

To obtain (6), assume that  $y' \in R_h \setminus T$ . If  $y' = \emptyset$ , then  $y' \subseteq R_h$ . Otherwise,  $y' \notin T'$ . Hence,  $r^{T'}(y') \geq 2$ . Then, if  $y' = h([y])$  for some  $[y] \in W$ , (1) gives  ${}^\circ r^{S'}([y]) \geq 2$ . Now, it follows from Definition 6.13 that  $y' \subseteq R_h$  in this case too.

Thus, (6) holds. ■

6.16. DEFINITIONS. We define the map  $*$  :  $H_\omega^{S'} \rightarrow H_\omega^{T'}$  by  $*a = h({}^\circ a)$ , for all  $a \in H_\omega^{S'}$ .

For any  $m \geq 0$ , and any rank consistent function  $f : (H_\omega^{S'})^m \rightarrow H_\omega^{S'}$ , we define the function  $*f : R_h^m \rightarrow R_h$  by  $(*f)(x'_1, x'_2, \dots, x'_m) = h(({}^\circ f)([x_1], [x_2], \dots, [x_m]))$ , for all  $x'_1, x'_2, \dots, x'_m \in R_h$ , where  $[x_k] \in W$  is chosen so that  $h([x_k]) = x'_k$  ( $1 \leq k \leq m$ ). (This is well defined, since  $h$  is injective, by (4) of Proposition 6.15.)

6.17. REMARKS. Functions such as  $*f$  in Definitions 6.16 are *not* of the same

type as the functions in Proposition 4.19. The latter functions are considered as elements in  $H_\omega^{T'}$ , while the functions here may be considered as *generalizations* of elements in  $R_h$ , identifying elements in  $R_h$  with functions with domain  $R_h^0$  and codomain  $R_h$ , in the same way as in Remarks 6.3. This makes the two usages in Definitions 6.16 of the symbol  $*$  consistent with each other.

We introduce this type of functions by the same reason as in Remarks 6.3.

#### 6.18. PROPOSITION.

- (1)  $*a = a$  for all  $a \in S'$ .
- (2) The map  $*$  :  $H_\omega^{S'} \rightarrow H_\omega^{T'}$  is injective.
- (3)  $r^{T'}(*a) = r^{S'}(a)$ , for all  $a \in H_\omega^{S'}$ .
- (4) If,  $f : (H_\omega^{S'})^m \rightarrow H_\omega^{S'}$  is rank consistent, for some  $m \geq 0$ , then,  $(*f)(*a_1, *a_2, \dots, *a_m) = *(f(a_1, a_2, \dots, a_m))$ , for all  $a_1, a_2, \dots, a_m \in H_\omega^{S'}$ .
- (5) Let  $f$  be as in (4), and let, for some  $k \geq 0$ ,  $g_1, g_2, \dots, g_m : (H_\omega^{S'})^k \rightarrow H_\omega^{S'}$  be rank consistent functions, and let the function  $q : (H_\omega^{S'})^k \rightarrow H_\omega^{S'}$  be defined by

$$q(x_1, x_2, \dots, x_k) = f(g_1(x_1, x_2, \dots, x_k), g_2(x_1, x_2, \dots, x_k), \dots, g_m(x_1, x_2, \dots, x_k)),$$

for all  $x_1, x_2, \dots, x_k \in H_\omega^{S'}$ .

Then,

$$(*q)(x'_1, x'_2, \dots, x'_k) = (*f)((*g_1)(x'_1, x'_2, \dots, x'_k), (*g_2)(x'_1, x'_2, \dots, x'_k), \dots, (*g_m)(x'_1, x'_2, \dots, x'_k)),$$

for all  $x'_1, x'_2, \dots, x'_k \in R_h$ .

- (6)  $*S = T$  and  $*S' = T'$

PROOF. (1) is immediately from the definitions of  $*$ ,  $\circ$ , and  $h$ , and (4) of Proposition 4.10.

(2) follows from the injectivities of  $\circ$  ((4) of Proposition 6.4) and  $h$  ((4) of Proposition 6.15), respectively, and the definitions.

(3) follows from (4) of Proposition 6.10, (1) of Proposition 6.15, and the definition.

(4) follows from the definitions and (1) of Proposition 6.5.

(5) follows from the definitions and (2) of Proposition 6.5.

(6) follows from the definitions and (5) of Proposition 6.15. ■

6.19. REMARKS. The reason that we used a separate clause in the definition of  $h$  for  $\circ a$  with  $a \in S'$ , in Definition 6.13, is, of course, that without that, (1) of Proposition 6.18, and hence Axiom 3, would not hold for  $*$ , and it would not be true that  $S \subseteq T$ .

Now, if we just had a single extension  $*$ , this would be no big deal. Axiom 3 would not be really necessary, it could be replaced with some injectivity assumption, and we could just work with an isomorphic copy  $*[H_\omega^{S'}]$  of  $H_\omega^{S'}$  in  $H_\omega^{T'}$ , instead of with  $H_\omega^{S'}$  itself.

However, in Theorem 7.6, we prove that an infinite sequence of successive extensions has a "limiting extension", a result we then use in the proof that every superstructure has a  $\kappa$ -saturated extension (where  $\kappa$  is a transfinite cardinal) (Theorem 7.13). The set of "urelements" in this "limiting" superstructure is the union of the sets of "urelements" of all the successive superstructures, and this union is not meaningful to form if the the sequence of sets is not increasing. It is thus difficult to see how any variant of Theorem 7.6 could be proved without Axiom 3.

Therefore, we must have this separate clause in the definition of  $h$ .

The following two propositions will be used in the proof of Theorem 6.22, when we verify that  $*$  satisfies the axioms in Definition 4.4.

6.20. PROPOSITION. *For any  $m \geq 0$ , and any rank consistent function  $f : (H_\omega^{S'})^m \rightarrow H_\omega^{S'} : \text{If } A_1, A_2, \dots, A_m \in H_\omega^{S'} \setminus S$ , then*

$$\begin{aligned} * \{f(a_1, a_1, \dots, a_m) \mid a_1 \in A_1, a_2 \in A_2, \dots, a_m \in A_m\} = \\ \{(*f)(a'_1, a'_2, \dots, a'_m) \mid a'_1 \in *A_1, a'_2 \in *A_2, \dots, a'_m \in *A_m\}. \end{aligned}$$

PROOF. Put

$$E = \{f(a_1, a_1, \dots, a_m) \mid a_1 \in A_1, a_2 \in A_2, \dots, a_m \in A_m\},$$

and

$$F = \{(*f)(a'_1, a'_2, \dots, a'_m) \mid a'_1 \in *A_1, a'_2 \in *A_2, \dots, a'_m \in *A_m\}.$$

Assume first that  $A_i = \emptyset$  for some  $i$  ( $1 \leq i \leq m$ ). Then,  $*A_i = \emptyset$ , by definitions 6.13 and 6.16, and hence  $E = F = \emptyset$ , and  $*E = F$ , by these definitions, again.

Thus, the desired conclusion holds in this case.

Assume, instead, that  $A_i \neq \emptyset$  for all  $i$  ( $1 \leq i \leq m$ ). Then, definitions 6.13 and 6.16, and (2) of Proposition 6.18, give that  $*A_i \neq \emptyset$  for all  $i$  ( $1 \leq i \leq m$ ), so  $E \neq \emptyset$  and  $F \neq \emptyset$ .

Then,  $E \notin S'$  and  $F \notin T'$ , so  $r^{S'}(E) \geq 2$  and  $r^{T'}(F) \geq 2$ . Hence, by (4) of Proposition 6.10,  ${}^\circ r^{S'}({}^\circ E) \geq 2$ .

Now, take  $y' \in *E$ . Then,  $y' = h([y])$  for some  $[y] \in W$  such that  $[y] \in {}^\circ E$ , by definition. This means that we can choose  $[b_1], [b_2], \dots, [b_m] \in W$  such that  $b_1(i) \in A_1, b_2(i) \in A_2, \dots, b_m(i) \in A_m$ , and  $f(b_1(i), b_2(i), \dots, b_m(i)) = y(i)$ , a.e. But then,  $[b_1] \in {}^\circ A_1, [b_2] \in {}^\circ A_2, \dots, [b_m] \in {}^\circ A_m$ , and  $({}^\circ f)([b_1], [b_2], \dots, [b_m]) = [y]$ , by the definitions of  ${}^\circ$ , and furthermore,  $h([b_1]) \in *A_1, h([b_2]) \in *A_2, \dots, h([b_m]) \in *A_m$ , and  $(*f)(h([b_1]), h([b_2]), \dots, h([b_m])) = y'$ , by the definitions and (2) of Proposition 6.15. This means that  $y' \in F$ .

Since this holds for all  $y' \in *E$ ,  $*E \subseteq F$ .

Conversely, take  $z' \in F$ . Then there are  $c'_1 \in *A_1, c'_2 \in *A_1, \dots, c'_m \in *A_m$  such that  $(*f)(c'_1, c'_2, \dots, c'_m) = z'$ .

Now, since  $A_1, A_2, \dots, A_m \notin S'$ , (4) of Proposition 6.10 gives that  ${}^\circ r^{S'}({}^\circ A_j) > 1$  for all  $j$  ( $1 \leq j \leq m$ ).

Then, by the definitions, there are  $[c_1], [c_2], \dots, [c_m] \in W$  such that  $h([c_j]) = c'_j$  and  $[c_j] \in {}^\circ A_j$  for all  $j$  ( $1 \leq j \leq m$ ), and such that, if we put  $[z] = ({}^\circ f)([c_1], [c_2], \dots, [c_m])$ , then  $h([z]) = z'$ . The definitions now give that  $c_1(i) \in A_1, c_2(i) \in A_2, \dots, c_m(i) \in A_m$ , and  $z(i) = f(c_1(i), c_2(i), \dots, c_m(i))$ , a.e. This

means that  $z(i) \in E$  a.e., which, by the definitions and (2) of Proposition 6.15, means that  $[z] \in {}^\circ E$ , and  $z' \in {}^*E$ .

Since this holds for all  $z' \in F$ ,  $F \subseteq {}^*E$ .

Thus, the desired conclusion,  ${}^*E = F$ , holds in this case too. ■

### 6.21. PROPOSITION.

- (1) For any  $m \geq 0$  and any  $k$  ( $1 \leq k \leq m$ ),  $({}^*\pi_k^m)(x'_1, x'_2, \dots, x'_m) = x'_k$ , for all  $x'_1, x'_2, \dots, x'_m \in R_h$ .
- (2) If  $f : (H_\omega^{S'})^2 \rightarrow H_\omega^{S'}$  is given by (for  $x, y \in H_\omega^{S'}$ ):

$$f(x, y) = \begin{cases} x \cup y & \text{if } x, y \notin S \\ \emptyset & \text{if } x \in S \vee y \in S, \end{cases}$$

then  $f$  is rank consistent, and, for all  $x', y' \in R_h$ :

$$({}^*f)(x', y') = \begin{cases} x' \cup y' & \text{if } x', y' \notin T \\ \emptyset & \text{if } x' \in T \vee y' \in T. \end{cases}$$

- (3) (2) remains true if  $x \cup y$  is replaced by  $x \setminus y$  and  $x' \cup y'$  by  $x' \setminus y'$ .
- (4) For any  $m \geq 0$ : If  $f : (H_\omega^{S'})^m \rightarrow H_\omega^{S'}$  is given by  $f(x_1, x_2, \dots, x_m) = \{x_1, x_2, \dots, x_m\}$ , for all  $x_1, x_2, \dots, x_m \in H_\omega^{S'}$ , then  $f$  is rank consistent, and  $({}^*f)(x'_1, x'_2, \dots, x'_m) = \{x'_1, x'_2, \dots, x'_m\}$ , for all  $x'_1, x'_2, \dots, x'_m \in R_h$ .
- (5) For any  $m \geq 0$ : If  $f : (H_\omega^{S'})^m \rightarrow H_\omega^{S'}$  is given by  $f(x_1, x_2, \dots, x_m) = \langle x_1, x_2, \dots, x_m \rangle$ , for all  $x_1, x_2, \dots, x_m \in H_\omega^{S'}$ , then  $f$  is rank consistent, and  $({}^*f)(x'_1, x'_2, \dots, x'_m) = \langle x'_1, x'_2, \dots, x'_m \rangle$ , for all  $x'_1, x'_2, \dots, x'_m \in R_h$ .
- (6) For any  $m \geq 0$  and any  $i$  ( $1 \leq i \leq m$ ): If  $f : (H_\omega^{S'})^1 \rightarrow H_\omega^{S'}$  is given by (for  $y \in H_\omega^{S'}$ ):

$$f(y) = \begin{cases} x_i, & \text{whenever } y \text{ is an } m\text{-tuple: } \langle x_1, x_2, \dots, x_m \rangle, \\ \emptyset & \text{otherwise,} \end{cases}$$

then  $f$  is rank consistent, and  $({}^*f)(y') = x'_i$  if  $y' \in R_h$  is an  $m$ -tuple:  $\langle x'_1, x'_2, \dots, x'_m \rangle$ .

PROOF. (1) follows immediately from the definitions and (3) of Proposition 6.5.

To prove (2), we first show that  $f$  is rank consistent: If  $x, y \in H_n^{S'}$ , for some  $n < \omega$ , and if  $x, y \notin S$ , then  $x, y \subseteq H_{n-1}$ , so  $f(x, y) = x \cup y \subseteq H_{n-1}^{S'}$ , and  $f(x, y) \in H_n^{S'}$ .

If  $x \in S$  or  $y \in S$ , then  $f(x, y) = \emptyset \in H_1^{S'}$ . Thus,  $f$  is rank consistent.

Now, pick  $x', y' \in R_h$ . Then,  $x' = h([x])$  and  $y' = h([y])$ , for some  $[x], [y] \in W$ . Then,  $({}^*f)(x', y') = h(({}^\circ f)([x], [y]))$ , by definition. Assume first that  $x', y' \notin T$ .

By (2) and (5) of Proposition 6.15, neither  $[x] \in {}^\circ S$  nor  $[y] \in {}^\circ S$  holds. Thus, by Proposition 5.6, and (4) of Definitions 5.1,  $x(i), y(i) \notin S$  a.e. Thus,  $f(x(i), y(i)) = x(i) \cup y(i)$  a.e. and since  $S$  is  $\omega$ -grounded,  $x(i) \cup y(i) \notin S$  a.e. Hence, by the definitions, and (2) and (5) of Proposition 6.15,  $\text{not-}({}^\circ f)([x], [y]) \in {}^\circ S$ , and  $({}^*f)(x', y') \notin T$ .

Pick  $z' \in ({}^*f)(x', y')$ . By (6) and (2) of Proposition 6.15,  $z' = h([z])$  for some  $[z] \in W$  such that  $[z] \in ({}^\circ f)([x], [y])$ , which means that  $z(i) \in x(i) \cup y(i)$  a.e.

By Corollary 5.7,  $z(i) \in x(i)$  a.e. or  $z(i) \in y(i)$  a.e. If  $z(i) \in x(i)$  a.e., then  $[z] \circ \in [x]$ , and, by (2) of Proposition 6.15,  $z' \in x'$ . Likewise, if  $z(i) \in y(i)$  a.e., then  $z' \in y'$ . Thus, in both cases,  $z' \in x' \cup y'$ . Since  $z' \in (*f)(x', y')$  was arbitrary,  $(*f)(x', y') \subseteq x' \cup y'$ .

Thus, if  $x', y' \notin T$ , then  $(*f)(x', y') \subseteq x' \cup y'$ .

Conversely, pick  $z' \in x' \cup y'$ . Assume first that  $z' \in x'$ . Then,  $x' \notin T'$ . By (6) and (2) of Proposition 6.15,  $z' = h([z])$  for some  $[z] \in W$  such that  $[z] \circ \in [x]$ , which means that  $z(i) \in x(i)$  a.e. But then,  $z(i) \in x(i) \cup y(i)$  a.e., which means that  $[z] \circ \in ({}^\circ f)([x], [y])$ , and, by the definitions and (2) of Proposition 6.15,  $z' \in (*f)(x', y')$ . If, instead,  $z' \in y'$ , then a similar argument gives  $z' \in (*f)(x', y')$  in this case too. Thus, in both cases,  $z' \in (*f)(x', y')$ . Since  $z' \in x' \cup y'$  was arbitrary,  $x' \cup y' \subseteq (*f)(x', y')$ .

Thus, if  $x', y' \notin T$ , then  $x' \cup y' \subseteq (*f)(x', y')$ .

It follows that,  $(*f)(x', y') = x' \cup y'$ , in the case  $x', y' \notin T$ .

Next, assume that  $x' \in T$ . By (2) and (5) of Proposition 6.15,  $[x] \circ \in {}^\circ S$ , which means that  $x(i) \in S$  a.e. Thus,  $f(x(i), y(i)) = \emptyset$  a.e., which means that  $({}^\circ f)([x], [y]) = {}^\circ \emptyset$ . By the definitions, then,  $(*f)(x', y') = \emptyset$ . If, instead,  $y' \in T$ , the same argument gives  $(*f)(x', y') = \emptyset$  in this case too.

Thus,  $(*f)(x', y') = \emptyset$  if  $x' \in T$  or  $y' \in T$ , which completes the proof of (2).

To prove (3), the rank consistency of  $f$  follows by a similar argument as in (2). So, take  $x', y' \in R_h$ . Again  $x' = h([x])$  and  $y' = h([y])$ , for some  $[x] \in W$  and  $[y] \in W$ .

Assume first that  $x', y' \notin T$ . Then, a similar argument as in (2) gives  $\text{not-}({}^\circ f) \circ \in {}^\circ S$  and  $(*f)(x', y') \notin T$ .

Pick  $z' \in (*f)(x', y')$ . Then, by (6) and (2) of Proposition 6.15,  $z' = h([z])$  for some  $[z] \in W$  such that  $[z] \circ \in ({}^\circ f)([x], [y])$ , which means that  $z(i) \in x(i) \setminus y(i)$  a.e. By (3) of Definitions 5.1,  $z(i) \in x(i)$  a.e. and  $z(i) \notin y(i)$  a.e. This means that  $[z] \circ \in [x]$  and  $\text{not-}[z] \circ \in [y]$ . By (2) of Proposition 6.15,  $z' \in x'$  and  $z' \notin y'$ , that is  $z' \in x' \setminus y'$ . Since  $z' \in (*f)(x', y')$  was arbitrary,  $(*f)(x', y') \subseteq x' \setminus y'$ .

Thus, if  $x', y' \notin T$ , then  $(*f)(x', y') \subseteq x' \setminus y'$ .

Conversely, pick  $z' \in x' \setminus y'$ . Then,  $x' \notin T'$ . By (6) and (2) of Proposition 6.15,  $z' = h([z])$  for some  $[z] \in W$  such that  $[z] \circ \in [x]$  and  $\text{not-}[z] \circ \in [y]$ . Then,  $z(i) \in x(i)$  a.e. and  $z(i) \notin y(i)$  a.e., which, by (4) and (3) of Definitions 5.1, implies that  $z(i) \in x(i) \setminus y(i)$  a.e., that is  $[z] \circ \in ({}^\circ f)([x], [y])$ , and thus, by (2) of Proposition 6.15 and the definitions,  $z' \in (*f)(x', y')$ . Since  $z' \in x' \setminus y'$  was arbitrary,  $x' \setminus y' \subseteq (*f)(x', y')$ .

Thus, if  $x', y' \notin T$ , then  $x' \setminus y' \subseteq (*f)(x', y')$ .

It follows that  $(*f)(x', y') = x' \setminus y'$ , in the case  $x', y' \notin T$ .

If  $x' \in T$  or  $y' \in T$ , the same argument as in (2) gives  $(*f)(x', y') = \emptyset$ , which concludes the proof of (3).

To prove (4), we first consider the case when  $m = 1$ . In this case,  $f$  is rank consistent: if  $x \in H_n^{S'}$ , then  $f(x) = \{x\} \in H_{n+1}^{S'}$ .

So, pick  $x' \in R_h$ . As usual,  $x' = h([x])$  for some  $[x] \in W$ , and  $(*f)(x') = h(({}^\circ f)([x]))$ . If we put  $[u] = ({}^\circ f)([x])$ , then  $u(i) = \{x(i)\}$  a.e. and so  ${}^\circ r^{S'}([u]) \geq 2$ , which gives  $\text{not-}[u] \circ \in {}^\circ S$ , by (3) of Proposition 6.10. Hence,  $(*f)(x') \notin T$ , by (2) and (5) of Proposition 6.15.

Clearly,  $x(i) \in \{x(i)\}$  a.e., so  $[x] \circ \in [u]$ , and, by (2) of Proposition 6.15 and the definition,  $x' \in (*f)(x')$ .

Conversely, if  $y' \in (*f)(x')$ , then, by (6) and (2) of Proposition 6.15,  $y' = h([y])$  for some  $[y] \in W$  such that  $[y]^\circ \in [u]$ , which means that  $y(i) \in \{x(i)\}$  a.e., that is,  $y(i) = x(i)$  a.e. Hence,  $[y] = [x]$  and  $y' = x'$ .

Hence,  $(*f)(x') = \{x'\}$  for all  $x' \in R_h$ , if  $m = 1$ .

We prove the general case by induction on  $m$ . If  $m = 0$ , it follows from the definition that the desired conclusion is just  $*\emptyset = \emptyset$ , which follows from (1) of Proposition 6.18. (Of course, a function of 0 variables with codomain  $H_\omega^{S'}$  is trivially rank consistent.)

Suppose that the proposition is true for some  $m \geq 0$ , and let  $f$  be as in the statement in the proposition, with this  $m$ , and let, and let  $f'$  be as  $f$  in the statement of the proposition, with  $m+1$  instead of  $m$ . We must prove that  $f'$  is rank consistent and that  $(*f')(x'_1, x'_2, \dots, x'_m, x'_{m+1}) = \{x'_1, x'_2, \dots, x'_m, x'_{m+1}\}$ .

Let  $g$  be the function called  $f$  in (2), and let  $q$  be the function called  $f$  in the previous ( $m = 1$ ) case.

Then, we see that

$$\begin{aligned} f'(x_1, x_2, \dots, x_m, x_{m+1}) = \\ g(f(\pi_1^{m+1}(x_1, x_2, \dots, x_m, x_{m+1}), \pi_2^{m+1}(x_1, x_2, \dots, x_m, x_{m+1}), \dots \\ \dots, \pi_m^{m+1}(x_1, x_2, \dots, x_m, x_{m+1})), q(\pi_{m+1}^{m+1}(x_1, x_2, \dots, x_m, x_{m+1}))), \end{aligned}$$

for all  $x_1, x_2, \dots, x_m, x_{m+1} \in H_\omega^{S'}$ .

Now, it follows from the induction hypothesis, the previous case, (1), (2), and several applications of (5) of Proposition 6.18, that  $f'$  has the desired property.

(4) now follows by induction.

To prove (5), we first show that  $f$  is rank consistent: If  $x_i \in H_n^{S'}$  for all  $i$  ( $1 \leq i \leq m$ ), then  $\langle x_1, x_2, \dots, x_m \rangle \in H_{n+3}^{S'}$ . This holds for all  $n < \omega$ . Thus,  $f$  is rank consistent.

Next, we note that  $*j = j$  for every natural number  $j$ . This follows from the fact that  $\mathbb{N} \subseteq S$ , and (1) of Proposition 6.18.

Next: Let each of  $g_1, g_2$ , and  $g_m$  be the function called  $f$  in (4), with 1, 2, and  $m$ , instead of  $m$ , respectively. For each  $j \in \mathbb{N}$ , let  $c_j : (H_\omega^{S'})^m \rightarrow H_\omega^{S'}$  be the constant function such that  $c_j(x_1, x_2, \dots, x_m) = j$ . Clearly, each  $c_j$  is rank consistent, since  $\mathbb{N} \subseteq S' = H_1^{S'}$ , and, by the definitions,  $(^\circ c_j)([x_1], [x_2], \dots, [x_m]) = ^\circ j$ , for all  $[x_1], [x_2], \dots, [x_m] \in W$ , and  $(*c_j)(x'_1, x'_2, \dots, x'_m) = j$ .

Now, for all  $x_1, x_2, \dots, x_m \in H_\omega^{S'}$ :

$$\begin{aligned} f(x_1, x_2, \dots, x_m) = \langle x_1, x_2, \dots, x_m \rangle = \\ g_m(g_2(g_1(c_1(x_1, x_2, \dots, x_m)), g_2(c_1(x_1, x_2, \dots, x_m), \pi_1^m(x_1, x_2, \dots, x_m))), \\ g_2(g_1(c_2(x_1, x_2, \dots, x_m)), g_2(c_2(x_1, x_2, \dots, x_m), \pi_2^m(x_1, x_2, \dots, x_m))), \dots \\ g_2(g_1(c_m(x_1, x_2, \dots, x_m)), g_2(c_m(x_1, x_2, \dots, x_m), \pi_m^m(x_1, x_2, \dots, x_m)))). \end{aligned}$$

(5) now follows from the above observation, several applications of (1) and (4), and several applications of (5) of Proposition 6.18.

To prove (6), let  $f$  satisfy the condition in the statement, for  $i$  ( $1 \leq i \leq m$ ), and let  $g$  be the function called  $f$  in (5).

It is clear that  $f$  is rank consistent: If  $x \in H_n^{S'}$ , then  $f(x) \in H_n^{S'}$  for all  $n < \omega$ .

Now,  $f(g(x_1, x_2, \dots, x_m)) = \pi_i^m(x_1, x_2, \dots, x_m)$ , for all  $x_1, x_2, \dots, x_m \in H_\omega^{S'}$ . Thus, by (1) and (5), and (5) of Proposition 6.18:

$$(*f)(\langle x'_1, x'_2, \dots, x'_m \rangle) = (*f)((*g)(x'_1, x'_2, \dots, x'_m)) = (*\pi_i^m)(x'_1, x'_2, \dots, x'_m) = x'_i,$$

for all  $x'_1, x'_2, \dots, x'_m \in R_h$ .

Now, repeated applications of (6) of Proposition 6.15, using Definitions 2.2, give that if  $\langle x'_1, x'_2, \dots, x'_m \rangle \in R_h$ , then  $x'_i \in R_h$  for all  $i$  ( $1 \leq i \leq m$ ).

It follows that (6) holds. ■

Now, we are ready to prove that  $*$  is a proper nonstandard extension. It is tedious, indeed, to verify that all twelve axioms are satisfied, but that's the way things are!

6.22. THEOREM. *The map  $*$  :  $H_\omega^{S'} \rightarrow H_\omega^{T'}$  is a proper nonstandard extension.*

PROOF.  $T$  is  $\omega$ -extendable and  $T' = T \cup \{\emptyset\} \neq \emptyset$ , by Remarks 6.12. By (3) of Proposition 3.31,  $\emptyset \notin T$ , so  $T = T' \setminus \{\emptyset\}$ . So,  $T$  and  $T'$  have the right properties, according to the first paragraph of Definition 4.4.

We must now verify the twelve axioms of Definition 4.4.

By (6) of Proposition 6.18,  $*S = T$ . Thus, Axiom 1 holds.

To prove Axiom 2, we use that the ultrafilter  $\mathcal{U}$  is countably incomplete.<sup>20</sup> Thus, let  $\{A_j\}_{j=1}^\infty$  be a countably infinite family of sets in  $\mathcal{U}$  such that  $\bigcap_{j=1}^\infty A_j = \emptyset$ . Put  $B_j = \bigcap_{l=1}^j A_l$  for  $j \geq 1$  and put  $B_0 = I$ . Then, by (4) of Definitions 5.1,  $\{B_j\}_{j=0}^\infty$  is a decreasing sequence of sets in  $\mathcal{U}$  with empty intersection. Now, we define  $x \in V$  by putting  $x(i) = \max\{j \mid i \in B_j\}$ . Then, clearly,  $[x]^\circ \in {}^\circ\mathbb{N}$ . But, if  $[x] = {}^\circ j$  for some  $j \in \mathbb{N}$ , then  $x(i) = j$  a.e., which means that  $B_j \setminus B_{j+1} \in \mathcal{U}$ . But since  $B_{j+1} \in \mathcal{U}$ , (4) of Definitions 5.1 implies  $\emptyset \in \mathcal{U}$ , which contradicts (2) of Definitions 5.1. Thus,  $[x] \neq {}^\circ j$  for all  $j \in \mathbb{N}$ . Thus, by the definitions and (2) and (4) of Proposition 6.15,  $h([x]) \in {}^*\mathbb{N}$  but  $h([x]) \neq j$  for all  $j \in \mathbb{N}$ . Thus,  ${}^*\mathbb{N} \neq \mathbb{N}$ , that is, Axiom 2 holds.

Axiom 3 follows from (1) and (2) of Proposition 6.18.

To verify Axiom 4, let  $x' \in *A \setminus T$  for some  $A \in H_\omega^{S'} \setminus S$ , and let  $y' \in x'$ .  $x' \notin *S$ , by (6) of Proposition 6.18. If  $r^{S'}(A) = n$ , then  $r^{T'}(*A) = n$ , by (3) of Proposition 6.18. Also,  $n \geq 1$  and  $A \subseteq H_{n-1}^{S'}$ . Since  $A \notin S$ ,  $\text{not-}^\circ A \in {}^\circ S$ , and likewise  $\text{not-}^\circ H_{n-1}^{S'} \in {}^\circ S$ , by (1) of Proposition 6.7. Hence, by (2) of Proposition 6.7, (3) of Proposition 6.15, and the definition,  $*A \subseteq *H_{n-1}^{S'}$ . Thus,  $x' \in *H_{n-1}^{S'}$ . By (1) of Proposition 6.18,  $*\emptyset = \emptyset$ , so  $H_{n-1}^{S'} \neq \emptyset$ . Hence,  $n \geq 2$  and  $r^{S'}(H_{n-1}^{S'}) \geq 2$ . Hence,  $*H_{n-1}^{S'} \notin T$ , by (3) of Proposition 6.18. Thus, by the definitions, and (4) of Proposition 6.10 and (6) and (2) of Proposition 6.15,  $x' = h([x])$  for some  $[x] \in W$  with  $[x]^\circ \in {}^\circ H_{n-1}^{S'}$ . This means that  $x(i) \in H_{n-1}^{S'}$  a.e. Furthermore, since  $x' \notin *S$ ,  $\text{not-}[x] \in {}^\circ S$ , by (2) of Proposition 6.15. Hence, by Propositions 5.6,  $x(i) \notin S$  a.e. Therefore,  $x(i) \subseteq H_{n-2}^{S'}$  a.e. Since also  $H_{n-2}^{S'} \notin S$ , the definitions and (3) of Proposition 6.15 give  $x' \subseteq *H_{n-2}^{S'}$ . Hence,  $y' \in *H_{n-2}^{S'}$ .

Since  $H_{n-2}^{S'} \in H_\omega^{S'} \setminus S$ , this completes the verification of Axiom 4.

Axiom 5 follows immediately from (3) of Proposition 6.18.

Let now establish the following:

- (\*) For all  $a \in H_\omega^{S'}$  and  $A \in H_\omega^{S'} \setminus S$ :  
 $a \in A$  if and only if  $*a \in *A$ .

<sup>20</sup>This is the only time we use the countable incompleteness of  $\mathcal{U}$ .

This follows from (1) of Proposition 6.7, (2) of Proposition 6.15, and the definitions.

To verify Axiom 6, if  $A, B \in H_\omega^{S'} \setminus S$ , then (\*) and (6) of Proposition 6.18 give  $*A, *B \notin T$ . Axiom 6 follows from this, (2) and (3) Proposition 6.21, and (4) of Proposition 6.18.

To verify Axiom 7, pick  $a, A \in H_\omega^{S'}$  such that  $a \in A$ . Then,  $A \notin S$ . Now, Axiom 7 follows from (\*).

To verify Axiom 8, let  $f$  be the function in (5) of Proposition 6.21, with  $m = 1$ . Then,  $A^1 = \{f(a) \mid a \in A\}$ . Thus, by Proposition 6.20 and (5) of Proposition 6.21,  $*(A^1) = \{(*f)(a') \mid a' \in *A\} = (*A)^1$ .

Thus, Axiom 8 is verified.

To verify Axiom 9, let  $f$  be the function in (5) of Proposition 6.21, with  $m + k$  instead of  $m$ . For each  $i, j$  with  $1 \leq i \leq j$ , let  $g_i^j$  be the function in (6) of Proposition 6.21, with  $j$  instead of  $m$ .

We define  $q : (H_\omega^{S'})^2 \rightarrow H_\omega^{S'}$  by

$$q(x_1, x_2) = f(g_1^m(\pi_1^2(x_1, x_2)), g_2^m(\pi_1^2(x_1, x_2)), \dots, g_m^m(\pi_1^2(x_1, x_2)), \\ g_1^k(\pi_2^2(x_1, x_2)), g_2^k(\pi_2^2(x_1, x_2)), \dots, g_k^k(\pi_2^2(x_1, x_2))).$$

Then  $q(\langle a_1, a_2, \dots, a_m \rangle, \langle b_1, b_2, \dots, b_k \rangle) = \langle a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_k \rangle$ , for all  $a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_k \in H_\omega^{S'}$ , and  $A \times B = \{q(x_1, x_2) \mid x_1 \in A, x_2 \in B\}$ .

Using that  $A$  and  $*A$  are  $m$ -ary relations, and that  $B$  and  $*B$  are  $k$ -ary relations, we obtain from Proposition 6.20 and (1), (5), and (6) of Proposition 6.21, and (5) of Proposition 6.18, that  $*(A \times B) = \{(*q)(x', y') \mid x' \in *A, y' \in *B\} = *A \times *B$ .

Thus, Axiom 9 is verified.

To verify Axiom 10, we must show that  $*(\text{memb } H_n^{S'}) = \text{memb } *H_n^{S'}$ , for all  $n < \omega$ .

First, we note that  $*H_n^{S'} \notin T$  and  $*(\text{memb } H_n^{S'}) \notin T$ , by (\*) and (6) of Proposition 6.18.

Now,  $\text{memb } H_n^{S'} \subseteq (H_n^{S'})^2$ . Also,  $(H_n^{S'})^2 \notin S$  and  $\text{memb } H_n^{S'} \notin S$ . By axioms 8 and 9, who are already verified,  $*((H_n^{S'})^2) = (*H_n^{S'})^2$ . Hence, by (2) of Proposition 6.7 and (3) of Proposition 6.15, and the definitions,  $*(\text{memb } H_n^{S'}) \subseteq (*H_n^{S'})^2$ . Now, let  $x' \in *( \text{memb } H_n^{S'} )$ . Then,  $x' = \langle a', A' \rangle$  for some  $a', A' \in *H_n^{S'}$ . By definition, there are  $[a], [A] \in W$  such that  $h([a]) = a'$  and  $h([A]) = A'$ . Let  $f$  be the function in (5) of Proposition 6.21, with  $m = 2$ . Then  $x' = (*f)(a', A')$  by (5) of Proposition 6.21. Then,  $(*f)(a', A') \in *( \text{memb } H_n^{S'} )$ , which implies that  $(\circ f)([a], [A]) \in \circ(\text{memb } H_n^{S'})$ , by (2) of Proposition 6.15. Hence,  $\langle a(i), A(i) \rangle = f(a(i), A(i)) \in \text{memb } H_n^{S'}$  a.e., so,  $a(i) \in A(i)$  a.e. This means that  $[a] \circ \in [A]$ , and, by (2) of Proposition 6.15,  $a' \in A'$ , so  $x' = \langle a', A' \rangle \in \text{memb } *H_n^{S'}$ . Since  $x' \in *( \text{memb } H_n^{S'} )$  was arbitrary,  $*( \text{memb } H_n^{S'} ) \subseteq \text{memb } *H_n^{S'}$ .

Conversely, assume that  $x' \in \text{memb } *H_n^{S'}$ . Then  $x' = \langle a', A' \rangle$ , for some  $a', A' \in *H_n^{S'}$  with  $a' \in A'$ . By (6) of Proposition 6.15,  $a' = h([a])$  and  $A' = h([A])$  for some  $[a], [A] \in W$ . Then, by (2) of Proposition 6.15,  $[a], [A] \circ \in \circ H_n^{S'}$  and  $[a] \circ \in [A]$ , which means that  $a(i), A(i) \in H_n^{S'}$  a.e. and  $a(i) \in A(i)$  a.e., and, by the (4) of Definitions 5.1,  $f(a(i), A(i)) = \langle a(i), A(i) \rangle \in \text{memb } H_n^{S'}$  a.e., which means that  $(\circ f)([a], [A]) \in \circ(\text{memb } H_n^{S'})$ , and, by (5) of Proposition 6.21, the definition, and (2) of Proposition 6.15,  $x' = \langle a', A' \rangle = (*f)(a', A') \in *( \text{memb } H_n^{S'} )$ . Since  $x' \in \text{memb } *H_n^{S'}$  was arbitrary,  $\text{memb } *H_n^{S'} \subseteq *( \text{memb } H_n^{S'} )$ .

Hence,  $*(\text{memb } H_n^{S'}) = \text{memb } *H_n^{S'}$ , for all  $n < \omega$ , and Axiom 10 is verified.

We now prove the following, from which axioms 11 and 12 follow as special cases:

*If  $\sigma : \{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, k\}$ , and if  $A \subseteq (H_n^{S'})^k$  ( $m, k \geq 0$ ), for some  $n < \omega$ , then*

$$*\{\langle x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(m)} \rangle \mid \langle x_1, x_2, \dots, x_k \rangle \in A\} = \{\langle x'_{\sigma(1)}, x'_{\sigma(2)}, \dots, x'_{\sigma(m)} \rangle \mid \langle x'_1, x'_2, \dots, x'_k \rangle \in *A\}.$$

To prove this, put  $E = \{\langle x_{\sigma(1)}, x_{\sigma(2)}, \dots, x_{\sigma(m)} \rangle \mid \langle x_1, x_2, \dots, x_k \rangle \in A\}$  and  $F = \{\langle x'_{\sigma(1)}, x'_{\sigma(2)}, \dots, x'_{\sigma(m)} \rangle \mid \langle x'_1, x'_2, \dots, x'_k \rangle \in *A\}$ . We must prove that  $*E = F$ .

First, by Axiom 8 and repeated applications of Axiom 9, which are already verified,  $*((H_n^{S'})^k) = (*H_n^{S'})^k$ . (This holds also for  $k = 0$ , since  $*\{\emptyset\} = \{\emptyset\}$ , by the definition, (4) of Proposition 6.21, and (1) of Proposition 6.18.) Since  $A \notin S$  and  $(H_n^{S'})^k \notin S$ , it follows from (2) of Proposition 6.7, the definitions, and (3) of Proposition 6.15, that  $*A \subseteq (*H_n^{S'})^k$ , so  $*A$  is a  $k$ -ary relation.

Let  $f$  be the function in (5) of Proposition 6.21, and for each  $j$  ( $1 \leq j \leq k$ ), let  $g_j$  be the function called  $f$  in (6) of Proposition 6.21, with  $k$  instead of  $m$  and  $j$  instead of  $k$ .

Define  $q : H_\omega^{S'} \rightarrow H_\omega^{S'}$  by  $q(x) = f(g_{\sigma(1)}(x), g_{\sigma(2)}(x), \dots, g_{\sigma(m)}(x))$ , for all  $x \in H_\omega^{S'}$ . Then,  $q(\langle a_1, a_2, \dots, a_k \rangle) = \langle a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(m)} \rangle$  for all  $a_1, a_2, \dots, a_k \in H_\omega^{S'}$ .

Then, it follows from Proposition 6.20, (6) and (5) of Proposition 6.21, and (5) of Proposition 6.18, that  $*E = \{(*q)(x') \mid x' \in *A\} = F$ .

Thus, the claim is proved. In addition, we saw that  $*((H_n^{S'})^k) = (*H_n^{S'})^k$ , for  $k \geq 0$  and  $n < \omega$ .

Axiom 11 follows from this by defining  $\sigma : \{1, 2, \dots, m\} \rightarrow \{1, 2, \dots, m\}$  by  $\sigma(l) = l$  if  $l \neq j$ , ( $1 \leq l \leq m$ ), and  $\sigma(j) = i$ , and putting  $A = (H_n^{S'})^m$ .

Axiom 12 follows by defining  $\sigma : \{1, 2, \dots, m-1\} \rightarrow \{1, 2, \dots, m\}$  by  $\sigma(l) = l$  for all  $l$  ( $1 \leq l \leq m-1$ ), and by noting that  $A \subseteq (H_n^{S'})^m$  for some  $n < \omega$ .

We have now verified all twelve axioms. Hence,  $* : H_\omega^{S'} \rightarrow H_\omega^{T'}$  is a proper nonstandard extension. ■

6.23. DEFINITION. A proper nonstandard extension  $* : H_\omega^{S'} \rightarrow H_\omega^{T'}$  constructed as above, using the ultrafilter  $\mathcal{U}$ , is called an *ultrapower extension of  $H_\omega^{S'}$ , using  $\mathcal{U}$* .

To summarize, we have:

6.24. PROPOSITION. *Assume that the axiom of choice holds.*

*Let  $S, Y \in H$  be sets such that  $S$  contains an  $\mathbb{N}$ -copy,  $S \cap Y = \emptyset$ , and  $S \cup Y$  is  $\omega$ -extendable. Also, let  $\mathcal{U}$  be a countably incomplete ultrafilter on an infinite index set  $I$ .*

*Then, there exists a set  $T \in H$  such that  $T \cap Y = \emptyset$  and such that  $T \cup Y$  and  $T$  are both  $\omega$ -extendable, and there exists an ultrapower extension  $* : H_\omega^{S'} \rightarrow H_\omega^{T'}$  using  $\mathcal{U}$ , where  $S' = S \cup \{\emptyset\}$  and  $T' = T \cup \emptyset$ .*

*In particular, this holds if  $S$  is  $\omega$ -extendable and contains an  $\mathbb{N}$ -copy, and  $Y = \emptyset$ .*

PROOF. By (1) of Proposition 3.31,  $H_\omega^{S'}$  is the superstructure over  $S$ .

Define  $\circ$  and  $W$  as in Definitions 6.2. Proceed as in Remarks 6.12, to obtain  $T$ ,  $T'$ , and  $h_1$  as there. Define  $h$  as in Definition 6.13, and  $*$  as in Definition 6.16.

The conclusion now follows from Remarks 6.12, Theorem 6.22, and Definition 6.23. ■

We also obtain the following characterization of the internal elements:

6.25. PROPOSITION. *Assume that the Axiom of Choice holds.*

*Let  $S$  and  $T$  be sets such that  $\emptyset \notin T$ . Put  $S' = S \cup \{\emptyset\}$  and  $T' = T \cup \{\emptyset\}$ . Let  $\mathcal{U}$  be a countably incomplete ultrafilter on an index set  $I$ , and assume that  $^* : H_\omega^{S'} \rightarrow H_\omega^{T'}$  is an ultrapower extension using  $\mathcal{U}$ .*

*Then, with the terminology of definitions 6.2 and 6.13,  $R_h = \{h([x]) \mid [x] \in W\}$  is the set of all internal elements in  $H_\omega^{T'}$ .*

PROOF. Pick  $[x] \in W$ . By Definitions 6.2, there is an  $n \geq 0$  such that  $x(i) \in H_n^{S'}$  a.e. which means that  $[x] \in {}^\circ H_n^{S'}$ .

Thus, by (2) of Proposition 6.15 and Definitions 6.16,  $h([x]) \in {}^* H_n^{S'}$ . Since  $H_n^{S'} \notin S$ ,  $h([x])$  is internal.

This holds for all  $[x] \in W$ .

Conversely, assume that  $x' \in H_\omega^{T'}$  is internal. This means that  $x' \in {}^* A$ , for some  $A \in H_\omega^{S'} \setminus S$ . Since  ${}^* A \neq \emptyset$ ,  $A \neq \emptyset$ , by (2) of Proposition 4.8, for  ${}^*$ , using Theorem 6.22. Hence,  $r^{S'}(A) \geq 2$ . (4) of Proposition 6.10 and definitions 6.13 and 6.16 then give that  $x' = h([x])$ , for some  $[x] \in W$ .

This holds for all internal  $x' \in H_\omega^{T'}$ , which completes the proof. ■

## CHAPTER 7

### Composite Extensions and Saturation

7.1. REMARKS. In this chapter, we will compose several proper nonstandard extensions, and prove that the result is a proper nonstandard extension. In Proposition 7.3 we have only two successive extensions, while in Proposition 7.4 and Theorem 7.6, we have a possibly infinite totally ordered sequence of successive extensions, and we prove that we can extend all the given superstructures to a superstructure larger than all of these.

We will then use these results to see that we can find a well ordered arbitrarily large, sequence of successive extensions of a given superstructure, such that all successive extensions are ultrapower extensions using the same ultrafilter. (Theorem 7.8)

The main reason for studying these successive extensions is that we can use these to construct a  $\kappa$ -saturated extension, for an arbitrary transfinite cardinal  $\kappa$ . (See Definitions 7.10 and Theorem 7.13). This saturation is a useful property in applications. The greater  $\kappa$ , the better!

We give a few important applications of saturation, such as comprehension (Theorem 7.15), and that every polysaturated extension is an *enlargement*. (See definitions 7.10 and 7.17, Theorem 7.20, and Corollary 7.21.)

7.2. REMARKS. *In this chapter, we will not always use the \*-notation for proper nonstandard extensions. Instead may use the letter  $t$  with and without indices.*

Thus, we may write  $t(A)$ ,  $t_1(A)$ ,  $t_\omega(A)$ ,  $t_\alpha^\beta(A)$ , etc. instead of  $*A$ . This is for notational convenience; the \*-notation is not very practical when dealing with several nonstandard extensions simultaneously.

*Also, whenever we, in this chapter, use "prime notation", such as  $S'$ , we assume that  $S' = S \cup \{\emptyset\}$ , where  $S$  is  $\omega$ -extendable and contains an  $\mathbb{N}$ -copy.*

We keep in mind that this means that  $H_\omega^{S'}$  is the superstructure over  $S$ , and that, since  $\emptyset \notin S$  by (3) of Proposition 3.31,  $S$  and  $S'$  have the properties  $T$  and  $T'$  have, respectively, in the first paragraph of Definitions 4.4.

We also keep in mind that every subset of an  $\omega$ -extendable set is  $\omega$ -extendable, by (1) of Proposition 3.31.

7.3. PROPOSITION. *Let  $S_0$ ,  $S_1$ , and  $S_2$  be  $\omega$ -extendable sets, all containing a common  $\mathbb{N}$ -copy.*

*Assume that  $t_1 : H_\omega^{S_0'} \rightarrow H_\omega^{S_1'}$  and  $t_2 : H_\omega^{S_1'} \rightarrow H_\omega^{S_2'}$  are proper nonstandard extensions.*

*Then,  $t : H_\omega^{S_0'} \rightarrow H_\omega^{S_2'}$ , defined by  $t = t_2 \circ t_1$ , is a proper nonstandard extension.*

PROOF. We must verify that  $t$  satisfies the twelve axioms of Definition 4.4.

To verify the axioms, we will often use that  $t_1(x) \in H_\omega^{S_1'} \setminus S_1$  whenever  $x \in H_\omega^{S_0'} \setminus S_0$ , which follows from (4) of Proposition 4.8 and Proposition 4.11 for  $t_1$ . Sometimes, we must also use (4) of Proposition 4.10, for  $t_1$ . We use these results freely below, without referring to them.

Remembering this, some of the axioms follow immediately, by applying the axiom twice, first for  $t_1$ , then for  $t_2$ .

In this way, axioms 5, 6, 7, 8, 9, and 12 are immediately obtained.

(For example, Axiom 7 is obtained thus: If  $a \in A$ , then  $t_1(a) \in t_1(A)$ , by Axiom 7 for  $t_1$ , and then  $t_2(t_1(a)) \in t_2(t_1(A))$ , by Axiom 7 for  $t_2$ . Hence,  $t(a) \in t(A)$ .)

It remains to verify axioms 1, 2, 3, 4, 10, and 11.

To verify Axiom 1, Proposition 4.11, for  $t_1$  and  $t_2$  respectively, gives  $t_1(S_0) = S_1$  and  $t_2(S_1) = S_2$ . Thus,  $t(S_0) = S_2$ .

Hence, Axiom 1 holds, for  $t$ .

To verify Axiom 2, (3) of Proposition 4.21, applied to  $t_1$  and  $t_2$  respectively, gives  $\mathbb{N} \subset t_1(\mathbb{N})$  and  $\mathbb{N} \subset t_2(\mathbb{N})$  (proper inclusions). Applying (3) of Proposition 4.8 to  $t_2$  and the first of these gives  $t_2(\mathbb{N}) \subseteq t(\mathbb{N})$ . Thus, by the second inclusion above,  $\mathbb{N} \subset t(\mathbb{N})$  (proper inclusion). It follows that Axiom 2 holds, for  $t$ .

Hence, Axiom 2 is verified.

To verify Axiom 3, we first apply (1) of Proposition 4.7 to  $t_1$  and  $t_2$  successively, using (2) of Proposition 4.7 for  $t_1$ , to obtain  $t(a) = a$  for all  $a \in S_0$ . Next, we use (1) of Proposition 4.9 twice and obtain that  $t$  is injective. It follows that Axiom 3 holds, for  $t$ .

Thus, Axiom 3 is verified.

To verify Axiom 4, we apply (1) of Theorem 4.25 to  $t_1$  and obtain that, for every  $n \geq 1$  ( $n < \omega$ ), the following sentence over  $S_1$  is true:

$$(*) \quad (\forall x_1 \in t_1(H_n^{S_0'}) \setminus S_1)(\forall x_2 \in H_n^{S_1'}) \\ (\langle x_2, x_1 \rangle \in \text{memb } H_n^{S_1'} \rightarrow \langle x_2 \rangle \in (t_1(H_n^{S_0'}))^1).$$

Thus, by (3) of the transfer principle, axioms 8 and 10, and Proposition 4.11, for  $t_2$ , the following sentence over  $S_2$  is true, for every  $n \geq 1$  ( $n < \omega$ ):

$$(**) \quad (\forall X_1 \in t(H_n^{S_0'}) \setminus S_2)(\forall X_2 \in t_2(H_n^{S_1'})) \\ (\langle X_2, X_1 \rangle \in \text{memb } t_2(H_n^{S_1'}) \rightarrow \langle x_2 \rangle \in (t(H_n^{S_0'}))^1).$$

Pick  $x \in t(A) \setminus S_2$  for some  $A \in H_\omega^{S_0'} \setminus S_0$ . Now,  $A \in H_n^{S_0'}$  for some  $n \geq 1$ , ( $n < \omega$ ). Then,  $A \subseteq H_{n-1}^{S_0'}$ . By (3) of Proposition 4.8, applied to first to  $t_1$ , then to  $t_2$ ,  $t(A) \subseteq t(H_{n-1}^{S_0'})$ . Hence,  $x \in t(H_{n-1}^{S_0'})$ . By (3) of Proposition 4.9 for  $t_1$  and (3) of Proposition 4.8 for  $t_2$ ,  $t(H_{n-1}^{S_0'}) \subseteq t_2(H_{n-1}^{S_1'})$ , and thus  $x \in t_2(H_{n-1}^{S_1'})$ . Then,  $n \geq 2$ , by (3) of Proposition 4.9, for  $t_2$ . Then, since  $x \notin S_2$ , (1) of Theorem 4.25 and (3) of Proposition 4.8, for  $t_2$  give  $x \subseteq t_2(H_{n-2}^{S_1'}) \subseteq t_2(H_{n-1}^{S_1'})$ .

Next, pick  $y \in x$ . Then  $y \in t_2(H_{n-2}^{S_1'}) \subseteq t_2(H_{n-1}^{S_1'})$ . Now we can apply (\*\*), with  $n-1$  instead of  $n$ , to obtain that  $y \in t(H_{n-2}^{S_0'})$ . This holds for all  $y \in x$ . Also,  $H_{n-2}^{S_0'} \in H_\omega^{S_0'} \setminus S_0$ .

It follows that Axiom 4 holds, for  $t$ .

The following argument will verify both axioms 10 and 11, for  $t$ :

Fix  $n < \omega$ . For Axiom 10, put  $E = \text{memb } H_n^{S_0'}$ ,  $E' = \text{memb } t_1(H_n^{S_0'})$ ,  $E'' = \text{memb } t(H_n^{S_0'})$ ,  $F = \text{memb } H_n^{S_1'}$ , and  $F' = \text{memb } t_2(H_n^{S_1'})$ .

For Axiom 11, put, instead, with  $1 \leq i < j \leq m$ :

$$E = \{\langle x_1, x_2, \dots, x_m \rangle \in (H_n^{S_0'})^m \mid x_i = x_j\}, \\ E' = \{\langle x_1, x_2, \dots, x_m \rangle \in (t_1(H_n^{S_0'}))^m \mid x_i = x_j\}, \\ E'' = \{\langle x_1, x_2, \dots, x_m \rangle \in (t(H_n^{S_0'}))^m \mid x_i = x_j\}, \\ F = \{\langle x_1, x_2, \dots, x_m \rangle \in (H_n^{S_1'})^m \mid x_i = x_j\}, \text{ and} \\ F' = \{\langle x_1, x_2, \dots, x_m \rangle \in (t_2(H_n^{S_1'}))^m \mid x_i = x_j\}.$$

By axioms 10 and 11, respectively, for both  $t_1$  and  $t_2$ , we have  $t_1(E) = E'$  and  $t_2(F) = F'$ , in both cases. We must prove that  $t(E) = E''$ .

With  $m = 2$  for Axiom 10, and  $m$  as above for Axiom 11, we now obtain from (3) of Proposition 4.9, for  $t_1$ , that  $E' = F \cap (t_1(H_n^{S_0'}))^m$ .

By (1) and (3) of Proposition 4.8, and (3) of Proposition 4.10, for  $t_2$ ,  $t(E) = t_2(E') = F' \cap (t(H_n^{S_0'}))^m = E''$ .

Thus, axioms 10 and 11 are verified.

We have now verified that  $t$  satisfies all twelve axioms. Hence,  $t$  is a proper nonstandard extension. ■

Let us now consider an infinite sequence of successive extensions.

The case of most interest to us is when the indices in the set  $J$  are ordinals, but it is not more difficult to prove the next two results for an arbitrary totally ordered set, so let us make that general assumption.

**7.4. PROPOSITION.** *Let  $(J, <)$  be a strictly totally ordered set, with a greatest element  $m$ .*

*Let  $\{S_j\}_{j \in J}$  be an indexed family of  $\omega$ -extendable sets, all containing a common  $\mathbb{N}$ -copy.*

*Assume that there is an doubly indexed family  $\{t_j^k\}_{j, k \in J, j < k}$ , of maps, such that for all  $j, k \in J$ , with  $j < k$ ,  $t_j^k : H_\omega^{S_j'} \rightarrow H_\omega^{S_k'}$  is a proper nonstandard extension.*

*Also, assume that if  $j < k < l$ :  $(j, k, l \in J)$ , then  $t_j^l = t_k^l \circ t_j^k$ .*

*Now, let  $S_r$  ( $r \notin J$ ) be an  $\omega$ -extendable set containing the same  $\mathbb{N}$ -copy, and assume that  $t_m^r : H_\omega^{S_m'} \rightarrow H_\omega^{S_r'}$  is a proper nonstandard extension.*

*Finally, define  $t_j^r = t_m^r \circ t_j^m$  for each  $j < m$  ( $j \in J$ ).*

*Then, for every  $j \in J$ ,  $t_j^r : H_\omega^{S_j'} \rightarrow H_\omega^{S_r'}$  is a proper nonstandard extension, and, for every  $k \in J$  with  $j < k$ ,  $t_j^r = t_k^r \circ t_j^k$ .*

**PROOF.** It is true by assumption that  $t_j^r$  is a proper nonstandard extension if  $j = m$ . If  $j < m$ , ( $j \in J$ ), then  $t_j^r = t_m^r \circ t_j^m$ , and Proposition 7.3 gives that  $t_j^r$  is a proper nonstandard extension. This proves the first part of the proposition.

To obtain the second part, assume that  $j < k$  ( $j, k \in J$ ). If  $m = k$ , then it is true by assumption that  $t_j^r = t_k^r \circ t_j^k$ . If  $k < m$ , then, by the assumptions,  $t_j^r = t_m^r \circ t_j^m = t_m^r \circ t_k^m \circ t_j^k = t_k^r \circ t_j^k$ . This proves the second part of the proposition. ■

**7.5. REMARKS** The previous proposition is rather trivial, and we also see that even if the extension  $t_m^r$  is not given, such an extension must exist: we could just take an ultrapower extension (provided that the axiom of choice holds).

But the next theorem, where we only change the assumptions so that  $J$  does not have a greatest element, is far from trivial. In this theorem, we must construct limiting extensions  $t_j^m$  for each  $j \in J$ , and prove that they have the right properties. Worst of all, we must go through an extremely tedious verification that all the axioms are satisfied, much worse than the axiom verification in the proof of Theorem 6.22.

**7.6. THEOREM.** *Let  $(J, <)$ ,  $\{S_j\}_{j \in J}$  and  $\{t_j^k\}_{j < k, j, k \in J}$ , be as in Proposition 7.4, except that here,  $J$  does not have a greatest element.*

*Let  $m \notin J$ . Put  $S_m = \cup_{j \in J} S_j$ , and assume that  $S_m$  is  $\omega$ -extendable.*

We define the family of maps  $\{t_j^m\}_{j \in J}$ , where  $t_j^m : H_\omega^{S_j'} \rightarrow H_\omega^{S_m'}$  for each  $j \in J$ , recursively on the  $S_j'$ -ranks of  $x \in H_\omega^{S_j'}$ , for all  $j \in J$ , thus:

If  $r^{S_j'}(x) = 1$  (that is, if  $x \in S_j'$ ), then  $t_j^m(x) = x$ .

If  $r^{S_j'}(x) = n > 1$ , we put

$$(*) \quad t_j^m(x) = \bigcup_{k \geq j, k \in J} \{t_k^m(y) \mid y \in t_j^k(x)\},$$

where we put  $t_j^j(x) = x$  for all  $x \in H_\omega^{S_j'}$  ( $j \in J$ ).

Then, for each  $j \in J$ ,  $t_j^m$  is a proper nonstandard extension, and for all  $k > j$  ( $k \in J$ ),  $t_j^m = t_k^m \circ t_j^k$ .

PROOF. First, notice that each  $t_j^m$  ( $j \in J$ ) is well defined. For if  $r^{S_j'}(x) = n > 1$  and  $k \geq j$  ( $k \in J$ ,  $x \in H_\omega^{S_j'}$ ), then  $r^{S_k'}(t_j^k(x)) = n$ , by (2) of Proposition 4.9, for  $t_j^k$ , so if  $y \in t_j^k(x)$  then  $y \in H_\omega^{S_k'}$  with  $r^{S_k'}(y) < n$ , so  $t_k^m(y)$  is already defined at that stage. (This is also true for  $k = j$  if we use the convention above.) We must also prove that  $t_j^m$  maps  $H_\omega^{S_j'}$  into  $H_\omega^{S_m'}$ . This will follow from (vi) below.

We also note that the formula occurring in (\*) holds also for  $x = \emptyset$ , since  $t_j^k(\emptyset) = \emptyset = t_k^m(\emptyset)$ , by definition and (2) of Proposition 4.8, for  $t_j^k$ , and the convention regarding  $k = j$ .

We must also make the following remark: In this proof, we will often use properties of the maps  $t_j^k$ , where  $j \leq k$ ,  $j, k \in J$ . We will then only talk about the case when  $j < k$ . We leave it to the reader to verify that also the identity maps  $t_j^j$  have the right properties. To do this is trivial in all cases, and therefore, we do not mention these cases explicitly below.

Now, we prove simultaneously by induction on  $n = r^{S_j'}(x)$ , for all  $x \in H_\omega^{S_j'}$  and all  $j \in J$ , that:

- (i) If  $n > 1$ , then  $\{t_k^m(y) \mid y \in t_j^k(x)\} \subseteq \{t_l^m(y) \mid y \in t_j^l(x)\}$ , for all  $k, l \in J$  such that  $j \leq k < l$ .
- (ii)  $t_j^m(x) = t_k^m(t_j^k(x))$  for all  $k \geq j$  ( $k \in J$ ).

If  $n = 1$ , there is nothing to prove for (i), and (ii) follows easily from the definitions of  $t_j^m$  and  $t_k^m$ , and (1) of Proposition 4.7 for  $t_j^k$ .

Assume now that  $n > 1$  and that (i) and (ii) hold for  $S_j'$ -ranks less than  $n$ , for all  $j \in J$ .

Pick  $y \in t_j^k(x)$ , for some  $k \geq j$  ( $j, k \in J$ ). Then, as we observed,  $r^{S_k'}(y) < n$ . Then, (ii) can be applied with  $y$  instead of  $x$ ,  $k$  instead of  $j$ , and  $l \geq k$  instead of  $k$ , which gives  $t_k^m(y) = t_l^m(t_k^l(y))$ . By Axiom 7 for  $t_k^l$ , and the assumptions,  $t_k^l(y) \in t_k^l(t_j^k(x)) = t_j^l(x)$ . It follows that  $t_k^m(y) \in \{t_l^m(z) \mid z \in t_j^l(x)\}$ . This holds for all  $y \in t_j^k(x)$ . Hence, (i) holds for all  $x \in H_\omega^{S_j'}$  with  $r^{S_j'}(x) = n$ , for all  $j, k, l \in J$  with  $j \leq k < l$ .

Now, if  $j \leq k$  ( $j, k \in J$ ), we get, using (i) and the definitions,

$$\begin{aligned} t_j^m(x) &= \bigcup_{l \geq j, l \in J} \{t_l^m(y) \mid y \in t_j^l(x)\} = \bigcup_{l \geq k, l \in J} \{t_l^m(y) \mid y \in t_j^l(x)\} = \\ &= \bigcup_{l \geq k, l \in J} \{t_l^m(y) \mid y \in t_k^l(t_j^k(x))\} = t_k^m(t_j^k(x)). \end{aligned}$$

Thus, (ii) holds for all  $x \in H_\omega^{S_j'}$  with  $r^{S_j'}(x) = n$ , for all  $j, k \in J$  with  $j \leq k$ .

By induction, (i) and (ii) hold for all  $x \in H_\omega^{S_j'}$ , for all  $j, k, l \in J$  with  $j \leq k < l$ . Now, (ii) gives  $t_j^m = t_k^m \circ t_j^k$ , for  $j, k \in J$  with  $j \leq k$ .

It remains to show that  $t_j^m$  maps  $H_\omega^{S_j'}$  into  $H_\omega^{S_{m'}}$ , and that it is a proper non-standard extension for each  $j \in J$ . In particular, we must verify the twelve axioms of Definition 4.4.

It is suitable to establish Axiom 7 first: If  $a, A \in H_\omega^{S_j'}$ , and  $a \in A$ , then  $r^{S_j'}(A) >$

1. Now, it follows easily from the definition that  $t_j^m(a) \in t_j^m(A)$ ; just take  $k = j$ .

Thus, Axiom 7 is verified for  $t_j^m$ .

Next, we first prove that the following hold for each  $j \in J$ :

(iii) If  $A, B \in H_\omega^{S_j'} \setminus S_j$  and  $A \subseteq B$ , then  $t_j^m(A) \subseteq t_j^m(B)$ .

(iv) If  $A \subseteq S_j$ , then  $A \subseteq t_j^m(A)$ .

(v) If  $a \in H_\omega^{S_j'}$ , then  $t_j^m(\{a\}) = \{t_j^m(a)\}$ .

(vi) For all  $x \in H_\omega^{S_j'} : t_j^m(x) \in H_\omega^{S_{m'}}$ , with  $r^{S_{m'}}(t_j^m(x)) = r^{S_j'}(x)$ .

Thus,  $t_j^m$  maps  $H_\omega^{S_j'}$  into  $H_\omega^{S_{m'}}$ .

To prove (iii), let  $z \in t_j^m(A)$ . By definition, there is a  $k \geq j$  ( $k \in J$ ) and a  $y \in t_j^k(A)$  such that  $z = t_k^m(y)$ . By (3) of Proposition 4.8 for  $t_j^k$ ,  $t_j^k(A) \subseteq t_j^k(B)$ , so  $y \in t_j^k(B)$ . Then,  $z = t_k^m(y) \in t_k^m(t_j^k(B)) = t_j^m(B)$ , by Axiom 7, for  $t_j^m$ , and (ii). Since this holds for all  $z \in t_j^m(A)$ ,  $t_j^m(A) \subseteq t_j^m(B)$ . Hence, (iii) holds.

To obtain (iv), Applying the definition and Axiom 7, for  $t_j^m$ , to  $A \subseteq S_j$  and  $x \in A$ , gives  $x = t_j^m(x) \in t_j^m(A)$ . This holds for all  $x \in A$ , so  $A \subseteq t_j^m(A)$ .

Hence, (iv) holds.

To obtain (v), if  $z \in t_j^m(\{a\})$ , then there is a  $k \geq j$  ( $k \in J$ ) and a  $y \in t_j^k(\{a\})$  such that  $z = t_k^m(y)$ . Then, by (2) of Proposition 4.10 for  $t_j^k$ ,  $y = t_j^k(a)$ , and, by (ii),  $z = t_j^m(a)$ . Also, Axiom 7, for  $t_j^m$ , gives that  $t_j^m(a) \in t_j^m(\{a\})$ . Thus,  $t_j^m(\{a\}) = \{t_j^m(a)\}$ . Hence, (v) holds.

We prove (vi) by induction on  $n = r^{S_j'}(x)$ , for all  $j \in J$  and all  $x \in H_\omega^{S_j'}$ .

If  $n = 1$ , then the definition gives  $t_j^m(x) = x \in S_j' \subseteq S_{m'}$ , and  $r^{S_{m'}}(t_j^m(x)) = r^{S_{m'}}(x) = 1$ , so (vi) holds in this case.

Now assume that  $n > 1$ , and that (vi) holds for all  $p < n$ . Let  $z \in t_j^m(x)$ . Then, there is a  $k \geq j$ , ( $k \in J$ ) and a  $y \in t_j^k(x)$  such that  $t_k^m(y) = z$ . By (2) of Proposition 4.9 for  $t_j^k$ ,  $r^{S_{k'}}(t_j^k(x)) = n$ , and hence  $r^{S_{k'}}(y) \leq n - 1$ . Thus, by the induction hypothesis (with  $k$  instead of  $j$ ),  $z = t_k^m(y) \in H_\omega^{S_{m'}}$  and  $r^{S_{m'}}(z) \leq n - 1$ . This holds for all  $z \in t_j^m(x)$ , so  $x \in H_\omega^{S_{m'}}$ , with  $r^{S_{m'}}(t_j^m(x)) \leq n$ .

Also, there is a  $y \in x$  such that  $r^{S_j'}(y) = n - 1$ . Then, by Axiom 7, for  $t_j^m$ ,  $t_j^m(y) \in t_j^m(x)$ , with  $r^{S_{m'}}(t_j^m(y)) = n - 1$ , by the induction hypothesis. Thus,  $r^{S_{m'}}(t_j^m(x)) \geq n$ .

Thus,  $r^{S_{m'}}(t_j^m(x)) = n$ .

(vi) now follows by induction.

In particular, we have now proved that  $t_j^m$  maps  $H_\omega^{S_j'}$  into  $H_\omega^{S_{m'}}$ , for each  $j \in J$ .

Now, let us verify that each  $t_j^m$  ( $j \in J$ ) satisfies the remaining eleven axioms of Definition 4.4.

To obtain Axiom 1: For each  $k \geq j$  ( $k \in J$ ),  $t_j^k(S_j) = S_k$ , by Proposition 4.11 for  $t_j^k$ . From this and the definition, it follows easily that if  $z \in t_j^m(S_j)$ , then  $z \in S_k$  for some  $k \geq j$ . Hence,  $t_j^m(S_j) \subseteq S_m$ .

Conversely, for all  $k \geq j$  ( $k \in J$ ), the definition and (ii), together with  $t_j^k(S_j) = S_k$ , give  $t_j^m(S_j) = t_k^m(S_k)$ . Then (iv), for  $t_k^m$ , gives  $S_k \subseteq t_j^m(S_j)$ . Also, by (2) of Proposition 4.7, for  $t_l^j$ ,  $S_l \subseteq S_j$ , for all  $l \leq j$  ( $l \in J$ ). It follows then from the definitions that  $S_m = \cup_{k \in J} S_k \subseteq t_j^m(S_j)$ .

Thus,  $t_j^m(S_j) = S_m$ .

It follows that Axiom 1 holds.

To obtain Axiom 2, pick  $k \in J$  such that  $j < k$ . Then, it follows from (3) of Proposition 4.21, (3) of Proposition 4.8, and Proposition 4.11, all for  $t_j^k$ , that  $\mathbb{N} \subset t_j^k(\mathbb{N}) \subseteq t_j^k(S_j) = S_k$ , the first inclusion being proper. Thus, applying (iv) for  $t_k^m$ , together with (ii), gives  $t_j^k(\mathbb{N}) \subseteq t_j^m(\mathbb{N})$ . Combining these inclusions gives  $\mathbb{N} \subset t_j^m(\mathbb{N})$  (proper inclusion). Thus, Axiom 2 holds.

To obtain Axiom 3, the definition gives  $t_j^m(x) = x$  for all  $x \in S_j$ , and  $t_j^m(\emptyset) = \emptyset \notin S_m$ . If  $x \in H_\omega^{S_j'} \setminus S_j'$ , then  $r^{S_j'}(x) > 1$ . Hence, by (vi),  $r^{S_m'}(t_j^m(x)) > 1$ . Since  $S_j \subseteq S_m$ , all this means that Axiom 3 holds.

To verify Axiom 4, assume that  $x \in t_j^m(A) \setminus S_m$  for some  $A \in H_\omega^{S_j} \setminus S_j$ . By the definitions, this means that there is a  $k \geq j$ , ( $k \in J$ ), and a  $z \in t_j^k(A) \setminus S_k$  such that  $x = t_k^m(z)$ . Now,  $A \subseteq H_n^{S_j'}$ , for some  $n \geq 1$  ( $n < \omega$ ). Thus, by (3) of Proposition 4.8, for  $t_j^k$ ,  $z \in t_j^k(H_n^{S_j'})$ , and, by (1) of Theorem 4.25 for  $t_j^k$ ,  $z \subseteq t_j^k(H_{n-1}^{S_j'})$ . Since  $H_{n-1}^{S_j'} \in H_\omega^{S_j'} \setminus S_j'$ , Axiom 7 and Proposition 4.11, both for  $t_j^k$ , give  $t_j^k(H_{n-1}^{S_j'}) \setminus S_k$ .

Then, (ii), and (iii) for  $t_k^m$ , give  $x \subseteq t_j^m(H_{n-1}^{S_j'})$ .

Thus, if  $y \in x$ , then  $y \in t_j^m(H_{n-1}^{S_j'})$ . It follows that Axiom 4 is holds for  $t_j^m$ .

Axiom 5 is contained in (vi).

Before we verify Axiom 6, we prove the following:

(vii) *Each  $t_j^m$  ( $j \in J$ ) is an injective map.*

To prove (vii), we prove that, for all  $j \in J$  and all  $a, b \in H_\omega^{S_j'}$ ,  $t_j^m(a) = t_j^m(b)$  implies  $a = b$ .

We prove this by induction on  $n = r^{S_j'}(a)$ , noting that if  $t_j^m(a) = t_j^m(b)$ , then  $r^{S_j'}(a) = r^{S_j'}(b)$ , by (vi).

For  $n = 1$ , then, the implication holds by definition.

Assume now that  $n > 1$  and that the implication holds for all  $p < n$ , and assume that  $t_j^m(a) = t_j^m(b)$ . Now, pick  $x \in a$ . Then, by Axiom 7 for  $t_j^m$ ,  $t_j^m(x) \in t_j^m(a)$ . Hence,  $t_j^m(x) \in t_j^m(b)$ , which, by definition, means that  $t_j^m(x) = t_k^m(y)$ , for some  $k \geq j$  ( $k \in J$ ), and some  $y \in t_j^k(b)$ . Since  $r^{S_j'}(a) = n$ ,  $r^{S_j'}(x) < n$ . Hence, by (2) of Proposition 4.9 for  $t_j^k$ ,  $r^{S_k'}(t_j^k(x)) < n$ . Now, by (ii),  $t_k^m(t_j^k(x)) = t_k^m(y)$ . Hence, by the induction hypothesis for  $t_k^m$ ,  $t_j^k(x) = y$ . Hence,  $t_j^k(x) \in t_j^k(b)$ . Since  $n > 1$ ,  $b \notin S_j$ , so (4) of Proposition 4.8, for  $t_j^k$ , gives  $x \in b$ . This holds for all  $x \in a$ , so  $a \subseteq b$ .

By an identical argument,  $b \subseteq a$ . Thus,  $a = b$ .

By induction, the implication holds for all  $a, b \in H_\omega^{S_j'}$ .

Hence, (vii) holds.

Now, let us verify Axiom 6 for each  $t_j^m$  ( $j \in J$ ).

Let  $A, B \in H_\omega^{S_j} \setminus S_j$ . We also assume that  $A \neq \emptyset$  and  $B \neq \emptyset$ , because otherwise, all of Axiom 6 holds trivially, since  $t_j^m(\emptyset) = \emptyset$ .

Then, by definition and Axiom 6 for each  $t_j^k$  ( $k \geq j$ ,  $k \in J$ ),

$$\begin{aligned} t_j^m(A \cup B) &= \cup_{k \geq j, k \in J} \{t_k^m(y) \mid y \in t_j^k(A \cup B)\} = \\ &\quad \cup_{k \geq j, k \in J} \{t_k^m(y) \mid y \in t_j^k(A) \cup t_j^k(B)\} = \\ &\quad \cup_{k \geq j, k \in J} (\{t_k^m(y) \mid y \in t_j^k(A)\} \cup \{t_k^m(y) \mid y \in t_j^k(B)\}) = \\ (\cup_{k \geq j, k \in J} \{t_k^m(y) \mid y \in t_j^k(A)\}) \cup (\cup_{k \geq j, k \in J} \{t_k^m(y) \mid y \in t_j^k(B)\}) &= t_j^m(A) \cup t_j^m(B). \end{aligned}$$

Thus, the first part of Axiom 6 is verified for  $t_j^m$ .

To verify the second part of Axiom 6, pick  $z \in t_j^m(A \setminus B)$ . By (iii),  $z \in t_j^m(A)$ . Assume, to get a contradiction, that  $z \in t_j^m(B)$ . Then, there are  $k, l \geq j$  ( $k, l \in J$ ), and  $y_1 \in t_j^k(A \setminus B)$ ,  $y_2 \in t_j^l(B)$ , such that  $z = t_k^m(y_1) = t_l^m(y_2)$ . Let  $p = \max_J(k, l)$ . Then, by (ii),  $z = t_p^m(t_k^p(y_1)) = t_p^m(t_l^p(y_2))$ . Thus, by (vii) for  $t_p^m$ ,  $t_k^p(y_1) = t_l^p(y_2)$ . By Axiom 7 for  $t_k^p$  and  $t_l^p$ , our assumptions, and Axiom 6 for  $t_j^p$ , we obtain  $t_k^p(y_1) \in t_k^p(t_j^p(A \setminus B)) = t_j^p(A \setminus B) = t_j^p(A) \setminus t_j^p(B)$ , and  $t_l^p(y_2) \in t_l^p(t_j^p(B)) = t_j^p(B)$ . Since  $t_k^p(y_1) = t_l^p(y_2)$ , we have reached a contradiction. It follows that  $z \in t_j^m(A) \setminus t_j^m(B)$ . Since this holds for all  $z \in t_j^m(A \setminus B)$ ,  $t_j^m(A \setminus B) \subseteq t_j^m(A) \setminus t_j^m(B)$ .

Conversely, assume that  $z \in t_j^m(A) \setminus t_j^m(B)$ . Then  $z \in t_j^m(A)$ . By definition, then, there is a  $k \geq j$  ( $k \in J$ ), and a  $y \in t_j^k(A)$  such that  $z = t_k^m(y)$ . Then, if  $y \in t_j^k(B)$ , Axiom 7, for  $t_k^m$ , and (ii), give  $z = t_k^m(y) \in t_k^m(t_j^k(B)) = t_j^m(B)$ , which was not the case. Hence  $y \notin t_j^k(B)$ , and, by Axiom 6 for  $t_j^k$ ,  $y \in t_j^k(A \setminus B)$ . Hence, by Axiom 7 for  $t_k^m$ , and (ii),  $z \in t_k^m(t_j^k(A \setminus B)) = t_j^m(A \setminus B)$ . Since this holds for all  $z \in t_j^m(A) \setminus t_j^m(B)$ ,  $t_j^m(A) \setminus t_j^m(B) \subseteq t_j^m(A \setminus B)$ .

Hence,  $t_j^m(A \setminus B) = t_j^m(A) \setminus t_j^m(B)$ .

Now, all of Axiom 6 is verified for  $t_j^m$ .

Now, by several applications of (v), Axiom 6, and the definitions, we obtain the following:

If  $j \in J$ ,  $q \geq 0$  and  $a_1, a_2, \dots, a_q \in H_\omega^{S_j'}$ , then

(viii)  $t_j^m(\{a_1, a_2, \dots, a_q\}) = \{t_j^m(a_1), t_j^m(a_2), \dots, t_j^m(a_q)\}$ , and

(ix)  $t_j^m(\langle a_1, a_2, \dots, a_q \rangle) = \langle t_j^m(a_1), t_j^m(a_2), \dots, t_j^m(a_q) \rangle$ .

(For  $q = 0$ , this follows immediately from the definitions.)

Axiom 7 is already verified.

To verify Axiom 8 for  $t_j^m$ , let  $A \in H_\omega^{S_j'} \setminus S_j$ , and assume that  $z \in t_j^m(A^1)$ . Then, by definition, there is a  $k \geq j$  ( $k \in J$ ) and a  $y \in t_j^k(A^1)$  such that  $z = t_k^m(y)$ . By Axiom 8 for  $t_j^k$ ,  $t_j^k(A^1) = (t_j^k(A))^1$ , so  $y = \langle x \rangle$  for some  $x \in t_j^k(A)$ . By (ix) for  $t_k^m$ ,  $z = \langle t_k^m(x) \rangle$ . Hence, by Axiom 7 for  $t_j^m$ , and (ii),  $z \in (t_k^m(t_j^k(A)))^1 = (t_j^m(A))^1$ . This holds for all  $z \in t_j^m(A^1)$ , so  $t_j^m(A^1) \subseteq (t_j^m(A))^1$ .

Conversely, assume that  $z \in (t_j^m(A))^1$ . Then  $z = \langle u \rangle$ , for some  $u \in t_j^m(A)$ . Then, there is a  $k \geq j$  ( $k \in J$ ) and an  $x \in t_j^k(A)$  such that  $u = t_k^m(x)$ . By Axiom 8 for  $t_j^k$ ,  $\langle x \rangle \in (t_j^k(A))^1 = t_j^k(A^1)$ . Now, by (ix) for  $t_k^m$ ,  $t_k^m(\langle x \rangle) = \langle u \rangle = z$ , and thus, by Axiom 7 for  $t_k^m$ , and (ii),  $z \in t_k^m(t_j^k(A^1)) = t_j^m(A^1)$ . This holds for all  $z \in (t_j^m(A))^1$ , so  $(t_j^m(A))^1 \subseteq t_j^m(A^1)$ .

Thus,  $t_j^m(A^1) = (t_j^m(A))^1$ , so Axiom 8 is verified for  $t_j^m$ .

To verify Axiom 9 for  $t_j^m$ , take  $A, B \in H_\omega^{S_j'}$ , and assume that  $A$  and  $t_j^m(A)$  are  $q$ -ary relations, and that  $B$  and  $t_j^m(B)$  are  $r$ -ary relations, where  $q, r \geq 0$ .

If  $q = 0$  or  $r = 0$ , then it follows immediately that  $t_j^m(A \times B) = t_j^m(A) \times t_j^m(B)$ , since  $t_j^m(\emptyset) = \emptyset$ , by definition, and  $t_j^m(\{\emptyset\}) = \{\emptyset\}$ , by this and (v).

So, assume that  $q, r \geq 1$ .

Pick  $z \in t_j^m(A \times B)$ . Then, there is a  $k \geq j$  ( $k \in J$ ) and a  $y \in t_j^k(A \times B)$  such that  $z = t_k^m(y)$ . Now, by (4) of Proposition 4.10 and Axiom 9, for  $t_j^k$ ,  $t_j^k(A)$  is a  $q$ -ary relation,  $t_j^k(B)$  is an  $r$ -ary relation, and  $t_j^k(A \times B) = t_j^k(A) \times t_j^k(B)$ . It follows that there is a  $q$ -tuple  $\langle u_1, u_2, \dots, u_q \rangle \in t_j^k(A)$  and an  $r$ -tuple  $\langle v_1, v_2, \dots, v_r \rangle \in t_j^k(B)$  such that  $y = \langle u_1, u_2, \dots, u_q, v_1, v_2, \dots, v_r \rangle$ . By (ix) and Axiom 7 for  $t_k^m$ , and (ii), we get  $\langle t_k^m(u_1), t_k^m(u_2), \dots, t_k^m(u_q) \rangle = t_k^m(\langle u_1, u_2, \dots, u_q \rangle) \in t_k^m(t_j^k(A)) = t_j^m(A)$ , and, by an identical argument,  $\langle t_k^m(v_1), t_k^m(v_2), \dots, t_k^m(v_r) \rangle \in t_j^m(B)$ . Since, again by (ix) for  $t_k^m$ ,  $z = t_k^m(y) = \langle t_k^m(u_1), t_k^m(u_2), \dots, t_k^m(u_q), t_k^m(v_1), t_k^m(v_2), \dots, t_k^m(v_r) \rangle$ , we obtain  $z \in t_j^m(A) \times t_j^m(B)$ . This holds for all  $z \in t_j^m(A \times B)$ , so  $t_j^m(A \times B) \subseteq t_j^m(A) \times t_j^m(B)$ .

Conversely, assume that  $z \in t_j^m(A) \times t_j^m(B)$ . Then, there is a  $\langle z_1, z_2, \dots, z_q \rangle \in t_j^m(A)$  and a  $\langle w_1, w_2, \dots, w_r \rangle \in t_j^m(B)$  such that  $z = \langle z_1, z_2, \dots, z_q, w_1, w_2, \dots, w_r \rangle$ . Then, there are  $k, l \geq j$ , ( $k, l \in J$ ), and  $x \in t_j^k(A)$ ,  $y \in t_j^l(B)$ , such that  $\langle z_1, z_2, \dots, z_q \rangle = t_k^m(x)$  and  $\langle w_1, w_2, \dots, w_r \rangle = t_l^m(y)$ . Let  $p = \max_J(k, l)$ , and put  $u = t_k^p(x) \in t_j^p(A)$ ,  $v = t_l^p(y) \in t_j^p(B)$ , using Axiom 7 for  $t_k^p$  and  $t_l^p$ , and the assumptions. By (4) of Proposition 4.10 for  $t_j^p$ , there are  $u_1, u_2, \dots, u_q, v_1, v_2, \dots, v_r \in H_\omega^{S_j'}$ , such that  $u = \langle u_1, u_2, \dots, u_q \rangle$  and  $v = \langle v_1, v_2, \dots, v_r \rangle$ . Now, by (ix) for  $t_p^m$ , and (ii),  $\langle t_p^m(u_1), t_p^m(u_2), \dots, t_p^m(u_q) \rangle = t_p^m(u) = t_p^m(t_k^p(x)) = t_k^m(x) = \langle z_1, z_2, \dots, z_q \rangle$ , and, by an identical argument  $\langle t_p^m(v_1), t_p^m(v_2), \dots, t_p^m(v_r) \rangle = \langle w_1, w_2, \dots, w_r \rangle$ . Also, by Axiom 9 for  $t_j^p$ ,  $\langle u_1, u_2, \dots, u_q, v_1, v_2, \dots, v_r \rangle \in t_j^p(A) \times t_j^p(B) = t_j^p(A \times B)$ . By (ix) and Axiom 7, for  $t_p^m$ , and (ii):

$$\begin{aligned} z &= \langle t_p^m(u_1), t_p^m(u_2), \dots, t_p^m(u_q), t_p^m(v_1), t_p^m(v_2), \dots, t_p^m(v_r) \rangle = \\ &= t_p^m(\langle u_1, u_2, \dots, u_q, v_1, v_2, \dots, v_r \rangle) \in t_p^m(t_j^p(A \times B)) = t_j^m(A \times B). \end{aligned}$$

This holds for all  $z \in t_j^m(A) \times t_j^m(B)$ , so  $t_j^m(A) \times t_j^m(B) \subseteq t_j^m(A \times B)$ .

It follows that  $t_j^m(A \times B) = t_j^m(A) \times t_j^m(B)$ . Thus, Axiom 9 is verified for  $t_j^m$ .

To verify Axiom 10, fix  $n < \omega$ ,

Pick  $z \in t_j^m(\text{memb } H_n^{S_j'})$ . Then, there is a  $k \geq j$  ( $k \in J$ ) and a  $y \in t_j^k(\text{memb } H_n^{S_j'})$  such that  $z = t_k^m(y)$ . By Axiom 10 for  $t_j^k$ , there are  $b, B \in t_j^k(H_n^{S_j'})$  such that  $y = \langle b, B \rangle$  and  $b \in B$ . By (ix) for  $t_k^m$ ,  $z = \langle t_k^m(b), t_k^m(B) \rangle$ . By Axiom 7 for  $t_k^m$ , and (ii),  $t_k^m(b) \in t_k^m(B)$  and  $t_k^m(b), t_k^m(B) \in t_k^m(t_j^k(H_n^{S_j'})) = t_j^m(H_n^{S_j'})$ . It follows that  $z \in \text{memb } t_j^m(H_n^{S_j'})$ . This holds for all  $z \in t_j^m(\text{memb } H_n^{S_j'})$ , so  $t_j^m(\text{memb } H_n^{S_j'}) \subseteq \text{memb } t_j^m(H_n^{S_j'})$ .

Conversely, pick  $z \in \text{memb } t_j^m(H_n^{S_j'})$ . Then,  $z = \langle a, A \rangle$ , for some  $a, A \in t_j^m(H_n^{S_j'})$  such that  $a \in A$ . Then, there are  $k, l \geq j$  ( $k, l \in J$ ), and  $b \in t_j^k(H_n^{S_j'})$ ,  $B \in t_j^l(H_n^{S_j'})$ , such that  $a = t_k^m(b)$  and  $A = t_l^m(B)$ . Put  $p = \max_J(k, l)$ . Then, by Axiom 7 for  $t_k^p$  and  $t_l^p$ , and the assumptions,  $t_k^p(b) \in t_k^p(t_j^k(H_n^{S_j'})) =$

$t_j^p(H_n^{S_{j'}})$  and  $t_l^p(B) \in t_l^p(t_j^l(H_n^{S_{j'}})) = t_j^p(H_n^{S_{j'}})$ . Assume, to get a contradiction, that  $t_k^p(b) \notin t_l^p(B)$ . Then  $t_k^p(b) \in t_j^p(H_n^{S_{j'}}) \setminus t_l^p(B)$ . By (ii), and axioms 7 and 6 for  $t_p^m$ , which are already verified, this implies that  $a = t_k^m(b) = t_p^m(t_k^p(b)) \in t_p^m(t_j^p(H_n^{S_{j'}})) \setminus t_p^m(t_l^p(B)) = t_j^m(H_n^{S_{j'}}) \setminus t_l^m(B) = t_j^m(H_n^{S_{j'}}) \setminus A$ , and hence  $a \notin A$ , which is a contradiction. Thus,  $t_k^p(b) \in t_l^p(B)$ . It follows by Axiom 10 for  $t_j^p$ , applied backwards, that  $\langle t_k^p(b), t_l^p(B) \rangle \in t_j^p(\text{memb } H_n^{S_{j'}})$ . By (ii), and (ix) and Axiom 7, for  $t_p^m$ , it follows that  $z = \langle a, A \rangle = \langle t_k^m(b), t_l^m(B) \rangle = \langle t_p^m(t_k^p(b)), t_p^m(t_l^p(B)) \rangle = t_p^m(\langle t_k^p(b), t_l^p(B) \rangle) \in t_p^m(t_j^p(\text{memb } H_n^{S_{j'}})) = t_j^m(\text{memb } H_n^{S_{j'}})$ . This holds for all  $z \in \text{memb } t_j^m(H_n^{S_{j'}})$ , so  $\text{memb } t_j^m(H_n^{S_{j'}}) \subseteq t_j^m(\text{memb } H_n^{S_{j'}})$ .

It follows that  $t_j^m(\text{memb } H_n^{S_{j'}}) = \text{memb } t_j^m(H_n^{S_{j'}})$ , and thus Axiom 10 is verified for  $t_j^m$ .

To verify Axiom 11 for  $t_j^m$ , fix  $q, r, s$  so that  $1 \leq q < r \leq s$  and fix  $n < \omega$ . Put

$$E = \{\langle a_1, a_2, \dots, a_s \rangle \in (H_n^{S_{j'}})^s \mid a_q = a_r\}, \text{ and}$$

$$F = \{\langle a_1, a_2, \dots, a_s \rangle \in (t_j^m(H_n^{S_{j'}}))^s \mid a_q = a_r\}.$$

We must prove that  $t_j^m(E) = F$ .

So, pick  $z \in t_j^m(E)$ . Then, there is a  $k \geq j$  ( $k \in J$ ) and a  $y \in t_j^k(E)$  such that  $z = t_k^m(y)$ . By Axiom 11 for  $t_j^k$ , there are  $b_1, b_2, \dots, b_s \in t_j^k(H_n^{S_{j'}})$  such that  $y = \langle b_1, b_2, \dots, b_s \rangle$  and  $b_q = b_r$ . By (ix) for  $t_k^m$ ,  $z = \langle t_k^m(b_1), t_k^m(b_2), \dots, t_k^m(b_s) \rangle$ . Then,  $t_k^m(b_q) = t_k^m(b_r)$ . By Axiom 7 for  $t_k^m$ , and (ii),  $t_k^m(b_i) \in t_k^m(t_j^k(H_n^{S_{j'}})) = t_j^m(H_n^{S_{j'}})$  for all  $i$  ( $1 \leq i \leq s$ ). It follows that  $z \in F$ . This holds for all  $z \in t_j^m(E)$ , so  $t_j^m(E) \subseteq F$ .

Conversely, pick  $z \in F$ . Then,  $z = \langle a_1, a_2, \dots, a_s \rangle$ , for some  $a_1, a_2, \dots, a_s \in t_j^m(H_n^{S_{j'}})$  such that  $a_q = a_r$ . Then, there are  $k_1, k_2, \dots, k_s \geq j$ , ( $k_i \in J$ ,  $1 \leq i \leq s$ ), and  $b_i \in t_j^{k_i}(H_n^{S_{j'}})$ , such that  $a_i = t_{k_i}^m(b_i)$ , for all  $i$  ( $1 \leq i \leq s$ ). Put  $p = \max_J(k_1, k_2, \dots, k_s)$ . Then, by Axiom 7 for  $t_{k_i}^p$ ,  $t_{k_i}^p(b_i) \in t_{k_i}^p(t_j^{k_i}(H_n^{S_{j'}})) = t_j^p(H_n^{S_{j'}})$ , ( $1 \leq i \leq s$ ). By (ii),  $t_p^m(t_{k_q}^p(b_q)) = t_{k_q}^m(b_q) = a_q = a_r = t_{k_r}^m(b_r) = t_p^m(t_{k_r}^p(b_r))$ . By (vii) for  $t_p^m$ ,  $t_{k_q}^p(b_q) = t_{k_r}^p(b_r)$ . Now, it follows by Axiom 11 for  $t_j^p$ , applied backwards, that  $\langle t_{k_1}^p(b_1), t_{k_2}^p(b_2), \dots, t_{k_s}^p(b_s) \rangle \in t_j^p(E)$ . By (ii), and (ix) and Axiom 7, for  $t_p^m$ , it follows that  $z = \langle a_1, a_2, \dots, a_s \rangle = \langle t_{k_1}^m(b_1), t_{k_2}^m(b_2), \dots, t_{k_s}^m(b_s) \rangle = \langle t_p^m(t_{k_1}^p(b_1)), t_p^m(t_{k_2}^p(b_2)), \dots, t_p^m(t_{k_s}^p(b_s)) \rangle = t_p^m(\langle t_{k_1}^p(b_1), t_{k_2}^p(b_2), \dots, t_{k_s}^p(b_s) \rangle) \in t_p^m(t_j^p(E)) = t_j^m(E)$ . This holds for all  $z \in F$ , so  $F \subseteq t_j^m(E)$ .

It follows that  $t_j^m(E) = F$ , and thus Axiom 11 is verified for  $t_j^m$ .

To verify Axiom 12 for  $t_j^m$ , let  $E \in H_\omega^{S_{j'}}$ , and assume that  $E$  and  $t_j^m(E)$  are  $q+1$ -ary relations, for some  $q \geq 0$ . We must prove that  $t_j^m(\pi_{q+1}^{q+1}[E]) = \pi_{q+1}^{q+1}[t_j^m(E)]$ .

First, there is an  $n < \omega$  such that  $E \subseteq (H_n^{S_{j'}})^{q+1}$ . By (iii), Axiom 8, and repeated applications of Axiom 9, all for  $t_j^m$ , which are already verified,  $t_j^m(E) \subseteq (t_j^m(H_n^{S_{j'}}))^{q+1}$ .

Now, pick  $z \in t_j^m(\pi_{q+1}^{q+1}[E])$ . Then, there is a  $k \geq j$  ( $k \in J$ ) and a  $y \in t_j^k(\pi_{q+1}^{q+1}[E])$  such that  $z = t_k^m(y)$ . By (4) of Proposition 4.10 for  $t_j^k$ ,  $t_j^k(E)$  is a  $q+1$ -ary relation, and by Axiom 12 for  $t_j^k$ , there is a  $\langle y_1, y_2, \dots, y_q, y_{q+1} \rangle \in t_j^k(E)$  such that  $\langle y_1, y_2, \dots, y_q \rangle = \pi_{q+1}^{q+1}(y_1, y_2, \dots, y_q, y_{q+1}) = y$ . By (ix) and Axiom 7, both

for  $t_k^m$ , and (ii),  $\langle t_k^m(y_1), t_k^m(y_2), \dots, t_k^m(y_q), t_k^m(y_{q+1}) \rangle = t_k^m(\langle y_1, y_2, \dots, y_q, y_{q+1} \rangle) \in t_k^m(t_j^k(E)) = t_j^m(E)$ , and  $\langle t_k^m(y_1), t_k^m(y_2), \dots, t_k^m(y_q) \rangle = t_k^m(\langle y_1, y_2, \dots, y_q \rangle) = t_k^m(y) = z$ . Then,  $\pi_{q+1}^{q+1}(t_k^m(y_1), t_k^m(y_2), \dots, t_k^m(y_q), t_k^m(y_{q+1})) = z$ , and so  $z \in \pi_{q+1}^{q+1}[t_j^m(E)]$ . This holds for all  $z \in t_j^m(\pi_{q+1}^{q+1}[E])$ , so  $t_j^m(\pi_{q+1}^{q+1}[E]) \subseteq \pi_{q+1}^{q+1}[t_j^m(E)]$ .

Conversely, pick  $z \in \pi_{q+1}^{q+1}[t_j^m(E)]$ . Then, there are  $z_1, z_2, \dots, z_q, z_{q+1} \in t_j^m(H_n^{S_j'})$  such that  $\langle z_1, z_2, \dots, z_q, z_{q+1} \rangle \in t_j^m(E)$  and  $z = \langle z_1, z_2, \dots, z_q \rangle$ . Then, there are  $k_1, k_2, \dots, k_q, k_{q+1} \in J$  such that  $k_i \geq j$ , and  $y_i \in t_j^{k_i}(H_n^{S_j'})$  such that  $z_i = t_{k_i}^m(y_i)$ , for all  $i$  ( $1 \leq i \leq q+1$ ). Put  $p = \max_J(k_1, k_2, \dots, k_q, k_{q+1})$ . Now,  $t_j^p(H_n^{S_j'}) \subseteq H_n^{S_p'}$ , by (3) of Proposition 4.9, for  $t_j^p$ . Hence,  $t_{k_i}^p(t_j^{k_i}(H_n^{S_j'})) \subseteq H_n^{S_p'}$  for all  $i$  ( $1 \leq i \leq q+1$ ). Then, since  $y_i \in t_j^{k_i}(H_n^{S_j'})$  for each  $i$  ( $1 \leq i \leq q+1$ ),  $\langle t_{k_1}^p(y_1), t_{k_2}^p(y_2), \dots, t_{k_q}^p(y_q), t_{k_{q+1}}^p(y_{q+1}) \rangle \in (H_n^{S_p'})^{q+1} \subseteq H_{n+3}^{S_p'}$ , by Axiom 7 for each  $t_{k_i}^p$ , and the assumptions.

Assume, to get a contradiction, that  $\langle t_{k_1}^p(y_1), t_{k_2}^p(y_2), \dots, t_{k_q}^p(y_q), t_{k_{q+1}}^p(y_{q+1}) \rangle \notin t_j^p(E)$ . Then  $\langle t_{k_1}^p(y_1), t_{k_2}^p(y_2), \dots, t_{k_q}^p(y_q), t_{k_{q+1}}^p(y_{q+1}) \rangle \in H_{n+3}^{S_p'} \setminus t_j^p(E)$ . Then, by (ii), and (ix) and axioms 7 and 6, for  $t_p^m$ , we obtain

$$\begin{aligned} \langle z_1, z_2, \dots, z_q, z_{q+1} \rangle &= \langle t_{k_1}^m(y_1), t_{k_2}^m(y_2), \dots, t_{k_q}^m(y_q), t_{k_{q+1}}^m(y_{q+1}) \rangle = \\ &= \langle t_p^m(t_{k_1}^p(y_1)), t_p^m(t_{k_2}^p(y_2)), \dots, t_p^m(t_{k_q}^p(y_q)), t_p^m(t_{k_{q+1}}^p(y_{q+1})) \rangle = \\ &= t_p^m(\langle t_{k_1}^p(y_1), t_{k_2}^p(y_2), \dots, t_{k_q}^p(y_q), t_{k_{q+1}}^p(y_{q+1}) \rangle) \in t_p^m(H_{n+3}^{S_p'} \setminus t_j^p(E)) = \\ &= t_p^m(H_{n+3}^{S_p'} \setminus t_p^m(t_j^p(E))) = t_p^m(H_{n+3}^{S_p'} \setminus t_j^m(E)). \end{aligned}$$

Hence,  $\langle z_1, z_2, \dots, z_q, z_{q+1} \rangle \notin t_j^m(E)$ , which is a contradiction.

Hence,  $\langle t_{k_1}^p(y_1), t_{k_2}^p(y_2), \dots, t_{k_q}^p(y_q), t_{k_{q+1}}^p(y_{q+1}) \rangle \in t_j^p(E)$ .

Now, by (ii), and (ix) and Axiom 7 for  $t_p^m$ , and Axiom 12 for  $t_j^p$ ,

$$\begin{aligned} z &= \langle t_p^m(t_{k_1}^p(y_1)), t_p^m(t_{k_2}^p(y_2)), \dots, t_p^m(t_{k_q}^p(y_q)) \rangle = \\ &= t_p^m(\langle t_{k_1}^p(y_1), t_{k_2}^p(y_2), \dots, t_{k_q}^p(y_q) \rangle) = \\ &= t_p^m(\pi_{q+1}^{q+1}(t_{k_1}^p(y_1), t_{k_2}^p(y_2), \dots, t_{k_q}^p(y_q), t_{k_{q+1}}^p(y_{q+1}))) \in \\ &= t_p^m(\pi_{q+1}^{q+1}[t_j^p(E)]) = t_p^m(t_j^p(\pi_{q+1}^{q+1}[E])) = t_j^m(\pi_{q+1}^{q+1}[E]). \end{aligned}$$

This holds for all  $z \in \pi_{q+1}^{q+1}[t_j^m(E)]$ , so  $\pi_{q+1}^{q+1}[t_j^m(E)] \subseteq t_j^m(\pi_{q+1}^{q+1}[E])$ .

Hence,  $t_j^m(\pi_{q+1}^{q+1}[E]) = \pi_{q+1}^{q+1}[t_j^m(E)]$ . Thus, Axiom 12 is verified for  $t_j^m$ .

We have now verified all twelve axioms. Hence, each  $t_j^m$  ( $j \in J$ ) is a proper nonstandard extension, which completes the proof. ■

**7.7. REMARKS.** In the next theorem, we prove that we can find an arbitrarily large, well ordered sequence of successive extensions of a given superstructure, such that all successive extensions are ultrapower extensions using the same ultrafilter.

One might believe that this a straightforward consequence of the two previous results and Proposition 6.24, but there are two rather difficult problems which must be overcome:

We want infinitely many successive ultrapower extensions, but at each step, there will be several possible ultrapower extensions to choose between. We must then use the axiom of choice to select one particular extension at each step. But to apply the axiom of choice, we must then define an appropriate family of extensions to choose from, and this is not trivial.

The other problem is that we must prove that for each limit ordinal  $\beta \leq \alpha$ ,  $S_\beta$  is  $\omega$ -extendable, so that we can apply Theorem 7.6. (Remember that in Theorem 7.6,  $S_m$  was assumed to be  $\omega$ -extendable.) It is at this point we need to use (1) of Theorem 3.35 with  $Y \neq \emptyset$ . In all other applications of (1) of Theorem 3.35 in this text, we may assume that  $Y = \emptyset$ . This is the only exception.

For each limit ordinal  $\beta \leq \alpha$ ,  $H_\omega^{S_\beta'}$ , constructed as in Theorem 7.8, is called an *ultralimit*, and the construction carried out is called the *ultralimit construction*.

**7.8. THEOREM.** *Assume that the axiom of choice holds.*

*Let  $\alpha$  be an ordinal, let  $S$  be an  $\omega$ -extendable set containing an  $\mathbb{N}$ -copy, and let  $\mathcal{U}$  be a countably incomplete ultrafilter on an infinite index set  $I$ .*

*Then, there exists a family  $\{S_\beta\}_{\beta=0}^\alpha$ , of  $\omega$ -extendable sets, and a doubly indexed family of proper nonstandard extensions  $\{t_\gamma^\beta\}_{\gamma < \beta \leq \alpha}$  as in Proposition 7.4, with  $\{\beta \mid \beta \leq \alpha\}$  instead of  $J$  (the indices are ordinals, and they are ordered in the usual way), such that:*

- (1)  $S_0 = S$ .
- (2) If  $\beta < \alpha$ , then  $t_\beta^{\beta+1}$  is an ultrapower extension using  $\mathcal{U}$ .
- (3) If  $\beta \leq \alpha$  is a limit ordinal, then each  $t_\gamma^\beta$  (with  $\gamma < \beta$ ) is obtained from all  $t_\gamma^\delta$ , with  $\gamma < \delta < \beta$ , as in Theorem 7.6, with  $\{\gamma \mid \gamma < \beta\}$  instead of  $J$ , and  $\beta$  instead of  $m$ .

**PROOF.** Let  $X, Y \in H$  be any sets such that  $X$  contains the given  $\mathbb{N}$ -copy,  $X \cap Y = \emptyset$ , and such that  $X \cup Y$  is  $\omega$ -extendable. By Proposition 6.24, with  $X$  instead of  $S$ , there exists a set  $T \in H$  such that  $T \cap Y = \emptyset$ ,  $T \cup Y$  and  $T$  are both  $\omega$ -extendable, and such that there exists an ultrapower extension  $*$  :  $H_\omega^{X'} \rightarrow H_\omega^{T'}$  using  $\mathcal{U}$ .

Now, for every ordinal  $\beta$ , it makes sense, by the property that a every set of ordinals is bounded (see Chapter 8), and (5) of Proposition 3.6, to put  $F(\beta)$  as the smallest ordinal  $\gamma \geq \beta$  such that for every pair of sets  $X, Y$  as above, such that  $X, Y \in H_\beta$ , there exists such a  $T$  and such an ultrapower extension  $*$  :  $H_\omega^{X'} \rightarrow H_\omega^{T'}$  as above, such that  $T \in H_\gamma$ . This  $F$  is then a well defined functional relation.

Now, by (1) of Theorem 3.35, with  $S$  as  $X$  and  $\emptyset$  as  $Y$ , there is a set  $U$  such that  $S \cap U = \emptyset$ ,  $S \cup U$  is  $\omega$ -extendable, and  $\text{card}(S \cup U) > \max(\text{card}(\alpha + 1), \text{card} S)$ . Then,  $\text{card} U = \text{card}(S \cup U) > \text{card}(\alpha + 1)$  (see Chapter 8). Then, we can choose an injection  $g : \{\beta \mid \beta \leq \alpha\} \rightarrow U$ . (Remember that the ordinals are defined so that  $\alpha + 1 = \{\beta \mid \beta \leq \alpha\}$ . See Chapter 8.)

Next, for each  $\beta \leq \alpha$ , put  $L_\beta$  as the set of all *limit ordinals*  $\gamma$  such that  $\beta < \gamma \leq \alpha$ , and put  $V_\beta = g[L_\beta]$ .

Now, we can define the following function  $G$  on  $\{\beta \mid \beta \leq \alpha\}$ , by transfinite recursion (see Chapter 8):

- (i)  $G(0) = r(S \cup U)$ .
- (ii)  $G(\beta + 1) = F(G(\beta))$ , if  $\beta < \alpha$ .
- (iii)  $G(\beta) = (\sup_{\gamma < \beta} G(\gamma)) + 1$ , if  $\beta \leq \alpha$  is a limit ordinal.

It is easy to prove with transfinite induction that  $\{G(\beta)\}_{\beta=0}^\alpha$  is an increasing sequence of ordinals, i.e. that  $G(\delta) \leq G(\beta)$  holds for all ordinals  $\beta$  and  $\delta$  such that  $\delta < \beta \leq \alpha$ :

If  $\beta = 0$ , there is nothing to prove. Assume that  $\beta < \alpha$  and that  $G(\gamma) \leq G(\beta)$  holds for all ordinals  $\gamma$  such that  $\gamma < \beta$ . Since  $F(G(\beta)) \geq \beta$ , it follows from this and (ii) that  $G(\gamma) \leq G(\beta + 1)$  for all  $\gamma \leq \beta$ . If  $\beta \leq \alpha$  is a limit ordinal, then (iii) gives immediately that  $G(\gamma) < G(\beta)$  for all  $\gamma < \beta$ .

By transfinite induction,  $\{G(\beta)\}_{\beta=0}^\alpha$  is an increasing sequence of ordinals.

Now, for any  $\beta < \alpha$ , if  $X$  and  $Y$  are as above, with  $X, Y \in H_{G(\beta)}$ , then there is an  $\omega$ -extendable set  $T$  and an ultrapower extension  $*$  :  $H_\omega^{X'} \rightarrow H_\omega^{T'}$  as above, such that  $T \in H_{F(G(\beta))}$ .

Also, for every fixed triple  $\langle \beta, X, Y \rangle$  as in the previous paragraph, the class of all pairs  $\langle T, * \rangle$  of such sets  $T$  and corresponding ultrapower extensions  $*$ , is a set. (See chapter 8. In particular, the graph of such an extension belongs to  $\mathcal{P}(H_\omega^{X'} \times H_\omega^{T'})$ .) Therefore, by the axiom of choice, there is a function  $b$  assigning such a pair  $\langle T, * \rangle$  to every such triple  $\langle \beta, X, Y \rangle$ . Also, we let  $c(\beta, X, Y)$  be the set  $T = \pi_1^2(b(\beta, X, Y))$  and we let  $f(\beta, X, Y)$  be the ultrapower extension  $* = \pi_2^2(b(\beta, X, Y))$ .

Now, let us define  $\{S_\beta\}_{\beta=0}^\alpha$  for all  $\beta \leq \alpha$ , by transfinite recursion:

- (a)  $S_0 = S$ .
- (b)  $S_{\beta+1} = c(\beta, S_\beta, V_\beta)$ , if  $\beta < \alpha$ .
- (c)  $S_\beta = \cup_{\gamma < \beta} S_\gamma$  if  $\beta \leq \alpha$  is a limit ordinal.

We must prove that this is well defined, i.e. that  $c(\beta, S_\beta, V_\beta)$  is defined, which means that also  $b(\beta, S_\beta, V_\beta)$  and  $f(\beta, S_\beta, V_\beta)$  are defined, for all  $\beta < \alpha$ . To prove this is in no way trivial, because it means that we must prove that  $S_\beta \in H_{G(\beta)}$  and  $V_\beta \in H_{G(\beta)}$ , that  $S_\beta$  contains the given  $\mathbb{N}$ -copy, that  $S_\beta \cap V_\beta = \emptyset$ , and that each  $S_\beta \cup V_\beta$  is  $\omega$ -extendable, for all  $\beta < \alpha$ . (But since  $S_\beta$  is not *a priori* well defined, we put e.g.  $S_\beta = \emptyset$  whenever  $c(\beta, S_\beta, V_\beta)$  is not defined, in order to get the recursive definition to work in this hypothetical case too.)  $S_\beta$  ( $\beta < \alpha$ ) will then also be  $\omega$ -extendable.

We prove this by transfinite induction on  $\beta$ . Simultaneously, we prove that  $S_\gamma \subseteq S_\beta$  for all  $\gamma < \beta$ . Actually, we prove this for all  $\beta \leq \alpha$ , that is, we include the case  $\beta = \alpha$ , so that we also obtain that  $S_\alpha$  is  $\omega$ -extendable.

For  $\beta = 0$ , we have  $S_0 = S \subseteq S \cup U$  and  $V_0 \subseteq U \subseteq S \cup U$ . By (i), and (12) of Proposition 3.6,  $S \in H_{G(0)}$  and  $V_0 \in H_{G(0)}$ . By our assumptions,  $S_0 = S$  contains the  $\mathbb{N}$ -copy. Also, by the above,  $S \cap V_0 \subseteq S \cap U = \emptyset$ , and since  $S \cup U$  is  $\omega$ -extendable, so is  $S \cup V_0$ .

Thus,  $c(0, S_0, V_0)$  is defined.

Suppose now that this holds for all  $\gamma \leq \beta$ , where  $\beta \leq \alpha$ . Then,  $S_{\beta+1} = c(\beta, S_\beta, V_\beta) \in H_{G(\beta+1)}$ , by definition and (ii).  $V_{\beta+1} = V_\beta \in H_{G(\beta)} \subseteq H_{G(\beta+1)}$ , by the induction hypothesis and (5) of Proposition 3.6, using that  $\{G(\gamma)\}_{\gamma=0}^\alpha$  is an increasing sequence of ordinals. Also,  $f(\beta, S_\beta, V_\beta)$  is defined, simultaneously with  $c(\beta, S_\beta, V_\beta)$ . Then, it follows from (2) of Proposition 4.7 for  $f(\beta, S_\beta, V_\beta)$  that  $S_\beta \subseteq S_{\beta+1}$  and that  $S_{\beta+1}$  contains the  $\mathbb{N}$ -copy. Also, by our definitions,  $S_{\beta+1} \cap V_{\beta+1} = c(\beta, S_\beta, V_\beta) \cap V_\beta = \emptyset$  and  $S_{\beta+1} \cup V_{\beta+1} = c(\beta, S_\beta, V_\beta) \cup V_\beta$  is  $\omega$ -extendable.

It follows that  $c(\beta + 1, S_{\beta+1}, V_{\beta+1})$  is defined. Also, since  $S_\beta \subseteq S_{\beta+1}$ , the induction hypothesis gives that  $S_\gamma \subseteq S_{\beta+1}$  for all  $\gamma < \beta$ .

Assume now that this holds for all  $\gamma < \beta$ , where  $\beta \leq \alpha$  is a limit ordinal.

Since  $S_\gamma \in H_{G(\gamma)}$  for  $\gamma < \beta$ , by the induction hypothesis, it follows from (c), and (5) and (6) of Proposition 3.16, that  $S_\beta \subseteq H$  and  $S_\beta \in H$ . Now, if  $x \in S_\beta$ , then  $x \in S_\gamma$  for some  $\gamma < \beta$ . Hence, (5) of Proposition 3.16 gives that  $r(x) < G(\gamma)$ . It follows that  $r(x) \leq \sup_{\gamma < \beta} G(\gamma)$ , for all  $x \in S_\beta$ . Then, (iii), (5) of Proposition 3.16, and (5) of Proposition 3.6, give that  $S_\beta \in H_{G(\beta)}$ .

Since  $V_\beta \subseteq V_\gamma$  for  $\gamma < \beta$ , the induction hypothesis, (iii), and (12) and (5) of Proposition 3.6, give  $V_\beta \in H_{G(\gamma)} \subseteq H_{G(\beta)}$ . Also, (c) gives that  $S_\gamma \subseteq S_\beta$ , for all  $\gamma < \beta$ , and hence, by the induction hypothesis, that  $S_\beta$  contains the  $\mathbb{N}$ -copy.

If  $x \in S_\beta \cap V_\beta$ , then  $x \in S_\gamma$  for some  $\gamma < \beta$ . Also, since  $V_\beta \subseteq V_\gamma$ ,  $x \in V_\gamma$ . Thus,  $x \in S_\gamma \cap V_\gamma$ , which contradicts the induction hypothesis. Hence,  $S_\beta \cap V_\beta = \emptyset$ .

To prove that  $S_\beta \cup V_\beta$  is  $\omega$ -extendable, we first notice that since  $S_\beta \in H$  and  $V_\beta \in H$ , (5) and (6) of Proposition 3.16 give  $S_\beta \cup V_\beta \in H$ , that is, (i) of Definition 3.29 holds, for  $S_\beta \cup V_\beta$  and  $\omega$ .

To verify (ii) of Definition 3.29, we put  $y = g(\beta)$ . Since  $y \in U$  and  $U \in H$ , (5) of Proposition 3.16 gives  $y \in H$ . Also,  $y \in V_\gamma$  for all  $\gamma < \beta$ , but  $y \notin V_\beta$ . Hence, by the induction hypothesis,  $y \notin S_\gamma$ , for all  $\gamma < \beta$ . By (c),  $y \in H \setminus (S_\beta \cup V_\beta)$ .

Now, since  $y \in S_\gamma \cup V_\gamma$  and  $S_\gamma \cup V_\gamma$  is  $\omega$ -extendable, for all  $\gamma < \beta$ , by the induction hypothesis, it follows from (3) of Proposition 3.31 that  $y \neq \emptyset$ . By Definition 3.29,  $r(v) > \omega$  for all  $v \in y$ . Hence, by (5) of Proposition 3.16,  $r(y) > \omega$ . Also, by Definition 3.29,  $r(z) > \omega$  for all  $z \in \cup_{x \in S_\gamma \cup V_\gamma} x \subseteq \cup_{x \in S_\gamma \cup V_\gamma} x$ , for all  $\gamma < \beta$ . Since  $S_\beta = \cup_{\gamma < \beta} S_\gamma$ , this implies that  $r(z) > \omega$  for all  $z \in (\cup_{x \in S_\beta \cup V_\beta} x) \cup \{y\}$ .

Assume now that there exists a finite sequence  $\{x_k\}_{k=0}^m$ , with  $m \geq 1$ , such that  $x_0, x_m \in S_\beta \cup V_\beta \cup \{y\}$ , and  $x_{k-1} \in x_k$  for all  $k$  ( $1 \leq k \leq m$ ). Then, since  $y \in V_\gamma$  for all  $\gamma < \beta$ , and since  $S_\delta \subseteq S_\gamma$  for  $\gamma$  and  $\delta$  such that  $\delta < \gamma < \beta$ , by the induction hypothesis, there is a  $\gamma < \beta$  such that  $x_0, x_m \in S_\gamma \cup V_\gamma$ . This means that  $S_\gamma \cup V_\gamma$  is not intransitive. But by (1) of Proposition 3.21, this contradicts the  $\omega$ -extendability of  $S_\gamma \cup V_\gamma$ , which holds by the induction hypothesis. It follows that no such sequence exists, that is,  $S_\beta \cup V_\beta \cup \{y\}$  is intransitive. Thus, (ii) in Definition 3.29 holds, for  $S_\beta \cup V_\beta$  and  $\omega$ .

Hence,  $S_\beta \cup V_\beta$  is  $\omega$ -extendable.

Hence  $c(\beta, S_\beta, V_\beta)$  is defined, and, by (c),  $S_\gamma \subseteq S_\beta$  for all  $\gamma < \beta$ .

Now, it follows by transfinite induction that  $c(\beta, S_\beta, V_\beta)$  is defined, and hence that  $S_\beta$  is well defined, indeed  $\omega$ -extendable, for all  $\beta \leq \alpha$ , and also that  $S_\gamma \subseteq S_\beta$  for all  $\gamma < \beta \leq \alpha$ .

But  $c(\beta, S_\beta, V_\beta)$  and  $f(\beta, S_\beta, V_\beta)$  are defined simultaneously, so  $f(\beta, S_\beta, V_\beta)$  is also defined, for all  $\beta \leq \alpha$ .

Now, we can define the proper nonstandard extensions  $t_\gamma^\beta$  for all  $\beta, \gamma$  such that  $\gamma < \beta \leq \alpha$ . This definition is also by transfinite recursion on  $\beta \leq \alpha$ . (We use the convention that  $t_\beta^\beta(x) = x$ , for all ordinals  $\beta \leq \alpha$  and all  $x \in H_\omega^{S_\beta}$ .)

(\*)  $t_\gamma^{\beta+1} = f(\beta, S_\beta, V_\beta) \circ t_\gamma^\beta$ , for all  $\beta < \alpha$ .

(\*\*)  $t_\gamma^\beta$  is defined as in Theorem 7.6, with  $\{\gamma \mid \gamma < \beta\}$  as  $J$ ,  $\beta$  as  $m$ , and where the ordinals are ordered in the usual way, if  $\beta \leq \alpha$  is a limit ordinal.

We now prove that all  $t_\delta^\beta = t_\gamma^\beta \circ t_\delta^\gamma$ , for all  $\beta, \gamma, \delta$  such that  $\delta \leq \gamma \leq \beta \leq \alpha$ , and that the family  $\{t_\gamma^\beta\}_{\gamma \leq \beta \leq \alpha}$  consists of proper nonstandard extensions, unless for  $\gamma = \beta$ , when the maps are identities.

This follows easily by transfinite induction on  $\beta$ . (Again, the extensions  $t_\gamma^\beta$  are not a priori well defined. So let us say that e.g.  $t_\gamma^\beta(x) = x$  in the hypothetical

situation, which actually never occurs, when  $t_\gamma^\beta(x)$  is not defined by the above.)

This is clear if  $\beta = 0$  or if  $\gamma = \beta$ . Assume that  $\beta < \alpha$ , that  $t_\delta^\beta = t_\gamma^\beta \circ t_\delta^\gamma$  holds, and that these are proper nonstandard extensions, for all  $\gamma$  and  $\delta$  such that  $\delta \leq \gamma \leq \beta$ . Then, by Proposition 7.4,  $t_\delta^{\beta+1} = t_\beta^{\beta+1} \circ t_\delta^\gamma$ , for  $\delta \leq \gamma \leq \beta$ , and these are proper nonstandard extensions.

Assume instead that  $\beta \leq \alpha$  is a limit ordinal, that  $t_\delta^\varepsilon = t_\gamma^\varepsilon \circ t_\delta^\gamma$ , and that these are proper nonstandard extensions, for all  $\gamma, \delta$ , and  $\varepsilon$  such that  $\delta \leq \gamma \leq \varepsilon < \beta$ . Then, by Theorem 7.6,  $t_\delta^\beta = t_\gamma^\beta \circ t_\delta^\gamma$ , for  $\delta \leq \gamma < \beta$ , and these are proper nonstandard extensions.

Now, it follows by transfinite induction that the family of maps defined above has the desired properties.

In particular,  $t_\beta^{\beta+1} = f(\beta, S_\beta, V_\beta)$ , for all  $\beta < \alpha$ , and these are ultrapower extensions using  $\mathcal{U}$ .

The proof of the theorem is now complete. ■

Now, we come to saturation.

7.9. DEFINITION. A nonempty family  $\{A_j\}_{j \in J}$  of sets has the *finite intersection property* if  $\bigcap_{j \in J'} A_j \neq \emptyset$  for every nonempty finite subfamily  $\{A_j\}_{j \in J'} \subseteq \{A_j\}_{j \in J}$ .

7.10. DEFINITIONS. Let  $\kappa$  be a transfinite cardinal. A proper nonstandard extension  $*$  :  $H_\omega^{S'} \rightarrow H_\omega^{T'}$  is  $\kappa$ -*saturated* if  $\bigcap_{j \in J} A_j \neq \emptyset$ , for every family  $\{A_j\}_{j \in J}$  of internal sets in  $H_\omega^{T'} \setminus T$ , whose cardinality is less than  $\kappa$ , and which has the finite intersection property.

The extension is called *polysaturated* if  $H_\omega^{S'}$  can be well ordered and is  $\kappa$ -saturated for  $\kappa = \text{card } H_\omega^{S'}$ . ( $\text{card } H_\omega^{S'}$  is transfinite, since  $S$  is infinite.)

Obviously:

7.11. PROPOSITION. Let  $*$  :  $H_\omega^{S'} \rightarrow H_\omega^{T'}$  be a  $\kappa$ -saturated extension, for some transfinite cardinal  $\kappa$ . Then  $*$  is  $\lambda$ -saturated, for every transfinite cardinal  $\lambda < \kappa$ .

7.12. REMARKS. Clearly, every proper nonstandard extension is  $\aleph_0$ -saturated, since only the finite cardinals are smaller than  $\aleph_0$ . The simplest nontrivial kind of saturation is therefore  $\aleph_1$ -saturation. The assumption of  $\aleph_1$ -saturation is sufficient for many applications.

It can be proved that every ultrapower extension is  $\aleph_1$ -saturated, provided that the axiom of choice holds (see Henson, [1], Th. 7.13, p. 48).

We can now prove the main theorem about saturation.

7.13. THEOREM. Assume that the axiom of choice holds.

Let  $S$  be an  $\omega$ -extendable set. Then, for every transfinite cardinal  $\kappa$ , there exists an  $\omega$ -extendable set  $T$  and a  $\kappa$ -saturated proper nonstandard extension  $*$  :  $H_\omega^{S'} \rightarrow H_\omega^{T'}$ .

PROOF. Let  $\kappa^+$  be the smallest cardinal such that  $\kappa^+ > \kappa$ . We will construct a  $\kappa^+$ -saturated extension of  $H_\omega^{S'}$ . This is also a  $\kappa$ -saturated extension, by Proposition 7.11.<sup>21</sup>

<sup>21</sup>The reason that we use  $\kappa^+$  instead of  $\kappa$  is that  $\kappa^+$  is a *regular* cardinal, which  $\kappa$  might not be. This means that the union over a family with cardinality less than  $\kappa^+$  of sets with cardinalities less than  $\kappa^+$ , has cardinality less than  $\kappa^+$  (see Chapter 8). This property of  $\kappa^+$  is used in the proof.

Let  $I$  be set of all ordinals  $\alpha$  such that  $\alpha \leq \kappa$ , where  $\kappa$  and  $\kappa^+$  are considered as ordinals. Let  $\mathcal{U}$  be the countably incomplete ultrafilter on  $\hat{I}$  constructed in Remarks 5.12, with  $I$  instead of  $X$ . Now, by Theorem 7.8, with  $\hat{I}$  instead of  $I$  and  $\kappa^+$  instead of  $\alpha$ , there is a family  $\{S_\beta\}_{\beta=0}^{\kappa^+}$  of sets, with  $S_0 = S$ , all containing the same  $\mathbb{N}$ -copy as  $S$ , and a doubly indexed family  $\{t_\gamma^\beta\}$ , where each  $t_\gamma^\beta : H_\omega^{S_\gamma'} \rightarrow H_\omega^{S_\beta}$  ( $\gamma < \beta \leq \kappa^+$ ) is a proper nonstandard extension, and where  $t_\beta^{\beta+1}$  is an ultrapower extension using  $\mathcal{U}$ , for each  $\beta < \kappa^+$ .

We will prove that the extension  $t_0^{\kappa^+}$  is  $\kappa^+$ -saturated. So, let  $\{A_j\}_{j \in J}$  be a nonempty family of internal sets (with respect to  $t_0^{\kappa^+}$ ) with cardinality  $\lambda < \kappa^+$ , which has the finite intersection property. Clearly, we may assume that  $J \subseteq I$ . In fact, by bringing in repetitions, if necessary, we will even assume that  $J = I$ .

Now, each  $A_j$  ( $j \in I$ ) is an internal set in  $H_\omega^{S_{\kappa^+}'} \setminus S_{\kappa^+}$ . This means that  $A_j \in t_0^{\kappa^+}(E_j)$  for some  $E_j \in H_\omega^{S_0'} \setminus S_0$ . By (2) of Proposition 4.8, for  $t_0^{\kappa^+}$ ,  $E_j \neq \emptyset$ . Since  $\kappa^+$  is a limit ordinal, theorems 7.8 and 7.6 imply that there is an ordinal  $\beta_j < \kappa^+$  and a set  $B_j \in t_0^{\beta_j}(E_j)$  such that  $A_j = t_{\beta_j}^{\kappa^+}(B_j)$ . This holds for all  $j \in I$ . Now  $\{\beta_j\}_{j \in I}$  is an indexed family with cardinality at most  $\kappa$ , consisting of ordinals  $\beta_j < \kappa^+$ . This implies that  $\text{card } \beta_j \leq \kappa$  for all  $j \in I$ . Put  $\alpha = \sup_{j \in I} \beta_j = \cup_{j \in I} \beta_j$ . It follows that  $\text{card } \alpha \leq \kappa$  (see Chapter 8). But a cardinal is an initial ordinal, so  $\alpha < \kappa^+$ .

Put  $C_j = t_{\beta_j}^\alpha(B_j)$ , for all  $j \in I$ .

Then, for all  $j \in I$ ,  $A_j = t_{\beta_j}^{\kappa^+}(B_j) = t_\alpha^{\kappa^+}(t_{\beta_j}^\alpha(B_j)) = t_\alpha^{\kappa^+}(C_j)$ , Theorem 7.8.

Now, for any nonempty finite subset  $I' \subseteq I$ ,  $\cap_{j \in I'} A_j \neq \emptyset$ . Hence, by repeated applications of (1) and (2) of Proposition 4.8 for  $t_\alpha^{\kappa^+}$ ,  $\cap_{j \in I'} C_j \neq \emptyset$ . Thus,  $\{C_j\}_{j \in I}$  has the finite intersection property, too. Thus, for every nonempty finite subset  $I' \subseteq I$ , we can, by the axiom of choice, choose an  $a(I') \in \cap_{j \in I'} C_j$ .

Now, for every  $j \in I$ , let  $\mathcal{I}_j$  be the set of all finite subsets  $I' \subseteq I$  such that  $j \in I'$ . Then,  $a(I') \in C_j$ , for every  $I' \in \mathcal{I}_j$ . But  $\mathcal{I}_j \in \mathcal{U}$  (see Remarks 5.12). Since  $t_\alpha^{\alpha+1}$  is an ultrapower extension using  $\mathcal{U}$ , we can use the terminology in Chapter 6 with the given  $\mathcal{U}$  and with  $t_\alpha^{\alpha+1}$  instead of  $*$ . Doing this, we see that  $a(I') \in C_j$  a.e., and if we define  $[a] \in W$  by these  $a(I')$  (using Definitions 6.2), we obtain  $[a]^\circ \in C_j$ . Then, by (2) of Proposition 6.15 and Definitions 6.16, for  $t_\alpha^{\alpha+1}$ ,  $h([a]) \in t_\alpha^{\alpha+1}(C_j)$ . Hence, by Axiom 7, for  $t_{\alpha+1}^{\kappa^+}$ , and Theorem 7.8,  $t_{\alpha+1}^{\kappa^+}(h([a])) \in t_{\alpha+1}^{\kappa^+}(t_\alpha^{\alpha+1}(C_j)) = t_\alpha^{\kappa^+}(C_j) = t_\alpha^{\kappa^+}(t_{\beta_j}^\alpha(B_j)) = t_{\beta_j}^{\kappa^+}(B_j) = A_j$ .

Thus,  $t_{\alpha+1}^{\kappa^+}(h([a])) \in A_j$ .

This holds for all  $j \in I$ . Hence,  $\cap_{j \in I} A_j \neq \emptyset$ . It follows that the proper nonstandard extension  $t_0^{\kappa^+} : H_\omega^{S'} \rightarrow H_\omega^{S_{\kappa^+}}$  is  $\kappa^+$ -saturated, and, as we said, it is also  $\kappa$ -saturated. ■

We conclude this chapter by giving a few important applications of saturation.

**7.14. PROPOSITION.** *Let  $*$  :  $H_\omega^{S'} \rightarrow H_\omega^{T'}$  be a  $\kappa$ -saturated extension, for some transfinite cardinal  $\kappa$ .*

*Then, for every infinite internal set  $A \in H_\omega^{T'} \setminus T$  which can be well ordered,  $\text{card } A \geq \kappa$ .*

**PROOF.** Assume that  $A \in H_\omega^{T'} \setminus T$  is internal and infinite, and that it can be well ordered. Assume, to get a contradiction, that  $\text{card } A = \lambda < \kappa$ . Then, for

each  $a \in A$ ,  $a$  is internal, by (2) of Proposition 4.24. Then, by (2) and (1) of Corollaries 4.28,  $A \setminus \{a\}$  is internal. The family  $\{A \setminus \{a\}\}_{a \in A}$  is then a family of internal sets with cardinality  $\lambda$ , and it has the finite intersection property. Since  $*$  is  $\kappa$ -saturated,  $\bigcap_{a \in A} \{A \setminus \{a\}\} \neq \emptyset$ . But this is obviously false. Therefore, we must have  $\text{card } A \geq \kappa$ . ■

7.15. THEOREM (Comprehension). *Let  $\kappa$  be a transfinite cardinal, and let  $*$  :  $H_\omega^{S'} \rightarrow H_\omega^{T'}$  be a  $\kappa$ -saturated proper nonstandard extension.*

*Let  $A, B \in H_\omega^{T'}$  be internal  $m$ -ary and  $k$ -ary relations, respectively, with  $B \neq \emptyset$  ( $m, k \geq 0$ ). Let  $C \subseteq A$  with  $\text{card } C < \kappa$ , and let  $f : C \rightarrow B$  be a function.*

*Then, there exists an internal function  $g : A \rightarrow B$  such that  $f$  is the restriction of  $g$  to  $C$ .*

PROOF. If  $A = \emptyset$ , then  $f = \langle \emptyset, \emptyset, B \rangle$ . This function is internal, by (3) of Proposition 4.24 and (7) (a) of Corollaries 4.28.

Also, if  $k = 0$  and  $A \neq \emptyset$ , then we can take  $g = \langle A, A, B \rangle$ , which is internal, by (7) (a) of Corollaries 4.28.

So, we assume that  $A \neq \emptyset$  and that  $k \geq 1$ .

Choose  $n \geq 1$  such that  $A \subseteq (H_n^{T'})^m$  and  $B \subseteq (H_n^{T'})^k$ . Since  $A$  and  $B$  are internal, (3) of Theorem 4.25 gives that  $A \subseteq (*H_n^{S'})^m$  and  $B \subseteq (*H_n^{S'})^k$ .

Put

$$D = \{\langle a_1, a_2, \dots, a_m, c \rangle \in (H_{n+4}^{S'})^{m+1} \mid \langle a_1, a_2, \dots, a_m \rangle \in c\},$$

$$E = \{\langle b_1, b_2, \dots, b_k, c \rangle \in (H_{n+4}^{S'})^{k+1} \mid \langle b_1, b_2, \dots, b_k \rangle \in c\},$$

and

$$F = \{\langle a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_k, c \rangle \in (H_{n+4}^{S'})^{m+k+1} \mid \langle a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_k \rangle \in c\}.$$

By Theorem 4.26 and (3) of Proposition 4.10,

$$*D = \{\langle a_1, a_2, \dots, a_m, c \rangle \in (*H_{n+4}^{S'})^{m+1} \mid \langle a_1, a_2, \dots, a_m \rangle \in c\},$$

and similarly for  $*E$  and  $*F$ .

Now, let  $\phi(x_1, x_2, \dots, x_{m+k+3})$  be the following formula over  $S$ :

$$\begin{aligned} & (\forall x_{m+k+4} \in H_{n+3}^{S'}) (\langle x_{m+k+4}, x_1 \rangle \in \text{memb } H_{n+4}^{S'} \rightarrow x_{m+k+4} \in (H_n^{S'})^{m+k}) \wedge \\ & (\forall x_{m+k+5} \in H_n^{S'}) (\forall x_{m+k+6} \in H_n^{S'}) \dots (\forall x_{2m+2k+4} \in H_n^{S'}) \\ & (\langle x_{m+k+5}, x_{m+k+6}, \dots, x_{2m+2k+4}, x_1 \rangle \in F \rightarrow \\ & (\langle x_{m+k+5}, x_{m+k+6}, \dots, x_{2m+k+4}, x_2 \rangle \in D \wedge \\ & (\langle x_{2m+k+5}, x_{2m+k+6}, \dots, x_{2m+2k+4}, x_3 \rangle \in E)) \wedge \\ & (\forall x_{m+k+5} \in H_n^{S'}) (\forall x_{m+k+6} \in H_n^{S'}) \dots (\forall x_{2m+k+4} \in H_n^{S'}) \\ & (\langle x_{m+k+5}, x_{m+k+6}, \dots, x_{2m+k+4}, x_2 \rangle \in D \rightarrow \\ & (\exists x_{2m+k+5} \in H_n^{S'}) (\exists x_{2m+k+6} \in H_n^{S'}) \dots (\exists x_{2m+2k+4} \in H_n^{S'}) \\ & (\forall x_{2m+2k+5} \in H_n^{S'}) (\forall x_{2m+2k+6} \in H_n^{S'}) \dots (\forall x_{2m+3k+4} \in H_n^{S'}) \\ & (\langle x_{m+k+5}, x_{m+k+6}, \dots, x_{2m+k+4}, x_{2m+2k+5}, x_{2m+2k+6}, \dots, x_{2m+3k+4}, x_1 \rangle \in F \leftrightarrow \\ & (\langle x_{2m+k+5}, x_{2m+2k+5} \rangle \in \text{diag } H_n^{S'} \wedge \langle x_{2m+k+6}, x_{2m+2k+6} \rangle \in \text{diag } H_n^{S'} \wedge \dots \\ & \dots \wedge \langle x_{2m+2k+4}, x_{2m+3k+4} \rangle \in \text{diag } H_n^{S'})) \wedge \langle x_4, x_5, \dots, x_{m+k+3}, x_1 \rangle \in F. \end{aligned}$$

If  $A_1 \subseteq (H_n^{S'})^m$ , with  $A_1 \neq \emptyset$ ,  $B_1 \subseteq (H_n^{S'})^k$ , and  $G_h \in H_{n+4}^{S'} \setminus S$ , then  $\phi(G_h, A_1, B_1, a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_k)$  holds if and only if  $G_h$  is the graph of a function  $h : A_1 \rightarrow B_1$ , such that  $\langle a_1, a_2, \dots, a_m \rangle \in A_1$  and  $h(a_1, a_2, \dots, a_m) = \langle b_1, b_2, \dots, b_k \rangle$ .

If, instead,  $G_h, A_1, B_1, a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_k$  are internal elements in  $H_\omega^{T'}$  such that  $A_1 \subseteq (H_n^{T'})^m$ ,  $A_1 \neq \emptyset$ ,  $B_1 \subseteq (H_n^{T'})^k$ , and  $G_h \in H_{n+4}^{T'} \setminus T$ , then, it follows from (3) and (2) of Theorem 4.25 that  $A_1 \subseteq (*H_n^{S'})^m$ ,  $B_1 \subseteq (*H_n^{S'})^k$ , and  $G_h \in *H_{n+4}^{S'} \setminus T$ . Then, it follows from (1) and (3) of Proposition 4.10, (2) of Proposition 4.24, (2) of Theorem 4.25, (3) of Proposition 4.8, and (7) (a) of Corollaries 4.28, that  $*\phi(G_h, A_1, B_1, a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_k)$  holds if and only if  $G_h$  is the graph of an internal function  $h : A_1 \rightarrow B_1$ , such that  $\langle a_1, a_2, \dots, a_m \rangle \in A$  and  $h(a_1, a_2, \dots, a_m) = \langle b_1, b_2, \dots, b_k \rangle$ .

Now, fix  $\langle a_1, a_2, \dots, a_m \rangle \in C$ . Put  $\langle b_1, b_2, \dots, b_k \rangle = f(a_1, a_2, \dots, a_m)$ , and

$$H(a_1, a_2, \dots, a_m) = \{G_h \in *H_{n+4}^{S'} \setminus T \mid *\phi(G_h, A, B, a_1, a_2, \dots, a_m, b_1, b_2, \dots, b_k)\}.$$

By Axiom 6, Proposition 4.11, (2) of Proposition 4.24, (2) and (7) (a) of Corollaries 7.28, (2) of the internal definition principle, and the above, that  $H(a_1, a_2, \dots, a_m)$  is an internal set whose elements are the graphs of all internal functions from  $A$  to  $B$  which map  $\langle a_1, a_2, \dots, a_m \rangle$  to  $f(a_1, a_2, \dots, a_m)$ .

Now, let  $\{\langle a_1^j, a_2^j, \dots, a_m^j, b_1^j, b_2^j, \dots, b_k^j \rangle\}_{j=1}^p$  be a finite subset of the graph  $G_f$  of  $f$ , and pick  $\langle d_1, d_2, \dots, d_k \rangle \in B$ .

The set

$$\begin{aligned} ((A \setminus \cup_{j=1}^p \{\langle a_1^j, a_2^j, \dots, a_m^j \rangle\}) \times \{\langle d_1, d_2, \dots, d_k \rangle\}) \cup \\ \cup_{j=1}^p \{\langle a_1^j, a_2^j, \dots, a_m^j, b_1^j, b_2^j, \dots, b_k^j \rangle\} \end{aligned}$$

is internal, by (2) of Proposition 4.24, and (1), (2), and (5) of Corollaries 4.28. But this set is the graph of a function  $h : A \rightarrow B$  such that  $h(a_1^j, a_2^j, \dots, a_m^j) = \langle b_1^j, b_2^j, \dots, b_k^j \rangle$  for all  $j$  ( $1 \leq j \leq p$ ). This function  $h$  is then internal, by (7) (a) of Corollaries 4.28.

It follows that the conclusion of the theorem holds if  $C = \emptyset$  (take  $p = 0$ ). If  $C \neq \emptyset$ , it follows, by taking suitable sets  $\{\langle a_1^j, a_2^j, \dots, a_m^j, b_1^j, b_2^j, \dots, b_k^j \rangle\}_{j=1}^p$  above, that the family  $\{H(a_1, a_2, \dots, a_m) \mid \langle a_1, a_2, \dots, a_m \rangle \in C\}$  has the finite intersection property.

Since this family has cardinality  $\text{card } C < \kappa$ , and since  $*$  is  $\kappa$ -saturated, this family has nonempty intersection. An element in this intersection is the graph of an internal function  $g : A \rightarrow B$ , such that  $f$  is the restriction of  $g$  to  $C$ .

It follows that the conclusion of the theorem is true whether  $C = \emptyset$  or not. ■

**7.16. REMARK** If  $* : H_\omega^{S'} \rightarrow H_\omega^{T'}$  is an  $\aleph_1$ -saturated extension, and if  $B \in H_\omega^{T'} \setminus T$  is an internal set, and if  $f : \mathbb{N}^1 \rightarrow B^1$  is a function, then, there is an internal function  $g : (*\mathbb{N})^1 \rightarrow B^1$  such that  $f$  is the restriction of  $g$  to  $\mathbb{N}^1$ .

This follows from the Comprehension Theorem, if we use Axiom 8, (3) of Proposition 4.21, (1) of Proposition 4.24, and (6) of Corollaries 4.28.

We express this by saying that every sequence in  $B$  can be extended to an internal hypersequence in  $B$ .

7.17. DEFINITION. A proper nonstandard extension  $*$  :  $H_\omega^{S'} \rightarrow H_\omega^{T'}$  is called an *enlargement* if, for every set  $A \in H_\omega^{S'} \setminus S$ , there exists a hyperfinite set  $B \in H_\omega^{T'} \setminus T$  such that  $\{^*a \mid a \in A\} \subseteq B \subseteq ^*A$ .

7.18. REMARK. It may seem weird that enlargements exist, but they do, as the following results show, and they are useful in applications.

As Nigel Cutland said:

”Nonstandard analysis is the art of making infinite sets finite by extending them!”

7.19. THEOREM. A proper nonstandard extension  $*$  :  $H_\omega^{S'} \rightarrow H_\omega^{T'}$  is an enlargement if and only if the following condition holds:

If  $A \in H_\omega^{S'} \setminus S$  is a nonempty family of sets in  $H_\omega^{S'} \setminus S$  with the finite intersection property, then  $\bigcap_{B \in A} ^*B \neq \emptyset$ .

PROOF. Assume first that  $*$  :  $H_\omega^{S'} \rightarrow H_\omega^{T'}$  is an enlargement. Let  $A \in H_\omega^{S'} \setminus S$  be a nonempty family of sets in  $H_\omega^{S'} \setminus S$  with the finite intersection property. Let  $C$  be the family of all nonempty finite subfamilies of  $A$ . Since  $\mathcal{P}(A) \cap S = \emptyset$ ,  $C$  can also be described as the set of all elements in  $(\mathcal{P}(A) \setminus \{\emptyset\}) \setminus S$  which are finite sets.

By Theorem 4.41, Axiom 6, (2) of Proposition 4.8, and (2) of Proposition 4.10,  $^*C$  is then the set of all elements in  $(^*\mathcal{P}(A) \setminus \{\emptyset\}) \setminus T$  which are hyperfinite sets. Since  $^*\mathcal{P}(A) \cap T = \emptyset$ , by (1) and (2) of Proposition 4.8, and Proposition 4.11, and since all hyperfinite sets are internal, by Proposition 4.39, (4) of Theorem 4.25 then gives that  $^*C$  is the family of all nonempty, hyperfinite subfamilies of  $^*A$ .

Also,  $A \cap S = \emptyset$ , together with (1) and (2) of Proposition 4.8 and Proposition 4.11, give that  $^*A \cap T = \emptyset$ .

Next, put  $n = r^{S'}(A)$ . Then  $n \geq 2$  and  $r^{S'}(C) = n + 1$ . Then, all elements in  $C$  lie in  $H_n^{S'}$ , all elements in  $A$  lie in  $H_{n-1}^{S'} \setminus S$ , and all elements in elements in  $A$  lie in  $H_{n-2}^{S'}$ .

Now, it follows from (3) of Proposition 4.9, (2) of Proposition 4.24, and (2) of Theorem 4.25, that  $r^{T'}(^*A) = n$ ,  $r^{T'}(^*C) = n + 1$ , and that all elements in  $^*A$  lie in  $^*H_{n-1}^{S'} \setminus T$ , and hence so does all elements in elements in  $^*C$ , while the elements in  $^*C$  themselves lie in  $^*H_n^{S'}$ . Likewise, all elements in elements in  $^*A$ , and thus all elements in elements in elements in  $^*C$ , lie in  $^*H_{n-2}^{S'}$ .

Now, the following sentence over  $S$  is true, since  $A$  has the finite intersection property:

$$(\forall x_1 \in C)(\exists x_2 \in H_{n-2}^{S'})(\forall x_3 \in A) \\ (\langle x_3, x_1 \rangle \in \text{memb } H_n^{S'} \rightarrow \langle x_2, x_3 \rangle \in \text{memb } H_{n-1}^{S'}).$$

Hence, by (3) of the transfer principle, its  $*$ -transform is true, and that is, by (1) of Proposition 4.10:

$$(\forall X_1 \in ^*C)(\exists X_2 \in ^*H_{n-2}^{S'})(\forall X_3 \in ^*A) \\ (\langle X_3, X_1 \rangle \in \text{memb } ^*H_n^{S'} \rightarrow \langle X_2, X_3 \rangle \in \text{memb } ^*H_{n-1}^{S'}).$$

Together with the above considerations, and (3) of Proposition 4.8, this means that every nonempty hyperfinite subfamily of  $^*A$  has a nonempty intersection.

Next, since  $*$  is an enlargement, there is a hyperfinite subfamily  $D$  of  $*A$  such that  $\{ *B \mid B \in A \} \subseteq D$ . Since  $A \neq \emptyset$ ,  $D \neq \emptyset$ . It follows that  $D$  has a nonempty intersection. Hence, so does  $\{ *B \mid B \in A \}$ .

This holds for all nonempty families of sets in  $H_\omega^{S'} \setminus S$  which have the finite intersection property.

Conversely, assume that the condition holds. Pick a set  $C \in H_\omega^{S'} \setminus S$ . If  $C = \emptyset$ ,  $C$  itself is hyperfinite, by Proposition 4.40. Also  $C = *C$ , by (2) of Proposition 4.8. Thus, the converse part of the theorem holds in this case.

So, assume that  $C \neq \emptyset$ . For each  $c \in C$ , let  $\mathcal{B}_c$  be the family of all finite subsets of  $D \subseteq C$  such that  $c \in D$ . Then,  $\mathcal{B}_c \in H_\omega^{S'} \setminus S$ . Since  $\mathcal{B}_c \subseteq \mathcal{P}(C)$ , (3) of Proposition 4.8 and (4) of Theorem 4.25 give  $*\mathcal{B}_c \subseteq *\mathcal{P}(C) \subseteq \mathcal{P}(*C)$ , for all  $c \in C$ .

The family  $\{\mathcal{B}_c \mid c \in C\}$  has the finite intersection property. Hence, by induction upon (1) and (2) of Proposition 4.8, the family  $\{*\mathcal{B}_c \mid c \in C\}$  has the finite intersection property. Thus, by our assumption, there exists a set  $E \in \bigcap_{c \in C} *\mathcal{B}_c \subseteq \mathcal{P}(*C)$ . But, by Corollary 4.42, every element in  $*\mathcal{B}_c$  ( $c \in C$ ) is hyperfinite set in  $H_\omega^{T'} \setminus T$ . Thus,  $E$  is a hyperfinite subset of  $*C$ .

Next, fix  $c \in C$ , and choose  $n \geq 1$  so that  $C \in H_n^{S'}$ . Let  $\phi(x_1)$  be the following formula over  $S$ :

$$(\forall x_2 \in \mathcal{B}_c) \langle x_1, x_2 \rangle \in \text{memb } H_n^{S'}.$$

Then,  $\phi(c)$  holds. Hence,  $*\phi(*c)$  holds, by (2) of the transfer principle. This and (1) of Proposition 4.10 now give that every element in  $*\mathcal{B}_c$  contains the element  $*c$ .

This holds for all  $c \in C$ .

Thus, since  $E \in \bigcap_{c \in C} *\mathcal{B}_c$ , it follows that  $*c \in E$  for all  $c \in C$ . Therefore,  $E$  has the desired properties.

Since  $C \in H_\omega^{S'} \setminus S$  was arbitrary, the converse part of the theorem is now proved, too. ■

**7.20. THEOREM.** *Every polysaturated proper nonstandard extension is an enlargement.*

**PROOF.** Let  $*$  :  $H_\omega^{S'} \rightarrow H_\omega^{T'}$  be a polysaturated extension. This means, by definition, that it is  $\kappa$ -saturated, where  $\kappa = \text{card } H_\omega^{S'}$ . (This  $\kappa$  is transfinite, and since it is defined,  $H_\omega^{S'}$  can be well ordered, see Chapter 8.)

Let  $A \in H_\omega^{S'} \setminus S$  be a nonempty family of sets in  $H_\omega^{S'} \setminus S$  with the finite intersection property. By iteration upon (1) and (2) of Proposition 4.8,  $\{ *B \mid B \in A \}$  has also the finite intersection property, and all these  $*B$  are internal, by (1) of Proposition 4.24.

Furthermore,  $A \subseteq H_n^{S'}$  for some  $n$ , and hence  $\text{card } A \leq \text{card } H_n^{S'} < \text{card } H_\omega^{S'} = \kappa$ .

Since  $*$  is  $\kappa$ -saturated,  $\bigcap_{B \in A} *B \neq \emptyset$ . This holds for all nonempty families  $A \in H_\omega^{S'} \setminus S$  of sets in  $H_\omega^{S'} \setminus S$  which have the finite intersection property.

Thus,  $*$  is an enlargement, by Theorem 7.19. ■

**7.21. COROLLARY.** *If the axiom of choice holds, then, every superstructure has an enlargement.*

**PROOF.** This follows immediately from theorems 7.13 and 7.20. ■

7.22. REMARK. Corollary 7.21 can also be proved in a simpler way, without using saturation. For this simpler proof, it is not necessary to know how to compose extensions as in propositions 7.3 and 7.4, and theorems 7.6 and 7.8. (See Henson, [1], Th. 7.12, p. 47 f.)

## CHAPTER 8

### Appendix: Set Theory

We will here mention, mostly without proofs, some set theoretic results, useful in this article. This is by no means a complete exposé; for this, we refer to the literature.

There are some different set theories used by mathematicians today. By far, the most popular one is ZFC, Zermelo-Fraenkel with the axiom of choice. Another rather common one is NBG, von Neumann-Bernays-Gödel. In NBG, there are two types of objects, sets and classes, while ZFC is simpler in the sense that it contains only one type of objects, sets. On the other hand, it is often necessary to talk about collections which cannot be sets (e.g. the collection of all sets, the collection of all ordinals, or of all cardinals, which are not sets, because if they were, this would lead to contradictions; the so called "paradoxes" of set theory) and this is more smoothly done in NBG.

It is worth to point out that neither ZFC nor NBG contain any "pure" elements (in German, "Urelementen"), which are not regarded as collections of other objects. This may seem strange, but it turns out that one can generate all sets one needs by starting from the empty set, then take its power set, then its power set, etc. and use all kinds of operations, described in the axioms, to form new sets from old ones. (In nonstandard analysis, however, one needs some kind of "urelements", see Chapter 1).

We will here follow ZFC. It is a *first order theory*, which means that its axioms and theorems are formulas in first order predicate logic (see e.g. Mendelson, [6], Ch. 2). These formulas are not of the same type as the formulas used in Chapter 4, although there are similarities between them. We will not go into the details here, but we formulate the axioms informally in semi-natural language. We interrupt the list of axioms at many places, with appropriate discussions.

*Axiom of Extensionality:* Two sets  $A$  and  $B$  are equal, if and only if they have the same elements, that is,  $A = B$  if and only if, for all  $x$ :  $x \in A \Leftrightarrow x \in B$ .

A consequence of this is that there are no urelements. There can only be one object with no elements: the empty set.

A formula  $\phi(x)$  can be regarded as a condition which an object, (in ZFC necessarily a set), for which all free occurrences of  $x$  in the formula should be substituted, may or may not satisfy. Sometimes, there exists a set (unique, by the Axiom of Extensionality) which contains precisely those sets which satisfy the condition. If so, we write this set as  $\{x \mid \phi(x)\}$ .  $a \in \{x \mid \phi(x)\}$  then means the same as  $\phi(a)$ . The formula  $\phi(x)$  may contain other free variables than  $x$ , which are then considered as *parameters* upon which the set  $\{x \mid \phi(x)\}$  depends.

That not all conditions of this kind define sets is clear if we consider the condition  $x \notin x$ , and, assuming that it defines a set  $A$  as above, we try to decide whether or not  $A \in A$ . We then obtain  $A \in A$  if and only if  $A \notin A$ . This is a contradiction, the famous "Russell's Paradox", which Bertrand Russell discovered in 1902. Thus, this condition cannot define a set. Another condition which can be proved to not define a set is  $x = x$ , which is satisfied by *all* sets. This means that there is no set of all sets.

However, whether the condition  $\phi(x)$  defines a set or not, we can still form the expression  $\{x \mid \phi(x)\}$ . We call any such expression a *class* (which also may depend upon parameters), and we may use any suitable letters or expressions to denote it. We then write  $a \in \{x \mid \phi(x)\}$  (or i.e.  $a \in A$ , if "A" denotes the given class) as a shorthand (or perhaps "longhand") for  $\phi(a)$ , which confirms with the situation when  $\{x \mid \phi(x)\}$  is a set.

A class which is not a set is called a *proper class*. These differ from sets in the sense that they cannot be elements in other classes. Thus, if  $A$  is a proper class, and if  $B$  is a class (either a set or a proper class), then the expression  $A \in B$  has no meaning. The Russell argument above then shows that  $\{x \mid x \notin x\}$  is a proper class. Other important proper classes are "the class of all sets", "the class of all ordinals", "the class of all cardinals", and the classes  $H^Y$  in Chapter 3 (which depends upon the parameter  $Y$ ). We use the binary operators such as  $\cup$ ,  $\setminus$ , and  $\cap$ , and the binary relations  $\subseteq$ ,  $\subset$ , and  $=$  on classes too, with the obvious interpretations.

In NBG, classes are actually objects themselves, not only notational conveniences, as here in ZFC.

The next three axioms in ZFC state conditions which are claimed to define sets. In all cases, the sets thus defined are unique, by the axiom of extensionality.

*Union Axiom:* For every set  $A$ , there exists a set  $B$  which contains precisely all elements in elements in  $A$ .

The corresponding condition  $\phi(x)$  is, informally, "There exists an element  $y \in A$  such that  $x \in y$ ." The set  $B$  is called the *union* of the elements in  $A$ . It is denoted  $\cup_{y \in A} y$ .

*Power Set Axiom:* For every set  $A$ , there exists a set  $B$ , which contains precisely all subsets of  $A$ .

The corresponding condition is " $x \subseteq A$ ". The set  $B$  is called the *power set* of  $A$ . It is denoted  $\mathcal{P}(A)$ .

We call a formula  $\phi(x, y)$  in ZFC a (binary) *relation* (possibly dependent upon parameters). A relation  $\phi(x, y)$  is called a *functional* relation if, for all  $a, b$ , and  $c$ :  $\phi(a, b)$  and  $\phi(a, c)$  together imply  $b = c$ . It is then meaningful to write  $F(a) = b$  instead of  $\phi(a, b)$ . As with classes,  $F$  need not be an object in itself, just a notational convenience. The expression  $F(a)$  is meaningful only if it is defined, that is, if there is a  $b$  such that  $\phi(a, b)$ . We say that the functional relation  $F$  is defined on the class  $A$  if  $F(a)$  is defined for every  $a \in A$ .

Now, we can state the next axiom thus:

*Axiom Schema of Replacement:* Let  $F$  be a functional relation. Then, for every set  $A$ , the class of all  $F(a)$  such that  $F(a)$  is defined and  $a \in A$ , is a set.

(We leave it to the reader to write this class in the standard form above.)

This is called an axiom *schema* because it is actually infinitely many statements, one for every functional relation  $F$ .

From this, we easily derive the following important consequences:

*Theorem Schema of Subsets:* For any condition  $\psi(x)$  and any set  $A$ , the class  $\{x \mid \psi(x) \wedge x \in A\}$  is a set.

To see this, apply the axiom schema of replacement to the functional relation defined by  $\psi(x) \wedge x = y$ , using the set  $A$ .

(Here, too every condition  $\psi(x)$  gives rise to one of infinitely many statements.)

So far, we have not seen any single example of a set. But now we can construct the empty set:

*Existence of the Empty Set:* There exists a (unique) set which has no elements. This is denoted  $\emptyset$ .

To see this, apply the theorem schema of subsets to some contradictory condition, e.g.  $x \neq x$ , and an arbitrary set  $A$ .

That there exists some set  $A$  to start from, i.e. that there exists any object at all, is not a particular property of ZFC, but it is an assumption that underlies the formalism of predicate logic.

Applying the power set axiom to  $\emptyset$ , we obtain the set  $\{\emptyset\}$ . A new application of the power set axiom then gives the set  $\{\emptyset, \{\emptyset\}\}$ .

This is used in next theorem:

*Pairing Theorem:* For all sets  $a$  and  $b$ , there exists a set whose elements are precisely  $a$  and  $b$ . This set is denoted  $\{a, b\}$ .

To see this, apply the axiom schema of replacement to the relation  $(x = \emptyset \wedge y = a) \vee (x = \{\emptyset\} \wedge y = b)$ , which is obviously a functional relation which depends upon the parameters  $a$  and  $b$ , using the set  $\{\emptyset, \{\emptyset\}\}$ .

We can now combine old sets to get new ones, using all the results hitherto. We can construct unions, differences, intersections, power sets, pairs, subsets given by conditions, etc. of old sets.

Given two sets  $a$  and  $b$ , we can construct the set  $\{\{a\}, \{a, b\}\}$ , which we denote by  $\langle a, b \rangle$ . Such a set is called an *ordered pair*. This is a more fundamental type of ordered pair than those in Definitions 2.1. We use the same notation, since there are little risk for confusion. A similar argument as in the proof of Proposition 2.4 yields that if  $\langle a, b \rangle = \langle c, d \rangle$ , then  $a = c$  and  $b = d$ . A set of the form  $\langle \langle a, b \rangle, c \rangle$  is called a *triple* and is denoted  $\langle a, b, c \rangle$ . Similarly for quadruples, quintuples, etc.

The *cartesian product* of two classes  $A$  and  $B$  is the class of all ordered pairs  $\langle a, b \rangle$  such that  $a \in A$  and  $b \in B$ . It is denoted  $A \times B$ . A subclass  $R \subseteq A \times B$  is called a (*binary*) *relation* on  $A \times B$ . If  $A = B$ , then we call it a (*binary*) relation on  $A$ . We usually write  $aRb$  instead of  $\langle a, b \rangle \in R$ . If  $C \subseteq A$  and  $D \subseteq B$ , then the relation  $R' = R \cap (C \times D)$  is called the *restriction* of the relation  $R$  to  $C \times D$  (or to  $C$ , if  $C = D$ ).

If  $A$  and  $B$  are sets, then so are  $A \times B$  and all relations on  $A \times B$ .

Here, too, we have terminology conflicting with the terminology in Chapter 2, and also with the terminology earlier in the present chapter. The latter conflict, however, may be resolved by simply identifying a relation given by a condition  $\phi(x, y)$  by the class of all ordered pairs  $\langle a, b \rangle$  such that  $\phi(a, b)$ . It can be considered to be a relation on the class of all sets, or on any class which contains all  $a$  and  $b$  such that  $\phi(a, b)$  holds.

We will now introduce functions, where we also get a conflict with the earlier definition.

A function  $f : A \rightarrow B$ , where  $A$  and  $B$  are sets, is a triple  $\langle A, G_f, B \rangle$ , where  $G_f$  is a binary relation on  $A \times B$ , such that for every  $a \in A$  there is a unique  $b \in B$  such that  $\langle a, b \rangle \in G_f$ . We write  $f(a) = b$  instead of  $\langle a, b \rangle \in G_f$ .  $A$  is called the *domain*,  $B$  the *codomain*, and  $G_f$  the *graph* of  $f$ . We write  $D_f = A$ . These sets are all uniquely determined by  $f$ . The *image* of a subset  $C \subseteq A$  under  $f$  is the set of all  $b \in B$  for which there exists an  $a \in A$  such that  $f(a) = b$ . It is denoted by  $f[C]$ . We put  $R_f = f[A]$ .  $R_f$  is called the *range* of  $f$ .  $f$  is said to be *injective* or

an *injection*, if there is at most one  $a \in A$  such that  $f(a) = b$  to every  $b \in B$ .  $f$  is called *surjective* or a *surjection* if  $R_f = B$ . It is called *bijective* or a *bijection* if it both injective and surjective. If  $f$  is bijective, then it has a (unique) *inverse*, i.e. a function  $f^{-1} : B \rightarrow A$  such that  $f^{-1}(b) = a$  if and only if  $f(a) = b$  for all  $a \in A$  and all  $b \in B$ . If  $f : A \rightarrow B$  and  $g : B \rightarrow C$  are functions, then we define the *composite function*  $g \circ f : A \rightarrow C$  by  $(g \circ f)(a) = g(f(a))$ , for all  $a \in A$ . If  $f$  and  $g$  both are injective (surjective, bijective), then so is  $g \circ f$ . If  $f : A \rightarrow B$  is bijective with inverse  $f^{-1} : B \rightarrow A$ , then  $f^{-1} \circ f = \text{id}_A$  and  $f \circ f^{-1} = \text{id}_B$ , where  $\text{id}_A : A \rightarrow A$  is the *identity function* on  $A$ , which is defined by  $\text{id}_A(a) = a$  for all  $a \in A$ , and similarly for  $\text{id}_B : B \rightarrow B$ . Clearly,  $f^{-1}$  has an inverse too, namely  $(f^{-1})^{-1} = f$ . If  $A \subseteq B$ , then the function  $f : A \rightarrow B$  defined by  $f(a) = a$  for all  $x \in A$  is called an *inclusion*. If  $A = D_f$  is an  $m$ -ary relation, for some  $m \geq 0$ , then we write  $f(a_1, a_2, \dots, a_m)$  instead of  $f(\langle a_1, a_2, \dots, a_m \rangle)$ , for  $\langle a_1, a_2, \dots, a_m \rangle \in A$  (see Definitions 2.2).

For any sets  $A$  and  $B$ , we write  $A \lesssim B$  if there exists an injection  $f : A \rightarrow B$ . We write  $A \sim B$  and say that  $A$  is *equinumerous* to  $B$ , if there exists a bijection  $f : A \rightarrow B$ . We write  $A \not\lesssim B$  if  $A \lesssim B$  does not hold, and  $A \not\sim B$  if  $A \sim B$  does not hold. We also write  $A \prec B$  if  $A \lesssim B$  and  $A \not\sim B$ , and  $A \not\prec B$  if not  $A \prec B$ . Clearly,  $A \sim B$  implies  $A \lesssim B$ . The following properties are also obvious ( $A, B$ , and  $C$  are arbitrary sets):

- (1) If  $A \subseteq B$ , then  $A \lesssim B$ .
- (2)  $A \sim A$ , and hence  $A \lesssim A$ .
- (3) If  $A \sim B$ , then  $B \sim A$ .
- (4) If  $A \lesssim B$  and  $B \lesssim C$ , then  $A \lesssim C$ .
- (5) If  $A \sim B$  and  $B \sim C$ , then  $A \sim C$ .

To see these, just apply appropriate inclusions, identity functions, inverses, and composite functions.

The following classical result, however, is by no means obvious. But it seems intuitively reasonable and it is very important. It is usually proved by defining a process jumping back and forth between  $A$  and  $B$ , and this process needs the natural numbers for its definition. There is however, another very elegant proof, due to J. Whitaker, which do not presuppose the natural numbers. I can't resist to give this proof here:

**THEOREM (Schröder-Bernstein).** *For any sets  $A$  and  $B$ : If  $A \lesssim B$  and  $B \lesssim A$ , then  $A \sim B$ .*

**PROOF.** Assume that  $A \lesssim B$  and  $B \lesssim A$ , and let  $f : A \rightarrow B$  and  $g : B \rightarrow A$  be injections. We must find a bijection  $h : A \rightarrow B$ .

Let  $\mathcal{T}$  be the family of all subsets  $V \subseteq A$  such that  $g[B \setminus f[V]] \subseteq A \setminus V$ . Put  $U = \cup_{V \in \mathcal{T}} V$ . Pick  $b \in B \setminus f[U]$ . Then, for any  $V \in \mathcal{T}$ ,  $b \in B \setminus f[V]$ , which yields that  $g(b) \in A \setminus V$ . This holds for all  $V \in \mathcal{T}$ , so  $g(b) \notin \cup_{V \in \mathcal{T}} V = U$ . Since this holds for all  $b \in B \setminus f[U]$ ,  $g[B \setminus f[U]] \subseteq A \setminus U$ , that is,  $U \in \mathcal{T}$ .

Next, pick  $a \in A \setminus g[B \setminus f[U]]$ . Put  $U' = U \cup \{a\}$ . Since  $U \subseteq U'$ ,  $g[B \setminus f[U']] \subseteq g[B \setminus f[U]] \subseteq (A \setminus U) \setminus \{a\} = A \setminus U'$ . It follows that  $U' \in \mathcal{T}$ , and hence that  $U' \subseteq U$ , which implies  $a \in U$ . Since  $a \in A \setminus g[B \setminus f[U]]$  was arbitrary,  $A \setminus g[B \setminus f[U]] \subseteq U$ . Hence,  $A \setminus U \subseteq g[B \setminus f[U]]$ .

Thus,  $g[B \setminus f[U]] = A \setminus U$ .

Therefore, we can now define a function  $h : A \rightarrow B$  by

$$h(a) = \begin{cases} f(a), & \text{if } a \in U, \\ g^{-1}(a), & \text{if } a \in A \setminus U, \end{cases}$$

where we, somewhat improperly, regard  $g^{-1}$  as a function with domain  $R_g$ .

If  $a, a' \in A$  and  $a \neq a'$ , then  $h(a) \neq h(a')$  if either both  $a, a' \in U$  or both  $a, a' \in A \setminus U$ . If  $a \in U$  and  $a' \in A \setminus U$ , then  $h(a) \in f[U]$  and  $h(a') \in B \setminus f[U]$ , and hence that  $h(a) \neq h(a')$ . Therefore,  $h$  is injective. Also,  $h[U] = f[U]$  and  $h[A \setminus U] = B \setminus f[U]$ , whence  $h$  is surjective.

Thus,  $h : A \rightarrow B$  is bijective, and this means that  $A \sim B$ . ■

**COROLLARY.** *If  $A \subseteq B$  and  $B \lesssim A$ , then  $A \sim B$ .*

**PROOF.** This follows immediately from the theorem and (1) above. ■

Another important question is: For arbitrary sets  $A$  and  $B$ , must either  $A \lesssim B$  or  $B \lesssim A$  hold? We postpone this question for the moment.

Let us, however, point out one important result: that  $\mathcal{P}(A) \not\sim A$ , for all sets  $A$ . (If there was a bijection  $f : \mathcal{P}(A) \rightarrow A$ , would then the set  $C = \{f(B) \mid B \subseteq A \wedge f(B) \notin B\}$  satisfy  $f(C) \in C$  or not?) On the other hand, it is easy to see that  $A \lesssim \mathcal{P}(A)$ . Hence,  $A \prec \mathcal{P}(A)$ .

A relation  $R$  on a class  $A$  is called

- reflexive*, if  $aRa$  for all  $a \in A$ ,
- irreflexive*, if  $aRa$  for no  $a \in A$ ,
- symmetric*, if  $aRb$  implies  $bRa$ , for all  $a, b \in A$ ,
- antisymmetric*, if  $aRb$  and  $bRa$  imply  $a = b$ , for all  $a, b \in A$ ,
- transitive* if  $aRb$  and  $bRc$  imply  $aRc$ , for all  $a, b, c \in A$ .

$R$  is called an *equivalence relation*, if it is reflexive, symmetric and transitive.

If  $R$  is an equivalence relation on a class  $A$ , then, for every  $a \in A$ , we define the class  $[a] = \{b \in A \mid aRb\}$ . This is called the *equivalence class* of  $a$ , given by  $R$ . Then,  $a \in [a]$ . For any  $a, b \in A$ , we have either  $[a] \cap [b] = \emptyset$  or  $[a] = [b]$ .

Two examples of equivalence relations are equinumerosity (on the class of all sets, see above), and similarity (on the class of all totally ordered sets, see below).

If  $A$  is a set, then set of all equivalence classes given by  $R$  forms a *partition* of  $A$ , i.e. a set of pairwise disjoint subsets of  $A$ , whose union is  $A$ .

Conversely, given a partition of a set  $A$ , we can define an equivalence relation  $R$  on  $A$  by stipulating that  $aRb$  if and only if  $a$  and  $b$  lie in the same subset, of those subsets forming the partition. These subsets will then become the equivalence classes given by  $R$ . Conversely, if we start from an equivalence relation  $R$  on a set  $A$ , then the equivalence relation defined by the set of equivalence classes given by  $R$  is  $R$  itself.

A relation  $<$  on a class  $A$  is called a (*strict*) *partial order* if it is both irreflexive and transitive.<sup>22</sup> If  $<$  is a partial order, we will often write  $a > b$  instead of  $b < a$ ,  $a \leq b$  instead of  $a < b$  or  $a = b$ , and  $a \geq b$  instead of  $a > b$  or  $a = b$ . A *strictly partially ordered set* is an ordered pair  $\langle A, < \rangle$ , where  $A$  is a set and  $<$  is a partial order on  $A$ . If  $\langle A, < \rangle$  is a strictly partially ordered set, if  $B \subseteq A$ , and if  $<'$  is

<sup>22</sup>" $<$ " and similar symbols are commonly used to denote order relations.

the restriction of  $<$  to  $B$ , then  $\langle B, <' \rangle$  is also a strictly partially ordered set. A (*nonstrict*) *partial order* on a class  $A$  is a relation  $\leq$  on  $A$  derived from a strict partial order  $<$  as above (with  $a \leq b$  meaning  $a < b$  or  $a = b$ ). Such a relation can also be characterized by saying that it is reflexive, antisymmetric, and transitive. We can then obtain, or retrieve, the strict partial order  $<$  from  $\leq$  by putting  $a < b$  as  $a \leq b$  and  $a \neq b$ , for all  $a, b \in A$ . A *nonstrictly partially ordered set* is an ordered pair  $\langle A, \leq \rangle$  where  $A$  is a set and  $\leq$  is a (nonstrict) partial order on  $A$ . Any restriction of a (nonstrict) partial order on  $A$  to a subclass  $B \subseteq A$  is itself a (nonstrict) partial order.

For every set  $A$ ,  $\langle \mathcal{P}(A), \subseteq_{\mathcal{P}(A)} \rangle$  is a (nonstrict) partially ordered set, where the partial order  $\subseteq_{\mathcal{P}(A)}$  is the restriction of the subset relation to  $\mathcal{P}(A)$ . The corresponding strict partial order is  $\subset_{\mathcal{P}(A)}$ , the restriction of the proper subset relation to  $\mathcal{P}(A)$ . If  $<$  is a strict partial order on a class  $A$ , then an element  $m \in A$  is called a *maximal* element (in  $A$  with respect to  $<$ ) if there is no  $a \in A$  such that  $m < a$ , and a *greatest element* (of  $A$  with respect to  $<$ ) if  $a \leq m$  for all  $a \in A$ . Clearly, a greatest element  $m$ , if it exists, is unique, and then,  $m$  is also a unique maximal element. But in general, several maximal elements may exist. If  $<'$  is the restriction of strict partial order on a class  $A$  to a subclass  $B \subseteq A$ , then an *upper bound* of  $B$  (in  $A$  with respect to  $<'$ ) is an element  $a \in A$  such that  $b \leq a$  for all  $b \in B$ . Similarly for *minimal* element, *smallest* element, and *lower bound* if we substitute  $>$  for  $<$ , and  $\geq$  for  $\leq$ . All these definitions can be made equally well for nonstrict partial orders, and this is also true for the succeeding definitions.

$<$  is (*strict*) *total order* on a class  $A$ , if  $<$  is a partial order on  $A$ , and, for all  $a, b \in A$ , either  $a < b$ ,  $a = b$  or  $a > b$  holds (it easy to see that at most one of these can hold). In this case, if  $A$  is a set, the pair  $\langle A, < \rangle$  is called a (strictly) *totally ordered set*. The corresponding relation  $\leq$  is then a (*nonstrict*) *total order*, and if  $A$  is a set, then  $\langle A, \leq \rangle$  is called a (*nonstrictly*) *totally ordered class*.

If  $<$  is a strict total order on the class  $A$ , then  $A$  can have at most one maximal element with respect to  $A$ , which is then the greatest element. Similarly for minimal element and smallest element.

If  $<$  is a strict total order on  $A$ , if  $B \subseteq A$ , and if  $<'$  is the restriction of  $<$  to  $B$ , then  $<'$  is also a strict total order (on  $B$ ). Let  $M \subseteq A$  be the class of all upper bounds of  $B$  (in  $A$  with respect to  $<$ ). If  $M$  has a (necessarily unique) smallest element  $m$ , this  $m$  is called the *supremum* of  $B$  with respect to  $<'$ , and it is denoted  $\sup_{b \in B} b$  or shorter  $\sup B$ . If  $\sup B = m \in B$ , then  $m$  is actually the greatest element of  $B$  with respect to  $<'$ , and it is denoted  $\max_{b \in B} b$  or  $\max B$ . The corresponding concepts for  $>$  are the *infimum* of  $B$ , denoted  $\inf_{b \in B} b$  or  $\inf B$ , and  $\min_{b \in B} b$  or  $\min B$ .

If  $<$  is a strict total order on the class  $A$ , then a (*proper*) *initial segment* of  $A$ , (with respect to  $<$ ), is a proper subclass  $B \subset A$  such that, for all  $a \in A$  and  $b \in B$ ,  $a \in B$  whenever  $a < b$ .

If  $<$  is a strict partial order on a class  $A$ , then a subclass  $B \subseteq A$  is called a *chain* in  $A$ , if the restriction of  $<$  to  $B$  is a strict *total* order.

If  $\langle A_1, <_1 \rangle$  and  $\langle A_2, <_2 \rangle$  are totally ordered sets, then a *similarity map* from  $A_1$  to  $A_2$  (with respect to  $<_1$  and  $<_2$ ) is a surjection  $f : A_1 \rightarrow A_2$  such that  $f(a) <_2 f(b)$  whenever  $a <_1 b$ , for all  $a, b \in A_1$ . (Here, it is essential the orders  $<_1$  and  $<_2$  are total.) Clearly, a similarity map is actually a bijection, with an inverse which is also a similarity map. Also, the composition of of two similarity maps is a similarity map. The identity function  $\text{id}_A$  on a set  $A$  is a similarity map from  $A$

to itself, with respect to any total order  $<$  on  $A$ . We say that a totally ordered set  $\langle A_1, <_1 \rangle$  is *similar* to a totally ordered set  $\langle A_2, <_2 \rangle$  if there exists similarity map from  $A_1$  to  $A_2$ , with respect to  $<_1$  and  $<_2$ . Then, for all strictly totally ordered sets  $\langle A_1, <_1 \rangle$ ,  $\langle A_2, <_2 \rangle$ , and  $\langle A_3, <_3 \rangle$ :

- (i)  $\langle A_1, <_1 \rangle$  is similar to itself.
- (ii) If  $\langle A_1, <_1 \rangle$  is similar to  $\langle A_2, <_2 \rangle$ , then  $\langle A_2, <_2 \rangle$  is similar to  $\langle A_1, <_1 \rangle$ .
- (iii) if  $\langle A_1, <_1 \rangle$  is similar to  $\langle A_2, <_2 \rangle$ , and  $\langle A_2, <_2 \rangle$  is similar to  $\langle A_3, <_3 \rangle$ , then  $\langle A_1, <_1 \rangle$  is similar to  $\langle A_3, <_3 \rangle$ .

This means that similarity is an equivalence relation on the class of totally ordered sets.

A strict partial order  $<$  on a class  $A$  is called a *well order* if every nonempty subclass  $B \subseteq A$ , has a smallest element, with respect to  $<$ . If  $A$  is a set, then  $\langle A, < \rangle$  is then called a *well ordered set*. By looking at subsets with two elements, we see that every well order is a total order, and every well ordered set is a totally ordered set. Also, for every  $B \subseteq A$ , the restriction  $<'$  of  $<$  to  $B$  is a well order.

Also, we see that if  $<$  is a well order on the class  $A$ , and  $B$  is a subclass of  $A$  which has an upper bound in  $A$  with respect to  $<$ , then  $\sup B$  exists. It is also obvious that a totally ordered set which is similar to a well ordered set, is itself well ordered.

It is convenient to introduce a class of "canonical" well ordered sets, such that every well ordered set is similar to exactly one of those. Because then, the problem of studying well ordered sets is reduced to the study of these "canonical" well ordered sets, which we call *ordinals*. There are several possible definitions. We will choose one of the most common, and elegant, definitions. We then define a *natural number* as a special case of ordinal.

For every set  $A$ , we let  $\in_A$  denote the relation  $\{\langle a, b \rangle \in A \times A \mid a \in b\}$ . A set  $A$  is called *transitive* if, for all  $a$ ,  $a \in A$  implies  $a \subseteq A$ . We define the *successor* of a set  $A$  as  $S(A) = A \cup \{A\}$ . We say that  $A$  is a *successor set* if there exists a set  $B$  such that  $A = S(B)$ .

We now define an *ordinal* as a transitive set  $\alpha$  such that  $\langle \alpha, \in_\alpha \rangle$  is a well ordered set. If  $\alpha$  and  $\beta$  are ordinals, then we write  $\alpha < \beta$  instead of  $\alpha \in \beta$ ,  $\alpha > \beta$  instead of  $\beta < \alpha$ ,  $\alpha \leq \beta$  instead of  $\alpha < \beta$  or  $\alpha = \beta$ , and  $\alpha \geq \beta$  instead of  $\alpha > \beta$  or  $\alpha = \beta$ . This is consistent with what we have done above, because it turns out that the restriction of the membership relation  $\in$  to the class of ordinals is a well order. Hence,  $\langle A, \in_A \rangle$  is a well ordered set if  $A$  is a set of ordinals. Moreover, the elements of an ordinal  $\alpha$  are all ordinals, in fact, precisely those ordinals  $\beta$  such that  $\beta < \alpha$ . If  $\alpha$  and  $\beta$  are ordinals such that  $\alpha < \beta$ , then  $\alpha$  is a (proper) initial segment of  $\beta$ . Any set  $A$  of ordinals has an upper bound. In fact,  $\sup A = \cup_{\alpha \in A} \alpha$ , which itself is an ordinal. Also, if  $\alpha$  is an ordinal, then so is  $S(\alpha)$ . Also, if  $\alpha < \beta$ , then  $S(\alpha) < S(\beta)$ .  $\emptyset$  is an ordinal (since all conditions in the definition are vacuously satisfied for  $\emptyset$ ), the smallest of all ordinals. There are two other types of ordinals: *successor ordinals*, i.e. ordinals which are successor sets, and *limit ordinals*, i.e. nonempty ordinals which are not successor ordinals. The most important property of the ordinals is that every well ordered set is similar to  $\langle \alpha, \in_\alpha \rangle$ , for some unique ordinal  $\alpha$ . It can be proved that the class of all ordinals is not set, but a proper class.

We define a *natural number* as an ordinal  $n$  such that every nonempty element in  $S(n)$  is a successor set. Then, every element in a natural number is a natural number,  $S(n)$  is a natural number if  $n$  is, and  $\emptyset$  is a natural number. Clearly, all

nonempty natural numbers are successor ordinals. The natural numbers are also called *finite ordinals*, while these ordinals which are not natural numbers are called *transfinite ordinals*.

We now define the natural numbers  $0, 1, 2, 3, \dots$  by  $0 = \emptyset$ ,  $1 = S(0) = \{0\}$ ,  $2 = S(1) = \{0, 1\}$ ,  $3 = S(2) = \{0, 1, 2\}$ , etc. We can define addition and multiplication of ordinals, but we shall not go into that in any depth, we just put  $\alpha + 1 = S(\alpha)$ ,  $\alpha + 2 = S(\alpha + 1)$ ,  $\alpha + 3 = S(\alpha + 2)$ , etc. for all ordinals  $\alpha$ . (In general,  $\alpha + (n + 1) = S(\alpha + n)$ , for all natural numbers  $n$ .)

A very important method of proof is *transfinite induction*, which generalizes ordinary induction to transfinite ordinals:

**TRANSFINITE INDUCTION.** Let  $\phi(x)$  be a condition. Assume that, for every ordinal  $\alpha$ , whenever  $\phi(\beta)$  holds for all ordinals  $\beta < \alpha$ , then  $\phi(\alpha)$  holds too.

Then,  $\phi(\alpha)$  holds for all ordinals  $\alpha$ .

It may seem that we must include a clause that  $\phi(0)$  should hold, but this is not necessary, since the condition "  $\phi(\beta)$  holds for all ordinals  $\beta < 0$ " is vacuously true. An alternative formulation is that the above condition is only assumed to hold if  $\alpha$  is a limit ordinal, and that we also assume that  $\phi(0)$  holds, and that, for all ordinals  $\alpha$ ,  $\phi(\alpha + 1)$  holds whenever  $\phi(\alpha)$  holds.

Related to this, there is the method of defining functional relations with transfinite recursion:

**TRANSFINITE RECURSION.** Let  $G$  be a functional relation defined on the class of all subsets of the class  $On \times A$ , where  $On$  is the class of all ordinals, and  $A$  is any class.

Then, there is a functional relation  $F$  defined on  $On$  such that  $F(\alpha) = G(\{\langle \beta, F(\beta) \rangle \mid \beta < \alpha\})$ , for all ordinals  $\alpha$ .

This  $F$  has the property that  $F(\alpha) \in A$  for every ordinal  $\alpha$ , and  $F$  is unique in the sense that if  $F'$  is another functional relation satisfying these conditions, then  $F(\alpha) = F'(\alpha)$  for all ordinals  $\alpha$ .

Here, too, we may have separate defining conditions for  $F(\alpha)$  if  $\alpha = 0$ , or  $\alpha$  is a successor ordinal, or if  $\alpha$  is a limit ordinal, respectively, as long as they can be summarized into a single one, as above.

Transfinite induction and recursion can both be modified so that instead of the class of all ordinals,  $On$ , we may use classes of the types  $\{\gamma \mid \gamma \geq \alpha\}$  and  $\{\gamma \mid \alpha \leq \gamma < \beta\}$ , for arbitrary ordinals  $\alpha$  and  $\beta$  ( $\alpha < \beta$ ). In the last case, transfinite recursion can be said to define a function instead of a functional relation.

The theory of transfinite ordinals looks very promising, and it seems to generate a lot of interesting results. But, with the axioms we have seen so far, it is, as far as we know, impossible to prove that there exist any transfinite ordinals at all. Indeed, we can only prove that there exists arbitrary large *finite sets*, not infinite sets. For this, we need a new axiom:

*Axiom of Infinity:* There exists a set  $A$  such that  $\emptyset \in A$ , and such that, for every set  $x$ ,  $S(x) \in A$  whenever  $x \in A$ .

It follows from this axiom that the class of natural numbers is a set. In fact, it is even an ordinal, which we denote by  $\omega$  when we want to emphasize that it is an ordinal, while we denote it by  $\mathbb{N}$  if we only want to view it as a collection of numbers.  $\omega$  is then a transfinite ordinal, the smallest one.

One can now prove that the set  $\mathbb{N}$ , with the successor functional relation  $S(x)$  restricted to  $\mathbb{N}$ , satisfies Peano's axioms (see e.g [6], p. 116), in particular that we can prove theorems with ordinary induction, and we can also define functions with ordinary recursion. Thus, arithmetic can be viewed as a subdiscipline of set theory.

We say that a set is *finite* if it is equinumerous with some natural number, and that it is *infinite* otherwise.

To get a more precise measure of the size of a set, we define a *cardinal* as an ordinal  $\alpha$  such that  $\alpha \not\sim \beta$  for all ordinals  $\beta$  such that  $\gamma < \beta$ . An cardinal is also called an *initial ordinal*, because of this definition.

It follows that if  $A$  is a set which can be well ordered, i.e. if there exists a well order  $<'$  on  $A$ , then there is a unique cardinal  $\kappa$  such that  $A \sim \kappa$ . This unique cardinal  $\kappa$  is called the *cardinality* of  $A$ , and it is denoted  $\text{card } A$ . Whenever we talk about the cardinality of a set, it is assumed that the set can be well ordered.

Every natural number is a cardinal, and a finite set with  $n$  distinct elements has cardinality  $n$ . From this, it follows that any set  $A$  which has a *proper subset*  $B \subset A$  such that  $A \sim B$ , must be infinite. Even more remarkable things are true for infinite sets. For example, it can be proved that if  $A$  is set such that  $\text{card } A \leq \kappa$ , and  $\text{card } a \leq \kappa$  for all  $a \in A$ , where  $\kappa$  is a transfinite cardinal, then  $\text{card } \cup_{a \in A} a \leq \kappa$ , provided that there exists a function  $f$  with domain  $A$  such that  $f(a)$  is an injection from  $a$  into  $\kappa$ , for every  $a \in A$ . (We can then use the function  $f$ , transfinite induction, and the Schröder-Bernstein theorem to define a well order  $<'$  on  $U = \cup_{a \in A} a$ , such that  $\langle U, <' \rangle$  is similar to  $\langle \alpha, \in_\alpha \rangle$ , for some ordinal  $\alpha \leq \kappa$ .) A simple consequence of this is that if  $A$  is a set with transfinite cardinality  $\kappa$ , and if  $B$  and  $C$  are sets such that  $\text{card } B < \text{card } A$  and  $\text{card } C \leq \text{card } A$ , then  $\text{card } (A \setminus B) = \text{card } (A \cup C) = \text{card } A = \kappa$ . Thus, if  $\alpha$  is a transfinite ordinal, then  $\text{card } (\alpha + 1) = \text{card } \alpha$ . It follows that every transfinite cardinal is a limit ordinal. In particular,  $\omega$  is the smallest transfinite cardinal, and we write  $\aleph_0$  instead of  $\omega$  when we want to emphasize that it is a cardinal, and not just an ordinal. By transfinite recursion, we can define a functional relation  $\aleph_\alpha$  such that for every ordinal  $\alpha$ ,  $\aleph_\alpha$  is the the smallest transfinite cardinal greater than all  $\aleph_\beta$  for which  $\beta < \alpha$ .  $\aleph_0$  is then as above, and the class of all  $\aleph_\alpha$ , where  $\alpha$  ranges over the class of all ordinals, is the same as the class of all transfinite cardinals. This class is a proper class.

Not all limit ordinals are (transfinite) cardinals. For example, the ordinal  $\omega + \omega$ , which is defined as the smallest ordinal which is greater than  $\omega + n$  for all natural numbers  $n$ , is a limit ordinal, but not a cardinal. Actually,  $\text{card } (\omega + \omega) = \aleph_0 = \text{card } \omega$ .

A set  $A$  is *countably infinite* if  $\text{card } A = \aleph_0$ , *countable* if  $\text{card } A \leq \aleph_0$  (i.e. if it is finite or countably infinite), and *uncountable* if it is not countable.

Now, if the sets  $A$  and  $B$  can be well ordered (and hence they have well defined cardinalities), then one of  $A \lesssim B$  and  $B \lesssim A$  must hold, because this is so for ordinals (since for every pair of distinct ordinals, one is a proper initial segment of the other) and every well ordered set  $\langle A, <' \rangle$  is similar to  $\langle \alpha, \in_\alpha \rangle$  for some ordinal  $\alpha$ . Thus, if we could establish that every set can be well ordered, then one of  $A \lesssim B$  and  $B \lesssim A$  must hold, for all sets  $A$  and  $B$ . Equivalently, this latter property can be expressed thus: Exactly one of  $A \prec B$ ,  $A \sim B$  and  $B \prec A$  holds, for all sets  $A$  and  $B$ . This property is called the *Trichotomy Law*.

If we use the next theorem, we can, conversely, prove that the trichotomy law implies that every set can be well ordered:

**HARTOGS' THEOREM.** *For every set  $A$ , there exists an ordinal  $\alpha$  such that  $\alpha \not\prec A$ .*

Now, if the trichotomy law holds, Hartogs' theorem implies that to every set  $A$ , there is an ordinal  $\alpha$  such that  $A \lesssim \alpha$ . This means that  $A$  is equinumerous to a subset of an ordinal, which can be well ordered. Thus,  $A$  can be well ordered.

Unfortunately, the axioms given so far are not sufficient (as far as we know) for establishing the *Well Ordering Principle*, i.e. that every set can be well ordered, and the trichotomy law. For this, we need the axiom of choice, which we give in two equivalent versions:

*Axiom of Choice, version 1:* Let  $A$  be a sets whose elements are mutually disjoint, nonempty sets. Then, there exists a set  $B$  which has exactly one element in common with each of the sets  $a \in A$ .

*Axiom of Choice, version 2:* For every set  $A$ , there exists a function  $f : \mathcal{P}(A) \setminus \{\emptyset\} \rightarrow A$  such that  $f(a) \in a$  for all  $a \in \mathcal{P}(A) \setminus \{\emptyset\}$ .

The axiom of choice is equivalent to a lot of properties. Let us list some of the most important ones:

**THEOREM.** *The following are equivalent:*

- (1) *Axiom of Choice, version 1.*
- (2) *Axiom of Choice, version 2.*
- (3) *Every set can be well ordered.*
- (4) *For all sets  $A$  and  $B$ : Either  $A \lesssim B$  or  $B \lesssim A$ .*
- (5) *For all sets  $A$  and  $B$ : One of  $A \prec B$ ,  $A \sim B$ , and  $B \prec A$  holds (*Trichotomy Law*).*
- (6) *Zorn's Lemma: If  $\langle A, <' \rangle$  is a partially ordered set such that every chain in  $A$  has an upper bound, then  $A$  has a maximal element.*
- (7) *Hausdorff's Maximality Principle: If  $\langle A, <' \rangle$  is a partially ordered set, then there is a maximal chain in  $A$ , i.e. a chain which is not a proper subset of any other chain in  $A$ .*

We will not prove this, but we see that the implication (3)  $\Rightarrow$  (2) is easy to prove: Just take a well order  $<'$  on  $A$ , and define  $f(a)$  as the smallest element in  $a$ .

Other consequences of the axiom of choice are:

- (1)  $\text{card } \cup_{a \in A} a \leq \kappa$  holds whenever  $\text{card } A \leq \kappa$  and  $\text{card } a \leq \kappa$  for all  $a \in A$ , as above, because the axiom of choice implies that a function such as  $f$  above exists.
- (2)  $\text{card } \mathcal{P}(A) > \text{card } A$ , for every set  $A$ .
- (3) If  $A \not\prec \kappa$ , then  $A$  has a subset with cardinality  $\kappa$ .

The axiom of choice seems simple and intuitive, but there are mathematicians who are skeptical against it. One reason is that it has some very weird consequences. The weirdest is probably the Banach-Tarski paradox: Every pair of bounded sets  $A, B \in \mathbb{R}^3$  with nonempty interiors, can be decomposed into a common finite number of disjoint subsets  $A_1, A_2, \dots, A_n$  and  $B_1, B_2, \dots, B_n$ , respectively such that for each  $i$  ( $1 \leq i \leq n$ ),  $A_i$  and  $B_i$  are geometrically congruent (even without using reflections). Here,  $A$  may be chosen as very small, and  $B$  may be chosen as very big! This may seem like an outright contradiction, but it is not: The parts  $A_i$

and  $B_i$  are chosen nonconstructively, with the axiom of choice, as sets on which no volume can be defined.

Our last axiom is:

*Axiom of Regularity.* Every nonempty set  $A$  has an element  $a \in A$  such that  $A \cap a = \emptyset$ .

It is also called the *Axiom of Restriction*, or the *Axiom of Foundation*.

This axiom is treated in Chapter 3. Let us mention one consequence:

If the axiom of regularity holds, then  $\alpha$  is an ordinal if and only if every element in  $S(\alpha)$  is transitive. This gives a somewhat simpler characterization of an ordinal than the definition given above.

ZFC is the theory with all these axioms. ZF is the theory with all these axioms except the axiom of choice. *However*, in this text, we *never* assume that the axiom of regularity holds, and, unless otherwise is stated, we do not assume that the axiom of choice holds either. The other axioms, we use freely.

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